

A generalization of Calderón-Vaillancourt's Theorem

東北大・理・数学 立澤 一哉 (Kazuya Tachizawa)

1 Introduction

In this paper we study L^2 boundedness of pseudodifferential operators with Weyl symbols. We give a generalization of Calderón-Vaillancourt's theorem.

For a symbol $\sigma \in \mathcal{S}'(\mathbf{R}^{2d})$ we associate a pseudodifferential operator L_σ . If $\sigma \in L^2(\mathbf{R}^{2d})$, then we can easily prove that L_σ extends to a bounded operator on $L^2(\mathbf{R}^d)$. On the other hand Calderón and Vaillancourt gave another condition for the L^2 boundedness of pseudodifferential operators([1], [2]). Their condition is about pseudodifferential operators with Kohn-Nirenberg symbols. Similar results hold for the Weyl symbol case([3], [10]). A generalization of their results is known([11]). This generalization does not contain the L^2 symbol case. In this paper we give a generalization of both results.

First we give the definition of pseudodifferential operators with Weyl symbols.

Let $W(f, g)$ be the Wigner transform of $f, g \in \mathcal{S}(\mathbf{R}^d)$, that is,

$$W(f, g)(x, \xi) = \int_{\mathbf{R}^d} e^{-2\pi i \xi \cdot p} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)} dp$$

for $x, \xi \in \mathbf{R}^d$.

For $\sigma \in \mathcal{S}'(\mathbf{R}^{2d})$ and $f \in \mathcal{S}(\mathbf{R}^d)$, we define $L_\sigma f \in \mathcal{S}'(\mathbf{R}^d)$ as

$$(L_\sigma f, g) = (\sigma, W(g, f))$$

for all $g \in \mathcal{S}$, where we use the notation $(F, f) = \langle F, \bar{f} \rangle$ for $F \in \mathcal{S}'$, $f \in \mathcal{S}$. It turns out that L_σ is a continuous linear operator from $\mathcal{S}$ to $\mathcal{S}'$. We call L_σ a pseudodifferential operator with Weyl symbol σ (cf. [10]).

In Folland [10] it is proved that if $\sigma \in C^{2d+1}(\mathbf{R}^{2d})$ and

$$\sum_{|\alpha|+|\beta| \leq 2d+1} \|\partial_x^\alpha \partial_\xi^\beta \sigma\|_\infty < \infty,$$

then L_σ extends to a bounded operator on $L^2(\mathbf{R}^d)$. This is the Calderón-Vaillancourt theorem for pseudodifferential operators with Weyl symbols.

In [11] Gröchenig and Heil proved a generalization of Calderón-Vaillancourt's theorem. If $\sigma \in M_{\infty,1}(\mathbf{R}^{2d})$, then L_σ extends to a bounded operator on $L^2(\mathbf{R}^d)$, where $M_{\infty,1}(\mathbf{R}^{2d})$ denotes the set of $F \in \mathcal{S}'(\mathbf{R}^{2d})$ satisfying

$$\int_{\mathbf{R}^{2d}} \sup_{x \in \mathbf{R}^{2d}} |(F, e^{i\xi \cdot -(\cdot - x)^2})| d\xi < \infty.$$

In this paper we give a generalization of these results.

Let

$$(1) \quad \varphi(x) = 2^{d/4} e^{-\pi|x|^2}, \quad x \in \mathbf{R}^d$$

and

$$\phi(x, y) = W(\varphi, \varphi)(x, y) = 2^d e^{-2\pi(|x|^2 + |y|^2)}, \quad (x, y) \in \mathbf{R}^{2d}.$$

For $\sigma \in \mathcal{S}'(\mathbf{R}^{2d})$ and $\alpha, \beta \in \mathbf{R}^{2d}$ we define

$$S_\phi(\sigma)(\alpha, \beta) = (\sigma, e^{2\pi i \beta \cdot} \phi(\cdot - \alpha)).$$

For $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2), \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbf{R}^d$, we set

$$N(\alpha, \beta) = \left(-\frac{\alpha_1 + \beta_1}{2}, \frac{\alpha_2 + \beta_2}{2}, \alpha_2 - \beta_2, \alpha_1 - \beta_1\right).$$

