

On shape derivative and free-boundary problems in
vortex dynamics

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Abstract

A new shape derivative formula for singular contour integrals with logarithmic kernels is proposed. This formula yields a simple numerical scheme to compute vortex patch equilibria. Owing to its simplicity, any steady configuration of point vortices can be extended to that of vortex patches. As a test problem, a doubly periodic array of vortex patches is considered to show the efficiency of the new formula. Many non-trivial families of stationary vortex patch lattices are presented.

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Chapter 1

Introduction

A shape calculus is a branch of analysis concerned with the functionalities of shapes, such as a shape derivative, a space of shapes and its topological structure among other things as in the standard calculus. It is derived from many areas of mathematics such as geometry, partial differential equations, shape optimization problems, free-boundary problems or fluid mechanics. In the shape calculus, the main concept is devoted to a geometrical object as a variable, whereas a scalar, a vector or a function play central roles in the classical analysis. In use of the shape differential calculus, the integral quantities can be differentiated with respect to shapes; for example a shape derivative of the domain integral type is a useful formula that represents the sensitivity of the integral quantity under perturbations of the integral domain.

Although a shape derivative of the boundary integral type is also well-known, to the author's best knowledge, there has been no such formula of the contour integral type in preceding works. In this paper, we present a new shape derivative formula of the contour integral type. It in turn gives rise to a clear geometrical interpretation on some fundamental facts in complex analysis from a shape differential point of view. We further develop a shape derivative formula of the singular contour integral type, which is originally motivated by a free-boundary problem in vortex dynamics.

Finite area regions with piecewise constant vorticity in two-dimensional spaces are called vortex patches. Since the existence of a unique weak solution to the 2D Euler equations is ensured globally in time for vortex patch initial data, the understanding of the evolution of vortex patches is considered to be a fundamental theme in mathematical fluid dynamics. Inviscid flows with vortex patch structures appear as theoretical models in many

aerodynamical, biomechanical, oceanography and atmospheric phenomena. The first attempt to deal with vortex patch evolutions as a dynamical system is finding steady states where their shapes remain unchanged. There are many preceding studies on the steady vortex patch problems physically as well as mathematically. The classical Rankine vortex is an example of such a stationary vortex patch. Pierrehumbert [1] has numerically computed a continuous one-parameter family of vortex patch pairs with the strengths of the opposite signs translating in one direction. Burbea [2] has exactly expressed a rigidly rotating vortex patch in an m -fold symmetry in terms of conformal mapping. Crowdy [3] has constructed analytic formulae of a family of vortex patches surrounded by a polygonal configuration of m point vortices. Elcrat [4, 5] has found various families of steady vortex patches trapped in the vicinity of a cylindrical obstacle in the presence of a background uniform flow.

There are three mathematical formulations of vortex patch equilibria. The first one is given by an elliptic free-boundary PDE for a streamfunction in which the boundaries of vortex patches are to be determined. Elcrat [4] has proposed an iterative method combining with a finite difference method. The second one is based on boundary integral equations for a boundary of the stationary vortex patch. Pierrehumbert [1] has proposed a relaxation method to solve the boundary integral equations. The third one, contour dynamics, is proposed by Zabusky and Deem [6, 7], in which another boundary integral equations with the logarithmic integral kernel appear. Elcrat [5] has proposed Newton method via contour dynamics formulation. It should be noted that, in the formulations by the boundary integral equations, the free-boundary problem is reduced to a one-dimensional problem, which makes the problem to be handled by numerical means effectively. Since the boundary of the vortex patch is unknown for those boundary integral equations, it is thus natural to consider the derivative of boundary integrals with respect to unknown boundaries in order to obtain its equilibrium state. However, a naive application of the standard shape calculus is unsuccessful to the problems considered in this paper, since the kernel of the boundary integrals contains logarithmic singularities. Elcrat and Protas [8] have overcome this difficulty, in which use of the integration by parts removes the logarithmic singularity so that the shape derivative formula is applicable. Although their framework is, in principal, designed for an arbitrary configuration of steady vortex patches, it is necessary to analytically deal with the logarithmic singularity depending on the problem, which makes the analysis complicated for general cases. Especially, when we consider a flow domain with various geometry, there appears a logarithmic kernel depending on its geometry.

Our objective in this paper is to construct an alternative shape derivative formula easily applicable to contour integrals with arbitrary logarithmic kernels.

As a test bed of our new formula, we compute doubly periodic arrays of stationary vortex patches. Lattice arrangements of vortex structures have been attracting many researchers. Tkachenko [9] has formulated rotating vortex lattices containing only one point vortex in each period cell. O’Neil [10] has derived a dynamical system of vortex lattices containing more than one point vortices in each period cell. In both formulations, the Weierstrass zeta function is utilised. Crowdy [11] has recently figured out an alternative derivation for the motion of rectangular vortex lattices in use of the P function which is the Schottky-Klein prime function associated with an annulus [12, 13]. In those preceding results, the lattice configurations of point vortices are concerned. On the other hand, to the author’s best knowledge, there exist no results in terms of vortex patch lattices, because the corresponding integral kernel is complicated. By utilising our shape derivative formula, non-trivial equilibria of vortex patch lattices are obtained, which is also a new result presented in this paper.

Following are the structure of this paper: in Chapter 2, we show a new shape derivative formula for singular contour integrals; in Chapter 3, we investigate vortex lattices equilibria; in Section 3.1, we propose a numerical procedure to compute vortex patch equilibria applicable to many problems; in Section 3.2, after reviewing the lattice configuration of point vortices described by the P function, we extend the formulation to the case of vortex patch lattices; we show some numerical results including non-trivial families of vortex patches depending on the aspect ratio of the lattice in Section 3.3; finally in Chapter 4, we conclude the present works and discuss some future works.

Chapter 2

Shape derivative theory

The shape derivative presented here is a directional derivative, namely a Gâteaux derivative, with respect to certain parametrisations of shapes. We first review the Gâteaux derivative in topological vector spaces. We refer to an elementary textbook [14] or [15] for the details on the shape calculus.

Def 1 (Gâteaux semiderivative) *Let X and Y be topological vector spaces and $\mathcal{F}$ be a map from a neighbourhood of $x \in X$ to Y . We say that $\mathcal{F}$ has a Gâteaux semiderivative at x in the direction $\delta x \in X$ if there exists the limit $\lim_{h \rightarrow +0} [\mathcal{F}(x + h\delta x) - \mathcal{F}(x)]/h$. Then, it is denoted by*

$$d\mathcal{F}(x; \delta x) := \lim_{h \rightarrow +0} \frac{\mathcal{F}(x + h\delta x) - \mathcal{F}(x)}{h} = \left. \frac{d}{dh} \right|_{h=0} \mathcal{F}(x + h\delta x).$$

2.1 Shape derivative for domain integrals

Consider a domain integral of integrand $\varphi \in C^1(\mathbf{R}^d; \mathbf{R})$ over a bounded domain Ω as follows:

$$\int_{\Omega} \varphi(x) d\mu(x),$$

where $d\mu$ denotes the Lebesgue measure in $\mathbf{R}^d$. We describe the shape derivative formula for such a domain integral in terms of the Gâteaux semiderivative.

For this aim, we need to introduce a set of integral domains equipped with a certain linear structure. In other words, a formulation for certain directional perturbations of shapes is required so that we can apply the Gâteaux semiderivative to the domain integral. This is accomplished by the

so-called velocity method. Let $V : \mathbf{R}^d \rightarrow \mathbf{R}^d$ be a C^1 vector field which will play a role in deformation of the integral domain. Consider the initial value problem of the ordinary differential equation

$$\frac{dx}{dh}(h) = V(x(h)), \quad x(0) = x_0.$$

We define the flow map by $T_h(x_0) := x(h)$ where a point $x_0 \in \mathbf{R}^d$ is advected by the vector field V in time h . Owing to the existence and uniqueness of a solution to the initial value problem, the flow map $T_h : \mathbf{R}^d \rightarrow \mathbf{R}^d$ is an isomorphism. We then obtain a new domain $\Omega_h(V) := T_h(\Omega)$ deformed by the advection, which is the origin of the “velocity method.” Note that the terminology “velocity” is ambiguous since we also deal with physical velocities in Chapter 3. We thus prefer to refer to V as an artificial velocity field. Similarly h is called an artificial time when we need to distinguish it from physical time.

We can now define, for a given integrand $\varphi \in C^1(\mathbf{R}^d; \mathbf{R})$ and an artificial velocity field $V \in C^1(\mathbf{R}^d; \mathbf{R}^d)$, a function $\mathcal{F}$ that maps an integral domain to a domain integral in terms of the velocity method as follows:

$$\mathcal{F}(\Omega_h(V)) := \int_{\Omega_h(V)} \varphi(x) d\mu(x),$$

where the domain of the definition of $\mathcal{F}$, called an admissible family, is given by

$$\mathcal{D}_{\text{adm}} := \left\{ \Omega_h(V) \mid h \in [0, 1], V \in C^1(\mathbf{R}^d; \mathbf{R}^d) \right\}.$$

Taking the derivative of $\mathcal{F}$ at the artificial time $h = 0$ yields the shape derivative of the domain integral type given by

$$d\mathcal{F}(\Omega; V) := \left. \frac{d}{dh} \right|_{h=0} \mathcal{F}(\Omega_h(V)) = \int_{\Omega} \operatorname{div}(\varphi V) dx.$$

The proof is by an elementary calculus. The change of variable $x = T_h(y)$ yields

$$\mathcal{F}(\Omega_h(V)) = \int_{\Omega} \varphi(T_h(y)) \det DT_h d\mu(y),$$

from which it follows

$$d\mathcal{F}(\Omega; V) = \left. \frac{d}{dh} \right|_{h=0} \int_{\Omega} \varphi(T_h(y)) \det DT_h d\mu(y).$$

Since $\frac{\partial T_h}{\partial h} = V \circ T_h$, we obtain

$$\begin{aligned} d\mathcal{F}(\Omega; V) &= \int_{\Omega} \nabla \varphi(y) \cdot V(y) + \varphi(y) \operatorname{div} V(y) d\mu(y) \\ &= \int_{\Omega} \operatorname{div}(\varphi(y) V(y)) d\mu(y). \end{aligned} \quad (2.1)$$

If Ω has a Lipschitzian boundary Γ , then owing to the Stoke's theorem, we have

$$d\mathcal{F}(\Omega; V) = \int_{\Gamma} \varphi V \cdot n d\Gamma, \quad (2.2)$$

where n denotes the outward unit normal vector of Γ . Note that these shape derivative formulae of the domain integral type are equivalent to the Reynolds transport theorem in continuum mechanics which describes the advection of the integral quantities. Thus some author [15] prefer to call them the material derivative method instead of the velocity method.

