

On certain L^2 -well posed mixed problems for
hyperbolic system of first order

by

Taira SHIROTA

Department of Mathematics Hokkaido University

1. Introduction and Theorem.

Let P be a x_0 -strictly hyperbolic $2p \times 2p$ -system of differential operators of first order defined over a C^∞ -cylinder $R^1 \times \Omega \subset R^{n+1}$. Let B be a $p \times 2p$ -system of functions defined on the boundary Γ of $R^1 \times \Omega$. We consider the following mixed problems under certain conditions:

$$P(x, D)u = f \quad x \in R^1 \times \Omega \quad (x_0 > 0),$$

$$B(x) u = g \quad x \in \Gamma \quad (x_0 > 0),$$

$$u = h \quad \text{on } x_0 = 0$$

$$\text{where } \sqrt{-1} D = \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right).$$

For the sake of simplicity of descriptions, we may only consider the case where $\Omega = \{x_n > 0\}$, by the localization process. Then our assumptions are the following:

(I). $\alpha)$ The coefficients of P and B are real, belong to $C^\infty(R^1 \times \bar{\Omega})$ and constant outside some compact set of $R^1 \times \bar{\Omega}$.

$\beta)$ For P , it satisfies the # condition with respect to Γ and for fixed (x, τ, σ) there is at most one real double
real

root λ of $|P|(x, \tau, \sigma, \lambda) = 0$ where $x \in \Gamma$.

Furthermore it is non-characteristic with respect to Γ and it is normal, i.e.

$$|P|(x, 0, \sigma, \lambda) \neq 0$$

for any real $(\sigma, \lambda) \neq 0$.

γ) The p row-vectors of $B(x)$ are linearly independent, where $x \in \Gamma$.

(II). α) If the Lopatinsky determinant $R(x_0, \tau_0, \sigma_0) = 0$ for a real point (x_0, τ_0, σ_0) such that there is no real double roots λ of $|P|(x_0, \tau_0, \sigma_0, \lambda) = 0$, then

$$|R(x_0, \tau_0 - i\gamma, \sigma_0)| \geq O(\gamma^1) \quad (\gamma > 0).$$

Furthermore if there is at least one real simple root $\lambda(x_0, \tau_0, \sigma_0)$, the zero set of $R(x, \tau \pm i\gamma, \sigma)$ in some neighbourhood $U(x_0, \tau_0, \sigma_0)$ is in the set $\{\gamma = 0\}$.

β) If $R(x_0, \tau_0, \sigma_0) = 0$ for a real point (x_0, τ_0, σ_0) such that there are real double roots λ of

$$|P|(x_0, \tau_0, \sigma_0, \lambda) = 0, \quad \text{then}$$

$$|R(x_0, \tau_0 - i\gamma, \sigma_0)| \geq O(\gamma^{\frac{1}{2}}) \quad (\gamma > 0).$$

Furthermore if there is at least one real simple root λ , the rank of the Hessian of $R(x, \tau, \sigma)$ at its zeros in some $U(x_0, \tau_0, \sigma_0)$ is equal to

$$\text{codim. of } \{R(x, \tau, \sigma) = 0\} \text{ in } R^{2n}.$$

Where the zero set of $R(x, \tau, \sigma)$ in some $U(x_0, \tau_0, \sigma_0)$ is preassumed to be a regular submanifold of R^{2n} .

γ) Moreover, if there is at least one non-real root λ of $|P|(x_0, \tau_0, \sigma_0, \lambda) = 0$ for the point (x_0, τ_0, σ_0) which satisfies the condition β), then for some smooth and non-singular matrix $S(x, \tau - i\gamma, \sigma)$ with $\gamma \geq 0$ defined on some $U(x_0, \tau_0, \sigma_0)$ the corresponding reflection coefficient $b_{II}(x, \tau, \sigma)$ is real whenever τ is real and $R(x, \tau, \sigma) \neq 0$ (For definitions, see §2).

(III). Any constant coefficients problems frozen the coefficient at boundary are L^2 -well posed.

Then we have the following

Theorem. Under assumptions (I), (II), (III), the mixed problem is L^2 -well posed.

The aim of the present note is to describe the outline of our proof of the above assertion. Here we use essentially the conception of reflection coefficients ([1], [2]) and modifying Kreiss' consideration [4] we make use of the localization of the characterization for L^2 -well posed mixed problem of order two. ([1], [3] and [7])

2. The outline of the proof.

Considering the assumption (I) let $S(x, \tau - i\gamma, \sigma)$ ($\gamma \geq 0$) be a smooth, non-singular matrix defined on some neighbourhood $U(x_0, \tau_0, \sigma_0)$ such that

$$S^{-1}PS = ED_n - A(x, \tau - i\gamma, \sigma)$$

where

$$A = \begin{pmatrix} \lambda_I^+ & & & & \\ & \lambda_I^- & & & \\ & & A_{II} & & \\ & & & A_{III}^+ & \\ & & & & A_{III}^- \end{pmatrix},$$

$$\lambda_I^\pm = \left(\lambda_{i_1}^\pm \right), i \in I, |I| = r,$$

$\lambda_i^\pm$ are real, and $\text{Im } \lambda_i^+ (\text{Im } \lambda_i^-) > 0 (< 0)$ respectively if $\gamma > 0$.