For $\xi, \eta \in \mathbf{R}^{2d}$ we set

$$F(\xi, \eta) = |S_\phi(\sigma)(N(\xi, \eta))|$$

and

$$(2) \quad k(\xi) = (1 + |\xi|)^{-2d-1}.$$

For each positive integer n and $a, b \in \mathbf{R}^{2d}$ we set

$$\begin{aligned} G_n(a, b) = & \left[\int_{\mathbf{R}^{2d}} \cdots \int_{\mathbf{R}^{2d}} k(a - \eta_1) F(\xi_1, \eta_1) k(\xi_1 - \xi'_1) F(\xi'_1, \eta'_1) k(\eta'_1 - \eta_2) \right. \\ & \times F(\xi_2, \eta_2) k(\xi_2 - \xi'_2) F(\xi'_2, \eta'_2) k(\eta'_2 - \eta_3) \cdots \\ & \left. \times F(\xi_n, \eta_n) k(\xi_n - \xi'_n) F(\xi'_n, \eta'_n) k(\eta'_n - b) d\xi_1 d\eta_1 \cdots d\xi'_n d\eta'_n \right]^{1/n}. \end{aligned}$$

Theorem 1.1 *We assume*

$$G_n(a, b) < \infty$$

for every $n \in \mathbf{N}$, $a, b \in \mathbf{R}^{2d}$ and

$$\sup_{n \in \mathbf{N}} \sup_{a \in \mathbf{R}^{2d}} G_n(a, a) < \infty.$$

Then L_σ extends to a bounded operator on $L^2(\mathbf{R}^d)$.

Corollary 1.1 *Let $F(\xi, \eta)$ be a function as above. For $1 \leq p \leq \infty$ and $p^{-1} + p'^{-1} = 1$ we assume*

$$\left\{ \int_{\mathbf{R}^{2d}} \left(\int_{\mathbf{R}^{2d}} F(\xi, \eta)^p d\xi \right)^{p'/p} d\eta \right\}^{1/p'} < \infty$$

and

$$\left\{ \int_{\mathbf{R}^{2d}} \left(\int_{\mathbf{R}^{2d}} F(\xi, \eta)^p d\eta \right)^{p'/p} d\xi \right\}^{1/p'} < \infty.$$

Then L_σ extends to a bounded operator on $L^2(\mathbf{R}^d)$.

Remark 1.1 *The case $p = 1$ or $p = \infty$ is mentioned in Gröchenig and Heil [11]. They used this fact to prove their result on L^2 boundedness of pseudodifferential operators. When $p = 2$, the conditions in the corollary is equivalent to saying $\sigma \in L^2(\mathbf{R}^{2d})$. When $1 < p < 2$ or $2 < p < \infty$, our corollary gives a new result.*

Remark 1.2 *We can prove similar results about the boundedness of pseudodifferential operators on Sobolev spaces.*

2 Proof of Theorem 1.1

First we recall the definition of the Weyl-Heisenberg frame. For $a = (x, y) \in \mathbf{R}^d \times \mathbf{R}^d$ and $f \in L^2(\mathbf{R}^d)$ we set

$$\rho(a)f(t) = \rho(x, y)f(t) = e^{\pi i x \cdot y} e^{2\pi i y \cdot t} f(t + x),$$

where $t \in \mathbf{R}^d$.

Let $\varphi(x)$ be the function defined by (1). Let $I = \mathbf{Z}^d \times \frac{1}{2\pi} \mathbf{Z}^d$. Then $\{\rho(a)\varphi\}_{a \in I}$ is a frame of $L^2(\mathbf{R}^d)$, that is, there exist positive constants C_1 and C_2 such that

$$C_1 \|f\|_2^2 \leq \sum_{a \in I} |(f, \rho(a)\varphi)|^2 \leq C_2 \|f\|_2^2$$

for all $f \in L^2(\mathbf{R}^d)$. The dual frame of $\{\rho(a)\varphi\}_{a \in I}$ is given by $\{\rho(a)\tilde{\varphi}\}_{a \in I}$, where $\tilde{\varphi}$ is the function in $\mathcal{S}(\mathbf{R}^d)$ which is constructed by φ . Furthermore we have

$$C_2^{-1} \|f\|_2^2 \leq \sum_{a \in I} |(f, \rho(a)\tilde{\varphi})|^2 \leq C_1^{-1} \|f\|_2^2$$

for all $f \in L^2(\mathbf{R}^d)$ (cf.[5]).