We can understand the formula (2.2) in terms of L^2 space as follows:

$$d\mathcal{F}(\Omega; V) = \langle \varphi|_{\Gamma}, V|_{\Gamma} \cdot n \rangle_{L^2(\Gamma; \mathbf{R})}, \quad (2.3)$$

that is to say, the trace of the integrand φ coincides with the gradient of $\mathcal{F}$ at Ω in the sense of $L^2(\Gamma; \mathbf{R}^d)$.

2.2 Shape derivative for boundary integrals

We also have the shape derivative formula of the boundary integral type in a similar manner, namely, the artificial velocity method (see [14] or [15] for the detail). Let $\Omega \subset \mathbf{R}^d$ be a bounded domain with a smooth boundary and $\varphi: \mathbf{R}^d \rightarrow \mathbf{R}$ be of C^1 . A function $\mathcal{F}$ is defined by

$$\mathcal{F}(\Gamma) := \int_{\Gamma} \varphi(x) d\Gamma,$$

where its admissible family is given by

$$\mathcal{D}_{\text{adm}} := \left\{ \partial\Omega_h(V) \mid h \in [0, 1], V \in C^1(\mathbf{R}^d; \mathbf{R}^d) \right\}.$$

Then we obtain, for $\Gamma = \partial\Omega$,

$$\begin{aligned} d\mathcal{F}(\Gamma; V) &:= \frac{d}{dh} \Big|_{h=0} \mathcal{F}(\partial\Omega_h(V)) \\ &= \int_{\Gamma} \nabla \varphi \cdot V + \varphi(\nabla \cdot V - DV(n) \cdot n) d\Gamma, \end{aligned} \quad (2.4)$$

where $DV(n)$ denotes the Jacobian matrix of V acting on the outward unit normal vector n of Γ . If the boundary Γ is of class C^2 , we can rewrite the latter term in the integrand of (2.4) in terms of the mean curvature $\overline{H}$ of the boundary as follows:

$$d\mathcal{F}(\Gamma; V) = \int_{\Gamma} [\nabla\varphi \cdot n + \varphi(d-1)\overline{H}] V \cdot n \, d\Gamma,$$

from which it follows

$$d\mathcal{F}(\Gamma; V) = \left\langle \frac{\partial\varphi}{\partial n} + \varphi|_{\Gamma}(d-1)\overline{H}, V|_{\Gamma} \cdot n \right\rangle_{L^2(\Gamma; \mathbf{R})}.$$

In other words, the gradient of $\mathcal{F}$ at Γ in the sense of $L^2(\Gamma; \mathbf{R})$ is given by the normal derivative of the integrand φ and its trace weighted by the curvature. Thus, from the shape differential point of view, it is essential to impose the regularity of the boundary of class C^2 , which is a much stronger assumption compared to that in the previous formula of the domain integral type.

2.3 Shape derivative for contour integrals

We now exploit the shape derivative formula of the contour integral type. Let D be an open connected set in $\mathbf{C}$. We take a curve C in D whose parametrisation is given by

$$z = f(s) \quad (s \in [0, 2\pi]),$$

where f is a C^1 mapping from $[0, 2\pi]$ to D . We denote $C(f) := C$ when we emphasize that the curve is parametrised by f .

We first introduce a conventional notation to represent the local perturbation of the shape in terms of the Gâteaux semiderivative. Let $s \in [0, 2\pi]$ be frozen. Take a C^1 function $\delta f: [0, 2\pi] \rightarrow \mathbf{C}$. The shape derivative of the application mapping $f \mapsto f(s)$ in the direction δf is denoted by

$$\delta z := \delta f(s).$$

The simplest example of the shape derivative in turn plays fundamental roles in what follows. Note that, if f is a one-to-one mapping, then we have an equivalent definition $\delta z = \delta f \circ f^{-1}(z)$.

A shape derivative of a contour integral along a smooth closed contour is given by the author [16] in terms of Gâteaux semiderivative. We here prove the same theorem under free of the assumption that the contour is closed.

Thm 2 (Shape derivative of a contour integral) *Let φ be a C^1 mapping from D to $\mathbf{C}$. A map $\mathcal{F}$ of a parametrisation f with $C(f) \subset D$ is defined as follows:*

$$\mathcal{F}(f) := \int_{C(f)} \varphi(z, \bar{z}) dz.$$

We then have the Gâteaux semiderivative of $\mathcal{F}$ at f in the direction δf

$$\begin{aligned} d\mathcal{F}(f; \delta f) &= \varphi(b, \bar{b}) \delta b - \varphi(a, \bar{a}) \delta a \\ &\quad + 2i \int_{C(f)} \frac{\partial \varphi}{\partial \bar{z}}(z, \bar{z}) \operatorname{Re}[\delta z \overline{(-i dz)}], \end{aligned}$$

where $a = f(0)$ and $b = f(2\pi)$ are the endpoints of $C(f)$ and $\delta a = \delta f(0)$ and $\delta b = \delta f(2\pi)$ are local perturbations at a and b , respectively.

The proof is straightforward. First, by the definition of contour integrals, we have

$$\begin{aligned} \left. \frac{d}{dh} \right|_{h=0} \mathcal{F}(f + h\delta f) &= \left. \frac{d}{dh} \right|_{h=0} \int_0^{2\pi} \varphi(f + h\delta f, \overline{f + h\delta f}) [f + h\delta f]' ds \\ &= \int_C d\varphi(z, \bar{z}; \delta z, \overline{\delta z}) dz + \int_0^{2\pi} \varphi(f, \bar{f}) \delta f' ds. \end{aligned}$$

The integration by parts in the second term yields

$$\begin{aligned} \int_0^{2\pi} \varphi(f, \bar{f}) \delta f' ds &= [\varphi(f, \bar{f}) \delta f]_0^{2\pi} - \int_0^{2\pi} d\varphi(f, \bar{f}; f', \bar{f}') \delta f ds \\ &= \varphi(b, \bar{b}) \delta b - \varphi(a, \bar{a}) \delta a - \int_C d\varphi(z, \bar{z}; dz, d\bar{z}) \delta z. \end{aligned}$$

The subtraction of two rest contour integrals is

$$\begin{aligned} &\int_C d\varphi(z, \bar{z}; \delta z, \overline{\delta z}) dz - d\varphi(z, \bar{z}; dz, d\bar{z}) \delta z \\ &= \int_C \frac{\partial \varphi}{\partial z} \delta z dz + \frac{\partial \varphi}{\partial \bar{z}} \overline{\delta z} dz - \frac{\partial \varphi}{\partial z} \delta z dz - \frac{\partial \varphi}{\partial \bar{z}} \delta z d\bar{z} \\ &= \int_C \frac{\partial \varphi}{\partial \bar{z}} [\overline{\delta z} dz - \delta z d\bar{z}] \\ &= \int_C \frac{\partial \varphi}{\partial \bar{z}} 2i \operatorname{Im}[\overline{\delta z} dz], \end{aligned}$$

which completes the proof.

The shape derivative of a contour integral is described by the normal directional perturbation and the partial derivative of the integrand $\varphi(z, \bar{z})$ with respect to $\bar{z}$. The notion that the shape derivative vanishes under certain assumptions leads to the following corollaries.