Next for $\tau_0 = \tau_0(x, \sigma)$

$$A_{II}(x, \tau_0, \sigma) = \begin{pmatrix} a(x, 0, \sigma) & 1 \\ 0 & a(x, 0, \sigma) \end{pmatrix}.$$

Here we may restrict ourself to the case where the eigenvalue of $A_{II}(x, \tau, \sigma)$ are described by the following form in some $U(x_0, \tau_0, \sigma_0)$

$$\lambda_{II}^\pm = a(x, \zeta, \sigma) \mp \sqrt{\zeta} b(x, \zeta, \sigma) \quad (\sqrt{1} = 1),$$

$a(x, \zeta, \sigma)$, $b(x, \zeta, \sigma)$ are real when ζ is real, $b(x, \zeta, \sigma) \neq 0$, $\tau_0 = \tau_0(x_0, \sigma_0)$, $\tau = \zeta + \tau_0(x, \sigma)$ and $\tau_0(x, \sigma)$ is real and positive.

Furthermore $A_{III}^\pm$ have only non-real eigenvalues for any $\gamma \geq 0$ and the ones of A_{III}^+ have positive imaginary parts.

$$\text{Let } BS' = (V_I^+, V_I^-, V_{II}^+, V_{II}^-, V_{III}^+, V_{III}^-).$$

Where $V_I^{\pm}$ are $(p \times r)$ -matrices, $V_{II}^{\pm}$ are p -vectors and $V_{III}^{\pm}$ are $(p \times s)$ -matrices respectively ($2r+2+2s = 2p$).

$$\text{Let } S_{II} = \begin{pmatrix} 1 & 0 \\ \frac{\lambda_{II}^+ - h_{11}\zeta - a}{1+h_{12}\zeta} & 1 \end{pmatrix}, a = a(x, 0, \sigma) \quad *$$

and let

$$S' = \begin{pmatrix} E_{2r} & & \\ & S_{II} & \\ & & E_{2s} \end{pmatrix},$$

where h_{ij} are the functions derived from $A_{II}(x, -i\gamma, \sigma)$.

Furthermore we denote $B \cdot S \cdot S'$ by

$$(V_I^+, V_I^-, V_{II}^+, V_{II}^-, V_{III}^+, V_{III}^-)(x, \tau, \sigma).$$

Then from our assumptions we obtain the following Lemmas.

In particular from (I). γ), (II). α) and (III), we see the following

Lemma 2. 1 If for real (x_0, τ_0, σ_0) there exist no real double roots λ , then there is a neighbourhood $U(x_0, \tau_0, \sigma_0)$ where

1) For some $V_{3,1}^-$ the determinant

$$|V_I^+, V_{31}^+, \dots, V_{3,i-1}^+, V_{3,i}^-, V_{3,i+1}^+, \dots, V_{3,s}^+| \neq 0$$

where $V_{III}^+ = (V_{3,1}^+, \dots, V_{3,s}^+)$, $s = p - \gamma$, $V_{3,1}^+$ are p -column vectors (Here after let $i = 1$).

ii) For some $V_{3,1}^+$ it belongs to the linear subspace $L(V_{3,2}^+, \dots, V_{3,s}^+)$ spanned by the vectors $V_{3,2}^+, \dots, V_{3,s}^+$.

iii) The column vectors of V_I^- belong to $L(V_I^+, V_{3,2}^+, \dots, V_{3,s}^+)$. But ii) and iii) are only valid at the points $\in U(x_0, \tau_0, \sigma_0)$ such that the Lopatinsky det. $|V_I^+, V_{II}^+| (x, \tau, \sigma) = c(\tau - \bar{\tau}(x, \sigma)) = 0$ ($c \neq 0$) and where $\tau(x, \sigma)$ is real whenever V_I^+ present.

From (II). β) we see the following

Lemma 2. 2 Let (x_0, τ_0, σ_0) be a real point such that there exists a real double root λ . Let $|V_I^+, V_{II}^+, V_{III}^+| (x_0, \zeta, \sigma_0) = 0$, where we consider ζ as a new variable instead of τ . Then

$$i) \quad \zeta = 0.$$

$$ii) \quad \text{Let } \zeta^{\frac{1}{2}} = \eta, \text{ then}$$

$$|V_I^+, V_{II}^+, V_{III}^+| = C(\eta - \eta(x, \sigma)) \quad (c \neq 0)$$

in some $U(x_0, \tau_0, \sigma_0)$, where $\eta(x, \sigma)$ may take complex values. / \

Under the assumption of Lemma 2. 2 we see the following Lemmas.