By the frame theory we have the following proposition.

Proposition 2.1 (i) *For every $f \in L^2(\mathbf{R}^d)$ we have*

$$(3) \quad f = \sum_{a \in I} (f, \rho(a)\varphi) \rho(a)\tilde{\varphi}$$

$$(4) \quad = \sum_{a \in I} (f, \rho(a)\tilde{\varphi}) \rho(a)\varphi$$

which converge in $L^2(\mathbf{R}^d)$.

(ii) *There exists a $K > 0$ such that*

$$\left\| \sum_{a \in I} c_a \rho(a) \varphi \right\|_2 \leq K \left(\sum_{a \in I} |c_a|^2 \right)^{1/2}$$

for all $\{c_a\} \in \ell^2(I)$.

(iii) *For every $f \in \mathcal{S}(\mathbf{R}^d)$ we have the expansions (3) and (4) in $\mathcal{S}$.*

(iv) *For every $f \in \mathcal{S}'(\mathbf{R}^d)$ we have the expansions (3) and (4) in $\mathcal{S}'$.*

The proofs of (i) and (ii) is in [5]. The properties (iii) and (iv) are consequences of Feichtinger and Gröchenig's result ([7], [8], [9], [14]).

Let $f \in \mathcal{S}(\mathbf{R}^d)$. By (iii) and (iv) of Proposition 2.1 we have

$$L_\sigma f = \sum_{a \in I} (L_\sigma f, \rho(a) \tilde{\varphi}) \rho(a) \varphi$$

in $\mathcal{S}'$.

If we show

$$\sum_{a \in I} |(L_\sigma f, \rho(a) \tilde{\varphi})|^2 < \infty,$$

then we conclude $L_\sigma f \in L^2(\mathbf{R}^d)$ and

$$\|L_\sigma f\|_2^2 \leq C \sum_{a \in I} |(L_\sigma f, \rho(a) \tilde{\varphi})|^2,$$

where we used (ii) of Proposition 2.1.

Here we have

$$\sum_{a \in I} |(L_\sigma f, \rho(a) \tilde{\varphi})|^2 = \sum_{a \in I} \left| \sum_{b \in I} (f, \rho(b) \tilde{\varphi}) (L_\sigma \rho(b) \varphi, \rho(a) \tilde{\varphi}) \right|^2.$$

If the infinite matrix $\{(L_\sigma \rho(b) \varphi, \rho(a) \tilde{\varphi})\}_{a,b \in I}$ is bounded on $\ell^2(I)$, then we conclude that

$$\|L_\sigma f\|_2^2 \leq C \sum_{a \in I} |(f, \rho(a) \tilde{\varphi})|^2 \leq C' \|f\|_2^2 < \infty.$$

Therefore L_σ extends to a bounded operator on $L^2(\mathbf{R}^d)$.

Now we use the following lemma to show the boundedness of an infinite matrix on ℓ^2 .

Lemma 2.1 ([4]) *Let $A = (a_{ij})$ be an infinite matrix which acts on the sequence space $\ell^2(\mathbf{N})$. Then the boundedness of A from $\ell^2(\mathbf{N})$ to $\ell^2(\mathbf{N})$ is equivalent to the following two conditions.*

(a) *For every $n \in \mathbf{N}$, $(A^*A)^n$ is well defined.*

(b)

$$\sup_{n \in \mathbf{N}} \sup_{i \in \mathbf{N}} |(A^*A)^n]_{ii}|^{1/n} < \infty,$$

where $[(A^*A)^n]_{ii}$ is the (i, i) component of $(A^*A)^n$.

Remark 2.1 In [4] Crone gave an additional condition. In [13] Maddox and Wickstead showed that the condition is unnecessary(cf.[12]).