Cor 3 *The contour integral along a contour $C(f)$ does not depend on its parametrisation f .*

Cor 4 *Assume that $\varphi_{\bar{z}} = 0$ on D , namely, φ is a holomorphic function. Let C_1 and C_2 be smooth paths from a to b in D . If C_1 and C_2 are homotopic, then the contour integrals of φ along C_1 and C_2 coincide.*

We only give the proof for Cor. 3. The key idea from point of view in the shape calculus is that the shape derivative (2.5) vanishes for any tangential perturbation δf . We consider a continuous deformation of parametrisations of C . Let $f_0, f_1: [0, 2\pi] \rightarrow D$ be parametrisations of the contour C such that there exists a change of variable $\theta: [0, 2\pi] \rightarrow [0, 2\pi]$ with $f_1 = f_0 \circ \theta$. We take a smooth homotopy $H: [0, 1] \times [0, 2\pi] \rightarrow [0, 2\pi]$ between the identity map I and θ such that $H_h(0) = 0$, $H_h(2\pi) = 2\pi$ and H_h is injective for all $h \in [0, 1]$. We denote $f_h := f_0 \circ H_h$, which gives a homotopy $(h, s) \mapsto f_h(s)$ between f_0 and f_1 . Then we have

$$\begin{aligned} \frac{\partial}{\partial h} f_h(s) &= f'(H_h(s)) \frac{\partial}{\partial h} H_h(s), \\ \frac{\partial}{\partial s} f_h(s) &= f'(H_h(s)) H'_h(s), \\ \frac{\partial}{\partial h} f_h(0) &= 0, \quad \frac{\partial}{\partial h} f_h(2\pi) = 0. \end{aligned}$$

It follows from (2.5) that

$$\begin{aligned} & \mathrm{d}\mathcal{F}\left(f_h; \frac{\partial f_h}{\partial h}\right) \\ &= 2i \int_0^{2\pi} \varphi_{\bar{z}} \operatorname{Re} \left[f'(H_h(s)) \frac{\partial}{\partial h} H_h(s) \overline{(-if'(H_h(s)) H'_h(s))} \right] \mathrm{d}s \\ &= 2i \int_0^{2\pi} \varphi_{\bar{z}} \frac{\partial}{\partial h} H_h(s) H'_h(s) \operatorname{Re} \left[f'(H_h(s)) \overline{(-if'(H_h(s)))} \right] \mathrm{d}s \\ &= 0. \end{aligned}$$

Since the derivative vanishes, we conclude that

$$\mathcal{F}(f_1) - \mathcal{F}(f_0) = \int_0^1 \mathrm{d}\mathcal{F}\left(f_h; \frac{\partial f_h}{\partial h}\right) \mathrm{d}h = 0,$$

namely, the contour integral does not depend on parametrisations.

Although both of Cor. 3 and Cor. 4 are well-known facts in complex analysis, to the author's best knowledge, the proofs are new and clear from the point of view in the shape calculus.

Assuming that $C(f)$ is a smooth closed contour, namely, f is a C^1 function from $S^1 := \mathbf{R}/(2\pi\mathbf{Z})$ to D , we immediately obtain the following theorem which has originally been given by the author in [16].

Cor 5 (Shape derivative of a closed contour integral) *Let φ be a C^1 mapping from D to $\mathbf{C}$. A map $\mathcal{F}$ of a parametrisation $f: S^1 \rightarrow D$ with $C(f) \subset D$ is defined as follows:*

$$\mathcal{F}(f) := \oint_{C(f)} \varphi(z, \bar{z}) dz.$$

For $\delta f: S^1 \rightarrow \mathbf{C}$, we have the following

$$d\mathcal{F}(f; \delta f) = 2i \oint_{C(f)} \frac{\partial \varphi}{\partial \bar{z}}(z, \bar{z}) \operatorname{Re} \left[\delta z \overline{-i dz} \right]. \quad (2.5)$$

The formula (2.5) is easy to understand when we apply it to the area of the region $A(f)$ surrounded by $C(f)$. When $C(f)$ is oriented counterclockwise, the area $\mathcal{A}(f)$ is given by

$$\mathcal{A}(f) := \iint_{A(f)} dz_1 dz_2 = \frac{1}{2i} \oint_{C(f)} \bar{z} dz.$$

The complex form of Green's formula is used in the second equality. The shape derivative formula (2.5) yields the following perturbation formula:

$$d\mathcal{A}(f; \delta f) = \oint_{C(f)} \operatorname{Re} \left[\delta z \overline{-i dz} \right], \quad (2.6)$$

which simply reads that the variation of the area occurs when the boundary deforms in the normal direction. Mathematically speaking, the gradient of $\mathcal{A}$ is given by $-i\delta f'$ in the sense of $L^2(S^1; \mathbf{R}^2)$:

$$d\mathcal{A}(f; \delta f) = \operatorname{Re} \langle \delta f, -i\delta f' \rangle_{L^2(S^1; \mathbf{C})}.$$

If the parametrisation f of C is consistent with the arclength parameter, we can rewrite it in terms of $L^2(C; \mathbf{R})$ as follows:

$$d\mathcal{A}(f; \delta f) = \langle 1, \delta f_n \rangle_{L^2(C; \mathbf{R})},$$

where $\delta f_n := \operatorname{Re}[\delta f(\overline{-if'})]$ denotes the outward normal component of δf . It is equivalent but the gradient is given by 1 in the sense of $L^2(C; \mathbf{R})$, whose notion is similar to (2.3).

In the same way, it immediately follows from (2.5) that

$$d\mathcal{F}(f; \delta f) = \left\langle 2i\varphi_{\bar{z}}|_{C(f)}, \delta f_n \right\rangle_{L^2(C(f); \mathbf{C})}.$$

We confirm that two shape derivative formulae (2.2) and (2.5) coincide under the assumption that the integrand φ is a C^1 function defined on a neighbourhood of the integral domain Ω . In the two-dimensional case, the complex form of Green's formula is given by

$$\oint_{\partial\Omega} \varphi dz = 2i \iint_{\Omega} \varphi_{\bar{z}} dx dy \quad (z = x + iy). \quad (2.7)$$

Owing to (2.2), the shape derivative of the right hand side of (2.7) is given by

$$2i \int_{\Gamma} \varphi_{\bar{z}} (V_1 n_1 + V_2 n_2) d\Gamma,$$

where (V_1, V_2) is an artificial velocity field and (n_1, n_2) is the outward unit normal vector of $\Gamma := \partial\Omega$. We take a counterclockwise parametrisation $z = f(l)$ of Γ with the arclength parameter l , namely, $|f'(l)| = 1$. Since $n_1 + in_2 = -if'(l)$, we obtain

$$2i \int_{\Gamma} \varphi_{\bar{z}} (V_1 n_1 + V_2 n_2) d\Gamma = 2i \oint_{\Gamma} \varphi_{\bar{z}} \operatorname{Re}[(V_1 + iV_2)(-i dz)].$$

We thus conclude that this coincides with (2.5) when we consider the perturbation of the form $\delta z = (V_1 + iV_2)|_z$.

Remark 6 *We introduced shape derivative formulae of the three types; the domain integral type (2.2), the boundary integral type (2.4) and the contour integral type (2.5). Owing to the Green's formula, the relation between the contour integral type and the domain integral type is found. We assume the lower regularity assumption for the contour integral type, whereas the higher one is assumed for the boundary integral type. The contour integral type is much simpler than that of the boundary integral type. Especially, the latter depends on the curvature, which makes the formula complicated. For these reasons, the contour integral type should be considered as a generalisation or a special case of the domain integral type rather than the boundary integral type.*

2.4 Shape derivative for singular contour integrals

The formula (2.5) holds true under the assumption that the integrand is smooth, which is not the case for the velocity field induced by a vortex patch since it has a logarithmic singularity. We now generalise so that it is applicable to the singular case. Let $\varphi(z, \bar{z}, w, \bar{w})$ be a C^1 function from $D \times D \setminus \{z = w\}$ to $\mathbf{C}$. For simplicity, we sometimes omit the dependency on the conjugate variables and just denote the map by $\varphi(z, w)$. We assume the following properties for the map φ .

Assumption 7 *For z and w along an arbitrary curve C in D , we assume that $\varphi(z, w)|z - w|$, $\varphi_w(z, w)|z - w|$ and $\varphi_{\bar{w}}(z, w)|z - w|$ have continuous modifications as $z \rightarrow w$.*

An arbitrary logarithmic singular kernel of the form $\log p(z - w)$ for a C^1 function $p: \mathbf{C} \rightarrow \mathbf{C}$ with a simple zero at the origin satisfies the Assumption 7.

In what follows, we assume that $C(f)$ is a Jordan curve, namely, f is a one-to-one map. We consider the contour integrals of φ as follows:

$$\begin{cases} \text{p.v.} \oint_C \varphi(z, \bar{z}, w, \bar{w}) dw, & (z \in C), \\ \oint_C \varphi(z, \bar{z}, w, \bar{w}) dw, & (z \in D \setminus C), \end{cases}$$

where the symbol p.v. means the Cauchy's principal value. The singularity appears only once at $w = z$ since C is a Jordan curve.

Take such a parametrisation f with $f'(s) \neq 0$ for all $s \in S^1$. Since $\min_s |f'(s)| > 0$, there exist a constant $C > 0$ such that

$$C|s - t| \leq |f(s) - f(t)|.$$

Define $g := f + h\delta f$ for C^1 function $\delta f: S^1 \rightarrow \mathbf{C}$. We denote the Lipschitz constant of δf by $L := \sup_{s \neq t} |\delta f(s) - \delta f(t)|/|s - t|$. We then have

$$\begin{aligned} |g(s) - g(t)| &\geq |f(s) - f(t)| - h|\delta f(s) - \delta f(t)| \\ &\geq (C - hL)|s - t|. \end{aligned}$$

If $h < C/L$, there exists a constant $C' > 0$ with $C'|s - t| \leq |g(s) - g(t)|$, from which it follows that g is injective. We conclude that, for sufficiently small h , the curve $C(f + h\delta f)$ is also a Jordan curve. Especially, the singularity in the contour integral for the contour $C(f + h\delta f)$ only occurs when $z + h\delta z = w + h\delta w$, where $\delta z := \delta f \circ f^{-1}(z)$ and $\delta w := \delta f \circ f^{-1}(w)$.

If we consider the contour integral as a function of z , it is difficult to deal with the singularities. We thus define a map $\mathcal{F}$ of a parametrisation f with $C(f) \subset D$ as follows:

$$\mathcal{F}(f) := \text{p.v.} \oint_{C(f)} \varphi(z, \bar{z}, w, \bar{w})|_{z=f(\cdot)} dw.$$

In this formulation, the function $\mathcal{F}$ maps a parametrisation f to the function $\mathcal{F}(f)(\cdot)$ defined on S^1 . The key point is that z is a variable dependent on the parametrisation f . Recall that, if f is replaced by $f + h\delta f$, then the variable z is replaced by $z + h\delta z$.