Lemma 2. 3 1) The reflection coefficient

$$\begin{aligned} b_{II, I}(x_0, -i\gamma, \sigma_0) &= \frac{|V_I^+, V_{II}^-, V_{III}^+|}{|V_I^+, V_{II}^+, V_{III}^+|} (x_0, -i\gamma, \sigma_0) \\ &= O(\gamma^{-\frac{1}{2}}) \quad (\gamma > 0). \end{aligned}$$

ii) Let $Q(x, \zeta, \sigma)$ be $\frac{a_{11} + a_{12}b_{II}}{a_{12} + a_{22}b_{II}}$, then

it is $\frac{|V_I^+, V_{II}^+, V_{III}^+|}{|V_I^+, V_{II}^-, V_{III}^+|}$, where $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = S_{II}^{-1}$.

Now from Lemma 2. 3 and (III) we obtain the following

Lemma 2. 4

- i) $|V_I^+, V_{II}^-, V_{III}^+| \neq 0$.
- ii) $V_{II}^+ \in L(V_{III}^+)$ on $\zeta = \eta(x, \sigma) = 0$.
- iii) $V_I^- \in L(V_I^+, V_{III}^+)$ on $\zeta = \eta(x, \sigma) = 0$.
- iv) $V_{II}^+ - QV_{II}^- \in L(V_I^+, V_{III}^+)$.

From (II), β , γ , (III) and the definition of Q we see the following

Lemma 2. 5. i) The above defined $Q(x, \zeta, \sigma)$ take only real values, when ζ is real.

ii) $\zeta = 0$, $Q(x, 0, \sigma) = 0$ are equivalent to $R(x, \zeta, \sigma) = 0$ for $\text{Im } \zeta \leq 0$.

iii) $-Q(x, 0, \sigma) \geq 0$.

From Lemma 2. 4 we obtain the following

Lemma 2. 6 For (x, ζ, σ) belonging to some $U(x_0, \tau_0, \sigma_0)$,

$$g = (V_I^+, V_{II}^+, V_{III}^+) \begin{pmatrix} U_I^+ + (\zeta K_{II}^I + K_{II}^{II})U' + K_{II}U_I^- \\ U_{II}^+ + QU_{II} + (\zeta K_{II}^I + K_{II}^{II})U_I^- \\ U_{III}^+ + K_{III}U_I^- + K_{III}U_{II}' \end{pmatrix} + V_{III}^- \begin{pmatrix} 0 \\ 0 \\ U_{III}^- \end{pmatrix},$$

where $u = (U_I^+, U_I^-, U_{II}^+, U_{II}^-, U_{III}^+, U_{III}^-)$.

Moreover the components of K_{II}^{II} and K_{II}^{II} are zero, whenever $\zeta = 0$ and $\eta(x, \sigma) = 0$.

From Lemma 2. 1 we obtain an a priori L^2 -estimate in the case where there is no double root λ . On the other hand if there is at least one double root λ , we see from Lemma 2. 5 and by some modifications of Kreiss' method that the problem $((D_n - A_{II})u = f, u'' + Qu' = g)$ has an a priori estimate

$$\|(D_n - A_{II})u\|_{0,\gamma} + \|g\|_{\frac{1}{2},\gamma} \geq C\gamma \|u\|_{0,\gamma} \quad (C > 0)$$

where $\text{supp } u \subset U(x_0)$, spectrum of u with respect to $x_0, \dots, x_{n-1} \subset U(\tau_0, \sigma_0)$. Then from the method of the proof of the above estimate and from Lemma 2. 6, we obtain a similar estimate in this case. Here we use the fact that the components k of K_{II}^I , K_{II}^I has the following form: in some $U(x_0, \tau_0, \sigma_0)$

$$k(x, \zeta, \sigma) = \tilde{k}(x, 0, \sigma) + \zeta \tilde{\tilde{k}}(x, 0, \sigma) + o(|\zeta|^2),$$

$$|\tilde{k}(x, 0, \sigma)|^2 \leq K|Q(x, 0, \sigma)| \quad (K > 0)$$

which follows from the last assumption of (II), (β). Furthermore our assumptions are valid for the dual problem and hence $\wedge_a$

priori estimate for that problem is also obtained. Thus our proof is complete ([6]).

Remark (1) The conditions (I), (II), (III) are invariant for certain coordinate transformation. Hence Theorem is applicable for problems defined on any smooth $R^1 \times \Omega$.

(2) The condition (II), γ should be omitted, but we have many examples which satisfy the condition.

References

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