We shall show that the infinite matrix $\{(L_\sigma \rho(b)\varphi, \rho(a)\tilde{\varphi})\}_{a,b \in I}$ satisfies the conditions of Lemma 2.1. In order to prove this we use the following lemma by Gröchenig and Heil [11].

Lemma 2.2 *For $\sigma \in \mathcal{S}'(\mathbf{R}^{2d})$ and $f, g \in \mathcal{S}(\mathbf{R}^d)$, we have*

$$(L_\sigma f, g) = \int_{\mathbf{R}^{2d}} \int_{\mathbf{R}^{2d}} S_\phi(\sigma)(N(\alpha, \beta)) e^{-2\pi i[\alpha, \beta]} (f, \rho(\beta)\varphi)(\rho(\alpha)\varphi, g) d\alpha d\beta,$$

where $[\alpha, \beta] = \frac{1}{2}(\alpha_2 \cdot \beta_1 - \alpha_1 \cdot \beta_2)$ for $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbf{R}^d \times \mathbf{R}^d$.

First we show the condition (a) of the Lemma 2.1.

Let $A = \{(L_\sigma \rho(b)\varphi, \rho(a)\tilde{\varphi})\}_{a,b \in I}$, $n \in \mathbf{N}$ and $c, d \in I$. The (c, d) component of the infinite matrix $(A^*A)^n$ is given by

$$\begin{aligned} & \sum_{a_1, \dots, a_n, b_1, \dots, b_{n-1} \in I} \overline{(L_\sigma \rho(c)\varphi, \rho(a_1)\tilde{\varphi})} (L_\sigma \rho(b_1)\varphi, \rho(a_1)\tilde{\varphi}) \overline{(L_\sigma \rho(b_1)\varphi, \rho(a_2)\tilde{\varphi})} \\ & \times (L_\sigma \rho(b_2)\varphi, \rho(a_2)\tilde{\varphi}) \cdots \overline{(L_\sigma \rho(b_{n-1})\varphi, \rho(a_n)\tilde{\varphi})} (L_\sigma \rho(d)\varphi, \rho(a_n)\tilde{\varphi}), \end{aligned}$$

where we will show that the series in the above sum absolutely converges.

By Lemma 2.2 we have

$$\begin{aligned}
& \overline{(L_\sigma \rho(c)\varphi, \rho(a_1)\tilde{\varphi})} (L_\sigma \rho(b_1)\varphi, \rho(a_1)\tilde{\varphi}) \overline{(L_\sigma \rho(b_1)\varphi, \rho(a_2)\tilde{\varphi})} \\
& \quad \times (L_\sigma \rho(b_2)\varphi, \rho(a_2)\tilde{\varphi}) \cdots \overline{(L_\sigma \rho(b_{n-1})\varphi, \rho(a_n)\tilde{\varphi})} (L_\sigma \rho(d)\varphi, \rho(a_n)\tilde{\varphi}) \\
= & \int \int S_\phi(\sigma)(N(\xi_1, \eta_1)) e^{2\pi i[\xi_1, \eta_1]} \overline{(\rho(c)\varphi, \rho(\eta_1)\varphi)} (\rho(\xi_1)\varphi, \rho(a_1)\tilde{\varphi}) d\xi_1 d\eta_1 \\
& \times \int \int S_\phi(\sigma)(N(\xi'_1, \eta'_1)) e^{-2\pi i[\xi'_1, \eta'_1]} (\rho(b_1)\varphi, \rho(\eta'_1)\varphi) (\rho(\xi'_1)\varphi, \rho(a_1)\tilde{\varphi}) d\xi'_1 d\eta'_1 \cdots \\
& \times \int \int S_\phi(\sigma)(N(\xi_n, \eta_n)) e^{2\pi i[\xi_n, \eta_n]} \overline{(\rho(b_{n-1})\varphi, \rho(\eta_n)\varphi)} (\rho(\xi_n)\varphi, \rho(a_n)\tilde{\varphi}) d\xi_n d\eta_n \\
& \times \int \int S_\phi(\sigma)(N(\xi'_n, \eta'_n)) e^{-2\pi i[\xi'_n, \eta'_n]} (\rho(d)\varphi, \rho(\eta'_n)\varphi) (\rho(\xi'_n)\varphi, \rho(a_n)\tilde{\varphi}) d\xi'_n d\eta'_n.
\end{aligned}$$