We now apply a similar calculus as in the previous derivation of (2.5).

$$\begin{aligned} & \left. \frac{d}{dh} \right|_{h=0} \mathcal{F}(f + h\delta f)(s) \\ &= \left. \frac{d}{dh} \right|_{h=0} \text{p.v.} \int_0^{2\pi} \varphi(f(s) + h\delta f(s), f(t) + h\delta f(t)) [f(t) + h\delta f(t)]' dt \\ &= \text{p.v.} \oint_C d\varphi(z, w; \delta z, \delta w) dw + \text{p.v.} \int_0^{2\pi} \varphi(f(s), f(t)) \delta f'(t) dt. \end{aligned}$$

The integration by parts in the second term yields

$$\begin{aligned} & \text{p.v.} \int_0^{2\pi} \varphi(f(s), f(t)) \delta f'(t) dt \\ &= [\varphi(f(s), f(t)) (\delta f(t) - \delta f(s))]_{t \in S^1} \\ &\quad - \int_0^{2\pi} d\varphi(f(s), f(t); 0, f'(t)) (\delta f(t) - \delta f(s)) dt \\ &= - \oint_C d\varphi(z, w; 0, dw) (\delta w - \delta z). \end{aligned}$$

Note that a constant $-\delta z$ can be added to remove the singularity owing Assumption 7. Hence, this contour integral is no longer taken in the sense

of principal value. We thus get:

$$\begin{aligned}
& \left. \frac{d}{dh} \right|_{h=0} F(f + h\delta f)(s) \\
&= \text{p.v.} \oint_C d\varphi(z, w; \delta z, \delta w) dw - \oint_C d\varphi(z, w; 0, dw) (\delta w - \delta z) \\
&= \text{p.v.} \oint_C [d\varphi(z, w; \delta z, \delta z) + d\varphi(z, w; 0, \delta w - \delta z)] dw \\
&\quad - \oint_C d\varphi(z, w; 0, dw) (\delta w - \delta z) \\
&= \text{p.v.} \oint_C d\varphi(z, w; \delta z, \delta z) dw + \oint_C \varphi_{\bar{w}}(\overline{\delta w - \delta z}) dw - \varphi_{\bar{w}}(\delta w - \delta z) d\bar{w} \\
&= \text{p.v.} \oint_C d\varphi(z, w; \delta z, \delta z) dw + \oint_C \varphi_{\bar{w}} 2i \operatorname{Im}[(\overline{\delta w - \delta z}) dw].
\end{aligned}$$

Consequently, we obtain the following theorem.

Thm 8 (Shape derivative of a singular contour integral) *Let D be a simply connected domain in $\mathbf{C}$ and $\varphi(z, w)$ be a map from $D \times D \setminus \{z = w\}$ to $\mathbf{C}$ satisfying Assumption 7. We define the map $\mathcal{F}$ of a one-to-one parametrisation f with $C(f) \subset D$ as follows:*

$$\mathcal{F}(f) := \text{p.v.} \oint_{C(f)} \varphi(z, \bar{z}, w, \bar{w})|_{z=f(\cdot)} dw.$$

We then have the following:

$$\begin{aligned}
d\mathcal{F}(f; \delta f) &= \text{p.v.} \oint_C d\varphi(z, w; \delta z, \delta z) dw \\
&\quad + 2i \oint_C \varphi_{\bar{w}} \operatorname{Re}[(\delta w - \delta z)(-i dw)], \tag{2.8}
\end{aligned}$$

where $\delta z := \delta f \circ f^{-1}(z)$ and $\delta w := \delta f \circ f^{-1}(w)$.

Cor 9 *If φ satisfies $\varphi_z = -\varphi_w$ and $\varphi_{\bar{z}} = -\varphi_{\bar{w}}$, the formula (2.8) is simply reduced to*

$$d\mathcal{F}(f; \delta f) = 2i \oint_C \varphi_{\bar{w}} \operatorname{Re}[(\delta w - \delta z)(-i dw)]. \tag{2.9}$$

Consider, for instance, a complex velocity field induced by a vortex patch D of vorticity ω in the unbounded plane which is given by

$$u - iv = \frac{-\omega}{4\pi} \text{p.v.} \oint_{\partial D} \log(z - w) d\bar{w}.$$

The derivation of the vortex patch is deferred to Chapter. 3. Let $\varphi(z, w) := \frac{-\omega}{4\pi} \overline{\log(z - w)}$. Then the function φ satisfies Assumption 7. Furthermore, the assumption of Cor. 9 is satisfied. Hence, we have

$$\begin{aligned}
 \mathcal{F}(f) &:= \frac{-\omega}{4\pi} \text{p.v.} \oint_{C(f)} \log(z - w) d\bar{w} = \text{p.v.} \oint_{C(f)} \overline{\varphi(z, w)} dw \\
 \implies d\mathcal{F}(f; \delta f) &= \frac{-2i\omega}{4\pi} \oint_{C(f)} \overline{\left(\frac{-1}{z - w}\right)} \text{Re}[(\delta w - \delta z)(-i dw)] \\
 &= \frac{-\omega}{2\pi i} \oint_{C(f)} \frac{\text{Re}[(\delta z - \delta w)(-i dw)]}{z - w}. \tag{2.10}
 \end{aligned}$$

The sensitivity of the induced velocity field with respect to shape perturbations is described by (2.10). We see that the discontinuity due to the denominator $z - w$ is removable owing to the presence of $\delta z - \delta w$ in the numerator. As a result, we can numerically evaluate this integral by the standard trapezoidal rule in high accuracy.

Chapter 3

Vortex patch equilibria

An incompressible and inviscid flow in a flow domain $\Omega \subset \mathbf{R}^2$ is governed by the Euler equation in the vorticity form as follows.

$$\frac{d\omega}{dt} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = 0 \quad \text{in } \Omega, \quad (3.1)$$

$$-\Delta \psi = \omega \quad \text{in } \Omega, \quad (3.2)$$

where the vorticity distribution $\omega(x, y, t)$ and the streamfunction $\psi(x, y, t)$ satisfy

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad \omega = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \quad (3.3)$$

Here (u, v) is the velocity at a particle position $(x, y) \in \Omega$. Since we have (3.1) namely that the material derivative of the vorticity is zero, the vorticity ω is conserved on any particle path. Consider the particle $(x(t), y(t))$ advected by the flow which satisfies the differential equation

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v. \quad (3.4)$$

It follows from (3.1) that

$$\frac{d}{dt}[\omega(x(t), y(t), t)] = \frac{d\omega}{dt} + \frac{\partial \omega}{\partial x} \frac{dx}{dt} + \frac{\partial \omega}{\partial y} \frac{dy}{dt} = 0.$$

Hence the vorticity $\omega(x(t), y(t), t)$ is constant in time for an arbitrary pathline $(x(t), y(t))$. We denote by $T_t: \Omega \rightarrow \Omega$ the flow map $T_t(x_0, y_0) := (x(t), y(t))$ which is defined as a solution to (3.4) with initial data $x(0) = x_0$

and $y(0) = y_0$. As a consequence, as long as the flow map exists, the vorticity distribution is invariant under the flow map in time, that is to say, $\omega(T_t(x_0, y_0), t) = \omega(x_0, y_0, 0)$. Yudovich [17] has proved the existence and uniqueness of the weak solution $\psi(\cdot, t)$ to the Euler equation for all time $t \in \mathbf{R}$ under the assumption that the initial vorticity distribution $\omega(\cdot, 0)$ belongs to $L^1 \cap L^\infty$.

A simply connected domain $D \subset \Omega$ of finite area with constant vorticity ω_0 is called a vortex patch. In other words, the vorticity distribution of the vortex patch is given by

$$\omega = \omega_0 \chi_D,$$

where χ_D denotes the characteristic function of D . Since $\omega_0 \chi_D$ belongs to $L^1 \cap L^\infty$, the global existence and uniqueness of the solution is ensured. Hence the understanding of the evolution of vortex patches is considered to be a fundamental theme in mathematical fluid dynamics. Consider the unbounded planar flow domain $\Omega = \mathbf{R}^2$. A function G given by

$$G(x, y, x', y') = \frac{1}{2\pi} \log \sqrt{(x - x')^2 + (y - y')^2} \quad (3.5)$$

is the Green's function of the Laplace operator Δ associated with Ω . It follows from (3.2) that

$$\psi(x, y) = - \iint G(x, y, x', y') \omega(x', y') dx' dy'.$$

A uniform vorticity distribution yields

$$\psi(x, y) = -\frac{\omega_0}{4\pi} \iint_D \log[(x - x')^2 + (y - y')^2] dx' dy'.$$

Differentiating with respect to x and y , we obtain

$$u = \frac{\partial \psi}{\partial y} = -\frac{\omega_0}{2\pi} \iint_D \frac{y - y'}{(x - x')^2 + (y - y')^2} dx' dy', \quad (3.6)$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\omega_0}{2\pi} \iint_D \frac{x - x'}{(x - x')^2 + (y - y')^2} dx' dy'. \quad (3.7)$$

Here (u, v) expressed by (3.6) and (3.7) is called the velocity field induced by the vortex patch D of vorticity ω_0 . The velocity field (u, v) depends on the integral domain D , whereas the vorticity region D is advected by the flow. If an initial vorticity region D_0 and a vorticity constant ω_0 are prescribed,

then there exists a unique vortex patch solution $D(t)$ with $D(0) = D_0$ for all time $t \in \mathbf{R}$. In terms of the flow map, we can express it as $D(t) = T_t(D_0)$ where the flow map T_t is associated with the induced velocity field (3.6) and (3.7). The area of $D(t)$ remains unchanged in time owing to the shape derivative formula (2.1) (or the Reynolds transport theorem). However, other quantities such as the length or the curvature of $\partial D(t)$ are not conserved. In general, the geometry of the shape $D(t)$ will be complicated after long time evolution. Nevertheless, Chemin [18] has proved that the boundary $\partial D(t)$ is free of loss of regularity in time, that is to say, if the initial boundary ∂D_0 is of Hölder class $C^{1,\mu}$ for $0 < \mu < 1$, then $\partial D(t)$ remains to belong to $C^{1,\mu}$ for all time $t \in \mathbf{R}$. The precise statement of the theorem and its proof can be found in [19] which is a translation from Chemin's original result [18].

If the vorticity distribution vanishes, namely, the flow is irrotational everywhere, then the streamfunction ψ is a harmonic function on Ω . The harmonic conjugate of $-\psi$ is called a velocity potential, denoted by φ , since its gradient coincides with the velocity field (u, v) . It is thus useful to introduce an underlying holomorphic function denoted by $W(z) = \varphi(x, y) + i\psi(x, y)$ for $z = x + iy$. Since the complex derivative of W becomes $u - iv$, we call W a complex velocity potential and $u - iv$ a complex velocity. In what follows, we may identify the planar flow domain $\mathbf{R}^2$ with the complex plane $\mathbf{C}$.

We can similarly express a vortex patch in terms of complex variables. First, we consider the complex potential associated with the Green's function (3.5) which is a fundamental solution to the Poisson's equation whose source term is the Dirac delta function $\delta(x - x', y - y')$. In other words, the vorticity is given by

$$\omega(x, y) = \gamma \delta(x - x', y - y')$$

in the sense of distributions. Here we put a real constant γ that represents a strength of a singularity. In fluid dynamics, such a vorticity distribution is called a point vortex at (x', y') of strength γ . Since it is supported on (x', y') , the flow is irrotational everywhere except for that single point. We can thus find the complex velocity potential

$$W = \frac{\gamma}{2\pi i} \log(z - z_0), \quad (z_0 = x' + iy'), \quad (3.8)$$

together with

$$\psi = \text{Im } W = -\frac{\gamma}{2\pi} \log|z - z_0| = -\gamma G(x, y, x', y'), \quad (3.9)$$

$$u - iv = \frac{dW}{dz} = \frac{\gamma}{2\pi i} \frac{1}{z - z_0}. \quad (3.10)$$

We say that a point vortex at z_0 of strength γ induces the complex velocity potential (3.8), the streamfunction (3.9) or the complex velocity field (3.10) respectively. In general, a contour integral of the complex velocity field along an arbitrary Jordan curve C gives

$$\oint_C (u - iv) dz = \oint_C \frac{dW}{dz} dz = [W]_C = [\varphi + i\psi]_C.$$

The quantity $[\varphi]_C$ coincides with a circulation around C , while $[\psi]_C$ coincides with a flow rate through C , respectively. For the induced velocity field given by (3.10), the circulation around z_0 is always γ . From the point of view of complex analysis, we can say that a complex function with the purely real residue γ at $z = z_0$ corresponds to the velocity field induced by the point vortex.

We now transfer the previous derivation of a point vortex to a vortex patch. Once we see that

$$\omega(x, y) = \omega_0 \chi_D(x, y) = \omega_0 \iint_D \delta(x - x', y - y') dx' dy',$$

we can obtain every corresponding things in a straightforward manner as follows:

$$\begin{aligned} W &= \frac{\omega_0}{2\pi i} \iint_D \log(z - w) dw_1 dw_2, & (w = w_1 + iw_2), \\ \psi &= \text{Im } W = -\frac{\omega_0}{2\pi} \iint_D \log|z - w| dw_1 dw_2, \\ u - iv &= \frac{dW}{dz} = \frac{\omega_0}{2\pi i} \iint_D \frac{dw_1 dw_2}{z - w}. \end{aligned} \quad (3.11)$$

Especially, the contour integral of the induced complex velocity along the patch boundary ∂D gives

$$\begin{aligned} \oint_{\partial D} (u - iv) dz &= \iint_D \frac{\omega_0}{2\pi i} \oint_{\partial D} \frac{dz}{z - w} dw_1 dw_2 \\ &= \iint_D \omega_0 \chi_D(w) dw_1 dw_2 = \omega_0 |D|. \end{aligned}$$

Thus the induced velocity field has a circulation $\omega_0 |D|$ around the patch boundary ∂D .

Zabusky and Deem [6, 7] have exploited a useful evolution equation for vortex patch problems in terms of boundary integro-differential equations, called a contour dynamics. Applying the complex form of Green's

formula (2.7) to the induced velocity field (3.11), we obtain

$$u - iv = \frac{-\omega_0}{4\pi} \oint_{\partial D} \log(z - w) d\bar{w}.$$

Thus we do not need the domain integral for evaluating the induced velocity anymore. Particularly, the evolution of a single vortex patch is governed by

$$\frac{\partial}{\partial t} \overline{z(s, t)} = (u - iv)|_{z=z(s, t)} = \frac{-\omega_0}{4\pi} \oint_{\partial D(t)} \log(z(s, t) - w) d\bar{w},$$

where $z(\cdot, t)$ denotes the parametrisation of the patch boundary $\partial D(t)$.

Finally we remark that there is another formulation for this problem via a free-boundary partial differential equation. Owing to (3.3), the velocity (u, v) at (x_0, y_0) is tangential to the streamline $\{(x, y) \in \Omega \mid \psi(x, y, t_0) = \psi_0\}$ for the level $\psi_0 := \psi(x_0, y_0)$ at time $t = t_0$. If the flow is steady, namely, the time derivative of ψ vanishes, then the streamline is unchanged in time. Hence a pathline $(x(t), y(t))$ tracked from (x_0, y_0) coincides with the streamline. The steady-state solution to the Euler equation is usually written in the following form:

$$\begin{aligned} -\Delta \psi &= F(\psi) && \text{in } \Omega, \\ \psi &= \psi_b && \text{on } \partial\Omega, \end{aligned} \quad (3.12)$$

where $\psi_b: \partial\Omega \rightarrow \mathbf{R}$ is a Dirichlet boundary data and $F: \mathbf{R} \rightarrow \mathbf{R}$ is a priori unknown function corresponding to the vorticity distribution.

The steady state of a vortex patch solution is expressed by

$$F(\psi) = \omega_0 H(\psi_0 - \psi) = \begin{cases} 0, & \psi \geq \psi_0, \\ \omega_0, & \psi < \psi_0, \end{cases} \quad (3.13)$$

where H denotes the Heaviside function and $\psi_0 \in \mathbf{R}$ is a constant representing the level of the streamfunction on the boundary ∂D of the vortex patch. The equations (3.12) and (3.13) can be written in an elliptic free-boundary problem of the following form.

$$-\Delta \psi_1 = \omega_0 \quad \text{in } D, \quad (3.14)$$

$$-\Delta \psi_2 = 0 \quad \text{in } \Omega \setminus \overline{D}, \quad (3.15)$$

$$\psi_1 = \psi_2 = \psi_0 \quad \text{on } \partial D, \quad (3.16)$$

$$\frac{\partial \psi_1}{\partial n} = \frac{\partial \psi_2}{\partial n} \quad \text{on } \partial D, \quad (3.17)$$

$$\psi_2 = \psi_b \quad \text{on } \partial\Omega, \quad (3.18)$$

where $\psi_1 := \psi|_D$, $\psi_2 := \psi|_{\Omega \setminus \overline{D}}$, $\psi_b \in \mathbf{R}$ and n denotes the outward unit normal vector on ∂D . Here ψ_1 is a streamfunction restricted to the rotational flow part, namely, the vortex patch D and ψ_2 is one restricted to the irrotational flow part, respectively. Note that the boundary of the vortex patch should coincide with the streamline in the steady state. Thus the free-boundary condition (3.16) is important so that $\partial D = \{\psi = \psi_0\}$ holds true. ∂D is called a free-boundary which should be determined together with unknown streamfunctions ψ_1 and ψ_2 . Finding the steady configuration of the vortex patch corresponds to the elliptic free-boundary problem (3.14), (3.15), (3.16), (3.17) and (3.18).

3.1 Numerical method for vortex patch equilibria

We consider steady configurations in a vortex flow with M vortex patches. Elcrat and Protas [8] applied the standard shape calculus to a single vortex patch in a planar flow. As is noted in [8], it is straightforward to apply it to general cases $M > 1$. We however utilise the shape derivative formulae (2.5) and (2.8) so that we can deal with several vortex patches easily.

We assume that the complex potential W induced by a vortex patch D of vorticity ω is of the following form

$$W(z) = \frac{\omega}{2\pi i} \iint_D \log p(z-w) dw_1 dw_2,$$

where $w = w_1 + iw_2$ and p is an entire function with a simple zero at the origin. Note that, for the vortex patch in the unbounded plane, we have $p(z) = z$. The formula (2.8) can be applied to more general kernels, but in this paper, we focus on the kernel of this special form for simplicity. Then, the induced velocity field is given by

$$u - iv = \frac{dW}{dz} = \frac{\omega}{2\pi i} \iint_D \frac{p'(z-w)}{p(z-w)} dw_1 dw_2.$$

The complex form of Green's formula yields

$$u - iv = \frac{-\omega}{4\pi} \oint_{\partial D} \log p(z-w) d\bar{w} =: V_{\omega, \partial D}(z).$$

Hence, if we apply the shape derivative formulae (2.5) and (2.9) to the

induced velocity field, we obtain

$$\begin{aligned} dV_{\omega, C(f)} &= \frac{\omega}{2\pi i} \oint_{C(f)} \frac{p'(z-w)}{p(z-w)} \operatorname{Re} \left[\delta w \overline{(-i dw)} \right], & (z \notin C(f)), \\ dV_{\omega, C(f)} &= \frac{\omega}{2\pi i} \oint_{C(f)} \frac{p'(z-w)}{p(z-w)} \operatorname{Re} \left[(\delta w - \delta z) \overline{(-i dw)} \right], & (z \in C(f)), \end{aligned}$$

respectively.