Let $k(\xi)$ be the function defined by (2). Since

$$|(\rho(b)\varphi, \rho(\eta)\varphi)| \leq |(\rho(b-\eta)\varphi, \varphi)| \leq ck(b-\eta)$$

and

$$|(\rho(\xi)\varphi, \rho(a)\tilde{\varphi})| \leq |(\rho(\xi-a)\varphi, \tilde{\varphi})| \leq ck(\xi-a)$$

for all $a, b \in I$ and $\xi, \eta \in \mathbf{R}^{2d}$, we have

$$\begin{aligned}
& \sum_{a_1, \dots, a_n, b_1, \dots, b_{n-1} \in I} \int \cdots \int |S_\phi(\sigma)(N(\xi_1, \eta_1))| |(\rho(c)\varphi, \rho(\eta_1)\varphi)| |(\rho(\xi_1)\varphi, \rho(a_1)\tilde{\varphi})| \\
& \times |S_\phi(\sigma)(N(\xi'_1, \eta'_1))| |(\rho(b_1)\varphi, \rho(\eta'_1)\varphi)| |(\rho(\xi'_1)\varphi, \rho(a_1)\tilde{\varphi})| \cdots \\
& \times |S_\phi(\sigma)(N(\xi_n, \eta_n))| |(\rho(b_{n-1})\varphi, \rho(\eta_n)\varphi)| |(\rho(\xi_n)\varphi, \rho(a_n)\tilde{\varphi})| |S_\phi(\sigma)(N(\xi'_n, \eta'_n))| \\
& \times |(\rho(d)\varphi, \rho(\eta'_n)\varphi)| |(\rho(\xi'_n)\varphi, \rho(a_n)\tilde{\varphi})| d\xi_1 d\eta_1 d\xi'_1 d\eta'_1 \cdots d\xi_n d\eta_n d\xi'_n d\eta'_n \\
& \leq C^n \sum_{a_1, \dots, a_n, b_1, \dots, b_{n-1} \in I} \int \cdots \int F(\xi_1, \eta_1) k(c-\eta_1) k(\xi_1-a_1) F(\xi'_1, \eta'_1) k(b_1-\eta'_1) \\
& \times k(\xi'_1-a_1) F(\xi_2, \eta_2) \cdots k(\xi_n-a_n) F(\xi'_n, \eta'_n) k(d-\eta'_n) k(\xi'_n-a_n) d\xi_1 \cdots d\eta'_n.
\end{aligned}$$

Since

$$\sum_{b \in I} k(\xi-b) k(\xi'-b) \leq ck(\xi-\xi')$$

for all $\xi, \xi' \in \mathbf{R}^d$, the above quantity is bounded by

$$\begin{aligned} & C^n \int \cdots \int k(c - \eta_1) F(\xi_1, \eta_1) k(\xi_1 - \xi'_1) F(\xi'_1, \eta'_1) k(\eta'_1 - \eta_2) F(\xi_2, \eta_2) \cdots \\ & \quad \times k(\xi_n - \xi'_n) F(\xi'_n, \eta'_n) k(d - \eta'_n) d\xi_1 \cdots d\eta'_n \\ & = C^n G_n(c, d)^n < \infty. \end{aligned}$$

Hence we conclude that the (c, d) component of the infinite matrix $(A^*A)^n$ is well defined.