We denote by D_m the vorticity region of m -th vortex patch and by ω_m its constant vorticity, for $1 \leq m \leq M$, respectively. Let $U_0(z)$ denote an external velocity field which may represent influences of some point vortices, a uniform flow, or may originate from transformation of the problem to a co-rotating frame in which a relative equilibrium is accomplished. Owing to the contour dynamics formulation by Zabusky and Deem [6, 7], the evolution of each boundary ∂D_m is governed by the velocity on the boundary as follows:

$$\begin{aligned} \frac{\partial}{\partial t} \overline{z_m(s, t)} &= (u - iv)|_{z=z_m(s, t)} \\ &= \sum_{m'=1}^M V_{\omega_{m'}, \partial D_{m'}}(z_m(s, t)) + U_0(z_m(s, t)), \quad (1 \leq m \leq M), \end{aligned} \tag{3.19}$$

where $z_m(s, t)$ is a certain parametrisation of the boundary ∂D_m at time t and s is a parameter in S^1 . Note that, only when $m = m'$, the principal value make sense, that is to say, the contour integral has a singular kernel for ∂D_m and non-singular ones otherwise. To find an equilibrium configuration of the M vortex patches, it is sufficient to solve

$$\operatorname{Re} \left[(u - iv)|_{z=f_m(s)} (-if'_m(s)) \right] = 0, \quad (s \in S^1, 1 \leq m \leq M), \tag{3.20}$$

where $f_m: S^1 \rightarrow \mathbf{C}$ is an unknown parametrisation of m -th patch boundary. Note that the left hand side of (3.20) is equivalent to the inner product of the velocity (u, v) and the normal vector on the boundary.

We now describe a numerical method based on the Newton method in which the derivative is obtained by the shape derivative formulae (2.5) and (2.9). We first introduce a map corresponding to (3.20). For parametrisations $f_m: S^1 \rightarrow \mathbf{C}$ of the patch boundaries $\partial D_m = C(f_m)$, $1 \leq m \leq M$, we define

$$\mathcal{F}_m(f_1, \dots, f_M) := \operatorname{Re} \left[(u - iv)|_{z=f_m(\cdot)} (-if'_m(\cdot)) \right].$$

We show how to derive the Gâteaux semiderivative of $\mathcal{F}_m$. First, we have the Gâteaux semiderivative of $U_0(f_m(s))$ at f_m in the direction δf_m in use of the standard directional differential calculus. If $m' \neq m$, then we can get the directional derivative of $V_{\omega_{m'}, C(f_{m'})}(f_m(s))$ at $f_m(s)$ in the direction $\delta f_m(s)$ similarly. We also obtain the Gâteaux semiderivative of $V_{\omega_{m'}, C(f_{m'})}(f_m(s))$ at $f_{m'}$ in the direction $\delta f_{m'}$ by the shape derivative formula (2.5). Finally, if $m' = m$, we can apply the shape derivative formula (2.9) to the singular contour integral $V_{\omega_m, C(f_m)}(f_m(s))$. Thus, the Gâteaux semiderivative of $\mathcal{F}_m$ is obtained.

We then use the following parametrisation of the m -th patch boundary,

$$z = f_{m,r}(s) := f_{m,0}(s) + r(s)\nu_m(s), \quad (s \in S^1), \quad (3.21)$$

where $f_{m,0}(s)$ is given parametrisation of the reference closed curve oriented counterclockwise, $\nu_m(s) := -if'_{m,0}(s)$ is its normal vector and $r: S^1 \rightarrow \mathbf{R}$ is a C^1 function with sufficiently small $|r(s)|$. Obviously the Gâteaux semiderivative of $f_{m,r}$ with respect to r is given by $\delta r \nu_m$. Using the chain rule, we can obtain the Gâteaux semiderivative of $\mathcal{F}_m(f_{1,r_1}, \dots, f_{M,r_M})$ with respect to $r_{m'}$ for $1 \leq m, m' \leq M$.

We use the discrete real Fourier series truncated up to $2N+1$ terms to approximate all periodic functions. Let $(\cdot)^\wedge$ denote the discrete Fourier transform from an $\mathbf{R}$ -valued 2π -periodic function to the corresponding Fourier coefficient in $\mathbf{R}^{2N+1}$. We thus have the following problem to find such an M -tuple of parametrisations $(f_{1,r_1}, \dots, f_{M,r_M})$ that

$$F(\hat{\mathbf{r}}) := \begin{pmatrix} \mathcal{F}_1(f_{1,r_1}, \dots, f_{M,r_M})^\wedge \\ \vdots \\ \mathcal{F}_M(f_{1,r_1}, \dots, f_{M,r_M})^\wedge \end{pmatrix} = 0,$$

where $\hat{\mathbf{r}}$ denotes the M -tuple of the Fourier coefficients $\hat{r}_m$. The trapezoidal rule is used to approximate the contour integrals in $\mathcal{F}_m$. Since the integrands in the shape derivative formulae are continuous, the Jacobian matrix DF of F can be evaluated by the trapezoidal rule in high accuracy. Note that each submatrix of the Jacobian matrix DF can be evaluated by the Gâteaux semiderivative $d\mathcal{F}_{m_1}(f_{1,r_1}, \dots, f_{M,r_M}; 0, \dots, 0, \delta f_{m_2, r_{m_2}}, 0, \dots, 0)$ for $1 \leq m_1, m_2 \leq M$. In general, the Jacobian matrix DF may not be invertible, in which case it has at least one singular value close to zero. Hence, we use the singular value decomposition to compute the pseudo inverse matrix $DF^\dagger$. We thus obtain the following Newton scheme in terms of Fourier coefficients,

$$\hat{\mathbf{r}}^{(n+1)} = \hat{\mathbf{r}}^{(n)} - DF^\dagger F(\hat{\mathbf{r}}^{(n)}), \quad \hat{\mathbf{r}}^{(0)} = 0.$$

We have several remarks on the implementation of the numerical procedure on computers. First, when we compute the pseudo inverse matrix $DF^\dagger$, we should carefully choose the tolerance for the minimum singular value to be cut off. The appropriate tolerance depends on the problem to be solved. Second, it generally takes too much time to compute the Jacobian matrix DF in each step. We may thus use pseudo the Newton method to save computational costs. But, in most cases described in what follows, it is enough to set $N = 32$ to approximate the periodic functions. This is a handful size to compute everything in each step.

In what follows, we show some reconstructed numerical results to confirm that our numerical scheme works well. The first example is the Pierrehumbert pair¹ of steadily translating vortex patches of the vorticity strengths of opposite signs [6, 1]. The induced velocity field is of the following form

$$u - iv = V_{\omega, \partial D_1}(z) + V_{-\omega, \partial D_2}(z) - U_T,$$

where $\partial D_2 := \overline{-\partial D_1}$ is oriented counterclockwise and $-U_T$ is a certain uniform velocity speed. Note that assuming the presence of the uniform flow of speed $-U_T$ is equivalent to considering the relative equilibria in a co-moving frame of the speed U_T . We show a reconstructed numerical result in Fig. 3.1. The normal component of the velocity on the boundary vanishes, which implies that vortex patches are steadily translating along the real axis. Although the axisymmetry configuration of the patches, namely, $\partial D_2 = \overline{-\partial D_1}$, is not considered in the numerical scheme, the symmetric solution is successfully reconstructed. Note that we may modify of the scheme so that the axisymmetry is taken in account.

The second example is a comparison with an analytic expression of a vortex patch surrounded by four point vortices in the 4-fold symmetry given by Crowdy [3], in which he has derived the family of the exact solutions by using a conformal mapping. The velocity field in a co-rotating frame of an angular velocity Ω is given by

$$u - iv = V_{\omega, \partial D}(z) + \sum_{m=0}^3 \frac{\gamma}{2\pi i} \frac{1}{z - i^m} - i\Omega \bar{z}.$$

¹Note that such a pair of vortex patches has been firstly found by Deem and Zabusky, while a continuous 1-parameter family of such pairs has been shown by Pierrehumbert, interestingly in which the steady solutions are obtained by continuation from a pair of translating point vortices.

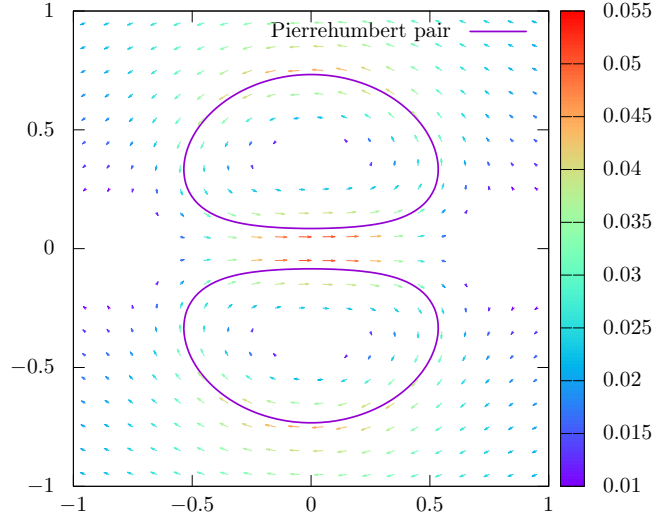


Figure 3.1: The Pierrehumbert pair of vortex patches whose boundaries are indicated by real lines. Their inducing velocity field is also plotted. This is shown in a co-moving frame translating with the same speed as the vortex patches.