Next we check the condition (b). By similar calculations we have

$$\begin{aligned} & \left| \sum_{a_1, \dots, a_n, b_1, \dots, b_{n-1} \in I} \overline{(L_\sigma \rho(c) \varphi, \rho(a_1) \tilde{\varphi}) (L_\sigma \rho(b_1) \varphi, \rho(a_1) \tilde{\varphi})} \overline{(L_\sigma \rho(b_1) \varphi, \rho(a_2) \tilde{\varphi})} \right. \\ & \quad \left. \times (L_\sigma \rho(b_2) \varphi, \rho(a_2) \tilde{\varphi}) \cdots \overline{(L_\sigma \rho(b_{n-1}) \varphi, \rho(a_n) \tilde{\varphi})} (L_\sigma \rho(d) \varphi, \rho(a_n) \tilde{\varphi}) \right| \\ & \leq C^n G_n(a, a)^n. \end{aligned}$$

Hence n -th root of the quantity of the left hand side is bounded by $CG_n(a, a)$. By the assumption we get the condition (b).

3 Proof of Corollary 1.1

We shall prove Corollary 1.1 for $1 < p < \infty$. The proof for $p = 1$ or $p = \infty$ is similar.

We set

$$\begin{aligned} & \left\{ \int_{\mathbf{R}^{2d}} \left(\int_{\mathbf{R}^{2d}} F(\xi, \eta)^p d\xi \right)^{p'/p} d\eta \right\}^{1/p'} = K_1, \\ & \left\{ \int_{\mathbf{R}^{2d}} \left(\int_{\mathbf{R}^{2d}} F(\xi, \eta)^p d\eta \right)^{p'/p} d\xi \right\}^{1/p'} = K_2 \end{aligned}$$

and $K = \max\{K_1, K_2\}$. We shall estimate

$$\begin{aligned} (5) \quad & \int \cdots \int k(a - \eta_1) F(\xi_1, \eta_1) k(\xi_1 - \xi'_1) F(\xi'_1, \eta'_1) k(\eta'_1 - \eta_2) F(\xi_2, \eta_2) \\ & \quad \times k(\xi_2 - \xi'_2) \cdots F(\xi_n, \eta_n) k(\xi_n - \xi'_n) F(\xi'_n, \eta'_n) k(\eta'_n - b) d\xi_1 d\eta_1 \cdots d\xi'_n d\eta'_n \end{aligned}$$

By Hölder's inequality we have

$$\begin{aligned}
& \int k(a - \eta_1) F(\xi_1, \eta_1) d\eta_1 \\
& \leq \left(\int k(a - \eta_1)^{p'} d\eta_1 \right)^{1/p'} \left(\int F(\xi_1, \eta_1)^p d\eta_1 \right)^{1/p} \\
& \leq C_1 \left(\int F(\xi_1, \eta_1)^p d\eta_1 \right)^{1/p},
\end{aligned}$$

where we set

$$C_1 = \left(\int k(a - \eta_1)^{p'} d\eta_1 \right)^{1/p'}.$$

Hence (5) is bounded by

$$\begin{aligned}
& C_1 \int \cdots \int \left(\int F(\xi_1, \eta_1)^p d\eta_1 \right)^{1/p} k(\xi_1 - \xi'_1) F(\xi'_1, \eta'_1) k(\eta'_1 - \eta_2) F(\xi_2, \eta_2) k(\xi_2 - \xi'_2) \\
& \quad \cdots F(\xi_n, \eta_n) k(\xi_n - \xi'_n) F(\xi'_n, \eta'_n) k(\eta'_n - b) d\xi_1 d\xi'_1 \cdots d\xi'_n d\eta'_n.
\end{aligned}$$

Now we have

$$\begin{aligned}
& \int \left(\int F(\xi_1, \eta_1)^p d\eta_1 \right)^{1/p} k(\xi_1 - \xi'_1) F(\xi'_1, \eta'_1) d\xi_1 d\xi'_1 \\
& \leq \left\{ \int \int \left(\int F(\xi_1, \eta_1)^p d\eta_1 \right)^{p'/p} k(\xi_1 - \xi'_1) d\xi_1 d\xi'_1 \right\}^{1/p'} \\
& \quad \times \left(\int \int k(\xi_1 - \xi'_1) F(\xi'_1, \eta'_1)^p d\xi_1 d\xi'_1 \right)^{1/p} \\
& \leq C_2 K \left(\int F(\xi'_1, \eta'_1)^p d\xi'_1 \right)^{1/p},
\end{aligned}$$