The boundary of the vorticity region is rigorously expressed by

$$z = R \left[\frac{1}{\zeta} + \frac{b\zeta^3}{\zeta^4 - a^4} \right], \quad \zeta = \exp(-is), \quad (s \in S^1), \quad (3.22)$$

where both R and b are real constants depending on the parameter $a > 0$ (see [3] for the detail). We obtain a numerical result shown in Fig. 3.2. The velocity vanishes inside the vortex patch, which implies that the vortex patch is rotating with the angular velocity Ω rigidly. The patch boundary of the approximate solution coincides with the closed curve (3.22) for $a = 3.634$. We also show quantitative evidence in Table 3.1. A moment of a shape is a certain integral quantity characterising its geometry. Since the moments only depend on its geometry, we can compare parametrisations of the two different form (3.21) and (3.22). The errors of the moments are sufficiently admissible.

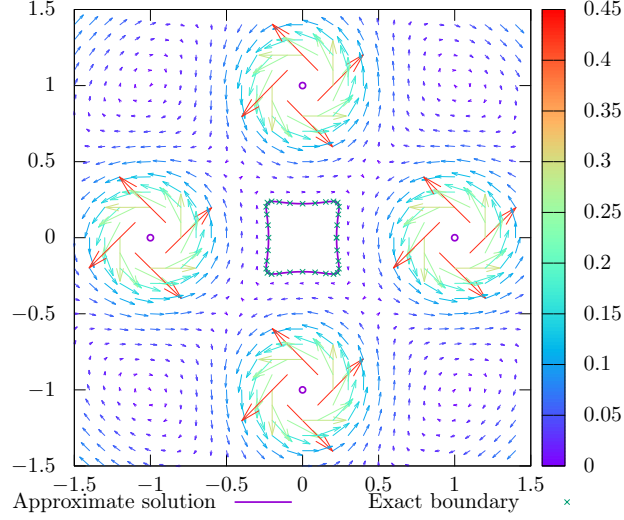


Figure 3.2: Crowdy's vortex patch surrounded by four point vortices and its induced velocity field are shown. This is shown in a co-rotating frame of a certain angular velocity. The markers indicate the exact boundary whose rigorous expression has been derived by Crowdy [3].

	Numerical sol.	Exact sol.	Error
0th moment	2.133e-01	2.133e-01	7.107e-07
4th moment	-2.043e-04	-2.039e-04	3.755e-07
8th moment	8.376e-07	8.359e-07	1.705e-09
12th moment	-4.779e-09	-4.804e-09	2.509e-11

Table 3.1: The moments of the vorticity region in Fig. 3.2 are listed. Each quantity is evaluated by the numerical integration. Since the configuration is in the 4-fold symmetry, only the m -th moments for $m = 0, 4, 8, 12$ are compared.

3.2 Vortex patch lattices

Tkachenko [9] represented an analytic formula of point vortex lattices rotating rigidly. We are going to extend this configuration to vortex patch lattices. Although the Weierstrass zeta function was originally used in the analytic formula, we formulate the vortex patch lattices in terms of the Schottky-Klein prime function P as Crowdy [11] did. This is because the convergence of the Weierstrass zeta function is slow, while the P function is expressed by a rapidly convergent Laurent series expansion and is easy to evaluate numerically. In order to use the numerical scheme in Section 3.1, we should determine an external flow so that the stationary lattice is rotating rigidly.

Let us first reformulate the analytic expression of point vortex lattices in terms of the P function. The P function for the annulus $\rho < |\zeta| < 1$ is given by

$$P(\zeta) = (1 - \zeta) \prod_{m=1}^{\infty} (1 - \rho^m \zeta)(1 - \rho^m / \zeta). \quad (3.23)$$

We here consider vortex lattices containing only one point vortex in each period cell, as is originally done by Tkachenko. Note that Crowdy [11] originally considered the case where vortex lattices contain more than one point vortices in each period cell. Under the conformal mapping $z = -i \log \zeta$ from the annulus to the rectangular period cell $0 < \operatorname{Re} z < 2\pi$ and $0 < \operatorname{Im} z < -\log \rho$, the function (3.23) has zeros at

$$z = z_{km} = 2\pi k - im \log \rho, \quad (k, m \in \mathbf{Z}).$$

We consider the complex potential given by

$$W(z) = \frac{\Gamma}{2\pi i} \log P(\zeta) + W_0(z), \quad (3.24)$$

where W_0 is a certain entire function to be determined. The potential (3.24) has log singularities of strength $\Gamma/(2\pi i)$ at $z = z_{km}$, in other words point vortices of the same strength Γ . The induced complex velocity is given by

$$u - iv = \frac{dW}{dz} = \frac{\Gamma}{2\pi i} \frac{d\zeta}{dz} \frac{P'(\zeta)}{P(\zeta)} + W_0'(z) = \frac{\Gamma}{2\pi} K(\zeta) + W_0'(z),$$

where $K(\zeta)$ is called K function defined by

$$K(\zeta) := \frac{\zeta P'(\zeta)}{P(\zeta)}.$$

It satisfies the following pseudo-periodicity property

$$K(\rho\zeta) = K(\zeta) - 1. \quad (3.25)$$

A point vortex is advected by the induced velocity of the flow in the absense of the contribution from the point vortex itself. We thus need to consider the regular part in the Laurent series expansion of $K(\exp(iz))$ at $z = 0$. We have

$$\begin{aligned} K(\zeta) - \frac{1}{\zeta - 1} &= 1 + \sum_{m=1}^{\infty} \left(\frac{\rho^m \zeta}{\rho^m \zeta - 1} - \frac{\rho^m / \zeta}{\rho^m / \zeta - 1} \right) \\ &\rightarrow 1 \quad (\text{as } \zeta \rightarrow 1). \end{aligned}$$

Under the conformal mapping $z = -i \log \zeta$, we get

$$\begin{aligned} \frac{1}{\zeta - 1} &= \frac{1}{\exp(iz) - 1} = \frac{1}{iz - z^2/2 + O(z^2)} \\ &= \frac{1}{iz} - \frac{1}{2} + O(z), \end{aligned}$$

from which it follows, as $z \rightarrow 0$,

$$K(\exp(iz)) \approx \frac{1}{iz} + \frac{1}{2}.$$

We thus have

$$(u - iv)|_{z=0} = \frac{\Gamma}{4\pi} + W'_0(0). \quad (3.26)$$

Note that we have the following relation between K function and the Weierstrass zeta function with half-periods π and $id/2$ denoted by $\mathcal{Z}(z)$:

$$K(\zeta) = \frac{1}{2} - i\mathcal{Z}(z) + \frac{i\mathcal{Z}(\pi)}{\pi}z,$$

in which we also see the regular part of K function at $z = 0$.

We seek such a stationary point vortex lattice configuration that the point vortex at the origin $z = 0$ is at rest and the others at $z = z_{km}$ ($k, m \neq 0$) are rotating rigidly with an angular velocity Ω . We thus assume that the velocity field satisfies

$$(u + iv)|_{z=z_{km}} = i\Omega z_{km} \quad (k, m \in \mathbf{Z}). \quad (3.27)$$

In such a lattice configuration, we need to choose W_0 carefully so that locations of the singularities are relatively fixed throughout their evolution. As in Tkachenko [9], we assume $W'_0(z) = \alpha z + \beta$ where α and β are certain complex constants. The constants Ω , α and β are determined as follows. Owing to (3.26) and (3.27), we have

$$\beta = -\frac{\Gamma}{4\pi}. \quad (3.28)$$

It follows from (3.25)

$$\begin{aligned} (u - iv)|_{z+z_{km}} &= \frac{\Gamma}{2\pi} K(\rho^m \zeta) + f'(z + z_{km}) \\ &= \frac{\Gamma}{2\pi} (K(\zeta) - m) + f'(z) + \alpha z_{km} \\ &= (u - iv)|_z - \frac{\Gamma}{2\pi} m + \alpha z_{km}. \end{aligned} \quad (3.29)$$

Combining (3.27) and (3.29), we have

$$(u - iv)|_{z_{km}} = -\frac{\Gamma}{2\pi} m + \alpha z_{km} = -i\Omega \overline{z_{km}}.$$

Substituting $(k, m) = (1, 0)$ and $(0, 1)$, we get

$$\alpha = -i\Omega, \quad (3.30)$$

$$\Omega = \frac{\Gamma}{4\pi d}. \quad (3.31)$$

Consequently, the complex velocity is represented by

$$u - iv = \frac{\Gamma}{2\pi} \left[K(\zeta) - \frac{1}{2} \right] + \frac{\Gamma}{4\pi di} z.$$

We apply the argument above to the vortex patch lattices. We consider a flow induced by infinitely many vortex patches of the constant vorticity ω arranged in lattices. One specific vortex patch is referred to as a centre vortex patch, denoted by D . We assume that the shape of any vortex patch is of the form $D + z_{km}$. In other words, the vorticity distribution is invariant with respect to the shift transformation $z \mapsto z + 2\pi k + idm$ for all $k, m \in \mathbf{Z}$. It is thus natural to consider the complex potential of the following form:

$$W(z) = \frac{\omega}{2\pi i} \iint_D \log P(\exp(iz - iw)) dw_1 dw_2 + W_0(z),$$

which is the domain integral of (3.24) with respect to the singular location w . The corresponding complex velocity is given by

$$u - iv = \frac{\omega}{2\pi} \iint_D K(\exp(iz - iw)) dw_1 dw_2 + \frac{dW_0}{dz}(z).$$