where

$$C_2 = \int k(t) dt.$$

Hence (5) is bounded by

$$\begin{aligned}
& C_1 C_2 K \int \cdots \int \left(\int F(\xi'_1, \eta'_1)^p d\xi'_1 \right)^{1/p} k(\eta'_1 - \eta_2) F(\xi_2, \eta_2) k(\xi_2 - \xi'_2) \\
& \quad \cdots F(\xi_n, \eta_n) k(\xi_n - \xi'_n) F(\xi'_n, \eta'_n) k(\eta'_n - b) d\eta'_1 d\xi_2 \cdots d\xi'_n d\eta'_n.
\end{aligned}$$

By repeating this calculation, we can estimate (5) by

$$\begin{aligned}
& C_1 C_2^{2n-1} K^{2n-1} \int \left(\int F(\xi'_n, \eta'_n)^p d\xi'_n \right)^{1/p} k(\eta'_n - b) d\eta'_n \\
& \leq C_1 C_2^{2n-1} K^{2n-1} \left\{ \int \left(\int F(\xi'_n, \eta'_n)^p d\xi'_n \right)^{p'/p} d\eta'_n \right\}^{1/p'} \left(\int k(\eta'_n - b)^p d\eta'_n \right)^{1/p} \\
& \leq C_1 C_2^{2n-1} C_3 K^{2n} < \infty,
\end{aligned}$$

where

$$C_3 = \left(\int k(\eta'_n - b)^p d\eta'_n \right)^{1/p}.$$

Hence the condition (a) of Theorem 1.1 is satisfied. Furthermore we have

$$\sup_{n \in \mathbf{N}} \sup_{a \in \mathbf{R}^{2d}} G_n(a, a) \leq CK^2.$$

Therefore we proved the L^2 boundedness by Theorem 1.1.

参考文献

- [1] A. Calderón and R. Vaillancourt, On the boundedness of pseudodifferential operators, J. Math. Soc. Japan, 23, 1971, 374-378.
- [2] A. Calderón and R. Vaillancourt, A class of bounded pseudodifferential operators, Proc. Nat. Acad. Sci. USA, 69, 1972, 1185-1187.
- [3] H.O. Cordes, On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators, J. Funct. Anal., 18, 1975, 115-131.
- [4] L. Crone, A characterization of matrix operators on ℓ^2 , Math. Z., 123, 1971, 315-317.
- [5] I. Daubechies, Ten lectures on wavelets, SIAM 1992.

- [6] H.G. Feichtinger, Atomic characterizations of modulation spaces through Gabor-type representations, *Rocky Mountain J. Math.*, 19, 1989, 113-126.
- [7] H.G. Feichtinger and K. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions I, *J. Funct. Anal.*, 86, 1989, 307-340.
- [8] H.G. Feichtinger and K. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions II, *Monatsh. Math.*, 108, 1989, 129-148.
- [9] H. G. Feichtinger and K. Gröchenig, Gabor wavelets and the Heisenberg Group: Gabor expansion and short time Fourier transform from the group theoretical point of view, C.K. Chui ed., *Wavelets-A tutorial in theory and applications*, Academic Press Inc., 1992, 359-397.
- [10] G.B. Folland, *Harmonic analysis in phase space*, *Annals of Mathematics Studies* No. 122, Princeton University Press, 1989.
- [11] K. Gröchenig and C. Heil, *Modulation spaces and pseudodifferential operators*, preprint.
- [12] I.J. Maddox, *Infinite matrices of operators*, *Lecture Notes in Mathematics*, 786, Springer, Berlin, 1980.
- [13] I.J. Maddox and A.W. Wickstead, Crone's theorem for operators, *Math. Colloq. Univ. Cape Town* XI, 1977, 33-45.

- [14] R. Rochberg and K. Tachizawa, Pseudodifferential operators, Gabor frames, and local trigonometric bases, H. Feichtinger and T. Strohmer ed., Gabor analysis and algorithms : Theory and applications, Birkhäuser, 1998, 171-192.

Mathematical Institute

Tohoku University

Sendai 980-8578, Japan

E-mail address: tachizaw@math.tohoku.ac.jp