We set $W'_0(z) = \alpha z + \beta$ with the same constants α, β and Ω given by (3.28), (3.30), (3.31), in which the circulation Γ is replaced by

$$\Gamma = \omega|D|. \quad (3.32)$$

We then obtain

$$u - iv = \frac{\omega}{2\pi} \iint_D \left[K(\exp(iz - iw)) - \frac{1}{2} \right] dw_1 dw_2 - i\Omega z.$$

We confirm that this velocity field leads to the double pseudo-periodicity similar to the case of point vortex lattices. We have

$$\begin{aligned} & (u - iv)|_{z+z_{km}} \\ &= \frac{\omega}{2\pi} \iint_D \left[K(\exp(iz - iw)) - m - \frac{1}{2} \right] dw_1 dw_2 - i\Omega(z + z_{km}) \\ &= (u - iv)|_z - \frac{\Gamma}{2\pi}m - i\frac{\Gamma}{4\pi d}(2\pi k + idm) \\ &= (u - iv)|_z - i\Omega\overline{z_{km}}, \end{aligned}$$

from which it follows

$$(u - iv)|_{z+z_{km}} - \overline{i\Omega(z + z_{km})} = (u - iv)|_z - \overline{i\Omega z}. \quad (3.33)$$

Eq. (3.33) reads that the velocity field induced by vortex patch lattices is doubly periodic in a co-rotating frame of the angular velocity Ω . Note that (3.32) is nothing but Kelvin's circulation theorem, which is an important constraint to extend the lattice configuration of point vortices to that of vortex patches.

3.3 Numerical results

Based on this formulation, we seek such a doubly periodic configuration of vortex patches where every vortex patch $D + z_{km}$ is rotating with the angular velocity Ω without changing its shapes; namely, for each $k, m \in \mathbf{Z}$, the patch boundary $\partial D + z_{km}$ is invariant in time in the co-rotating frame

of the angular velocity Ω . Consequently, it follows from (3.33) that (3.19) is reduced to the equation only for the centre vortex patch D as follows.

$$\operatorname{Re} \left[(u - iv - \overline{i\Omega z})|_{z=f(\cdot)} (-if') \right] = 0 \quad (z \in \partial C(f)), \quad (3.34)$$

where f denotes the parametrisation of the unknown patch boundary ∂D . We need to solve (3.32) for a given circulation Γ around the centre vortex patch.

We modify our numerical scheme so that we can solve the equations (3.32) and (3.34) simultaneously. For given Γ and ω with unknown shape D , the corresponding maps are given by

$$\begin{pmatrix} \mathcal{F}_1(f) \\ \mathcal{F}_2(f) \end{pmatrix} := \begin{pmatrix} \omega \mathcal{A}(f) - \Gamma \\ \operatorname{Re} \left[(u - iv - \overline{i\Omega z})|_{z=f(\cdot)} (-if') \right] \end{pmatrix}.$$

Note that the Gâteaux semiderivative of $\mathcal{F}_1$ can be obtained by (2.6). Substituting the parametrisation of the form $f_r = f_0 + r\nu$ as in (3.21), we have the equations for the Fourier coefficient $\hat{r}$

$$F(\hat{r}) := \begin{pmatrix} \mathcal{F}_1(f_r) \\ \mathcal{F}_2(f_r) \end{pmatrix}^\wedge = 0.$$

The reference parametrisation f_0 is given by a circle of certain radius. Since $F: \mathbf{R}^{2N+1} \rightarrow \mathbf{R}^{2N+2}$, the Jacobian matrix DF of F at $\hat{r}$ becomes a $(2N+2) \times (2N+1)$ matrix. Hence, we always use the singular value decomposition to compute the pseudo inverse matrix $DF^\dagger$. We thus have the following Newton scheme in terms of Fourier coefficients,

$$\hat{r}^{(n+1)} = \hat{r}^{(n)} - DF^\dagger F(\hat{r}^{(n)}), \quad \hat{r}^{(0)} = 0.$$

An interesting parameter we focus on in this problem is the aspect ratio of the lattice, defined by $a := d/(2\pi)$, since it is not obvious how the shape of the steady vortex patch depends on the lattice structure.

We first consider vortex patches arranged in regular rectangular lattices, that is, $a = 1$. We set $N = 32$ and $\Gamma = 1$ in computations. We show numerical results in Fig. 3.3. We observe that, as ω gets smaller, the enlarged shape becomes rectangular. The limiting shape as $\omega \approx 0.0518$ looks like a regular rectangle with the diagonal length $\approx 2\pi$, that is to say, its vertices touch nearly to those of neighbouring vortex patches. For ω smaller than 0.0518, the Newton iterations do not converge.

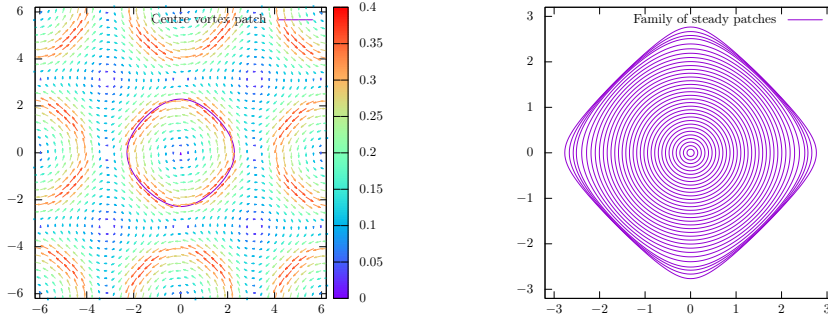


Figure 3.3: The left figure shows a velocity field induced by steady vortex patch lattices in the co-rotating frame. The real line indicates the boundary of the centre vortex patch. The right figure shows a family of centre vortex patches with different vorticity strength $0.0518 \leq \omega \leq 51.2$. The outermost vortex patch is a rectangular shape, while the innermost one is a circular shape.

Second, we consider the case of the non-regular lattice, that is, $a \neq 1$. We show centre vortex patches for the case $0.85 \leq a < 1$ and $1 < a \leq 1.15$. Fig. 3.4 shows the cases $a = 0.85$ and 1.15 . In this case, the shape becomes elliptic. When $a < 1$ (resp. $a > 1$), the shapes are elongated to the imaginary axis (resp. the real axis). The limiting value of ω is much larger than that we observe in the regular lattice case even when a is close to 1.

If the lattice is nearly regular, namely $a \approx 1$, another kind of families of vortex patches are observed as ω is small enough. Fig. 3.5 shows the cases $a = 0.97$ and 1.03 . The shapes resemble those in Fig. 3.3. Interestingly the rectangular shapes are not obtained by a continuation from a point vortex. Especially the one-parameter family of ω is discontinuous; nevertheless the two-parameter family is still continuous with respect to the parameter a . It is also interesting that the rectangular shapes flatten out along the real axis (resp. the imaginary axis) when $a < 1$ (resp. $a > 1$), which is the opposite behavior compared to the family in Fig. 3.4.

Convergence histories are plotted in the left side of Fig. 3.6. Note that we can always choose the initial guess so that $\mathcal{F}_1 = 0$ at the initial step, namely, $|D| = \Gamma/\omega$, which can be satisfied by a certain magnification of the initial shape D . This leads to attain sufficiently small residues $|\mathcal{F}_1(f_{r(m)})|$ through the iterations. Hence, it is enough to check only the rest residues $\|\mathcal{F}_2(f_{r(m)})\|_{L^2}$ which is shown in Fig. 3.6. For each solution, both the residues reached an order of $O(10^{-16})$ which is a machine epsilon order

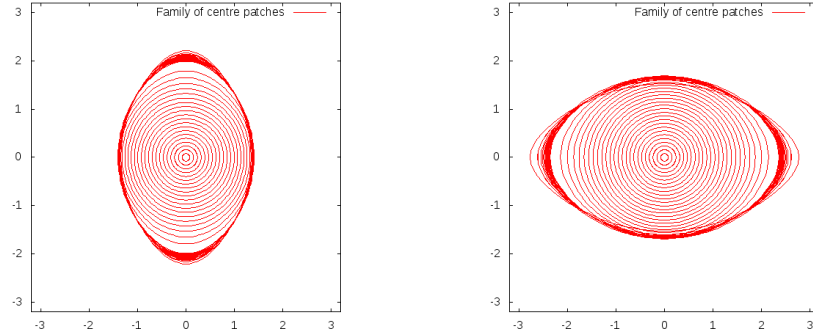


Figure 3.4: The left (resp. right) figure shows the family of elliptic vortex patches with various ω for $a = 0.85$ (resp. 1.15). The outermost shape nearly touches the boundary of the period cell.

in the double precision.

The right side of Fig. 3.6 concerns the continuous variation of the shapes in terms of moments. It indicates both of the behavior shown in Figs. 3.4 and 3.5, namely, the elliptic deformations and the rectangular deformations throughout the families. The most important quantity is the second moment representing the ellipticity of the shape. As the shape becomes larger, the second moment increases at first, and it in turn goes to decrease after a gap. We also see that the second moment is bounded above by the other moments on the right segment, which indicates that the shape is less elliptic as it becomes larger. As a consequent, it is appropriate to conclude that the family of the shapes can not be continued between the gap.

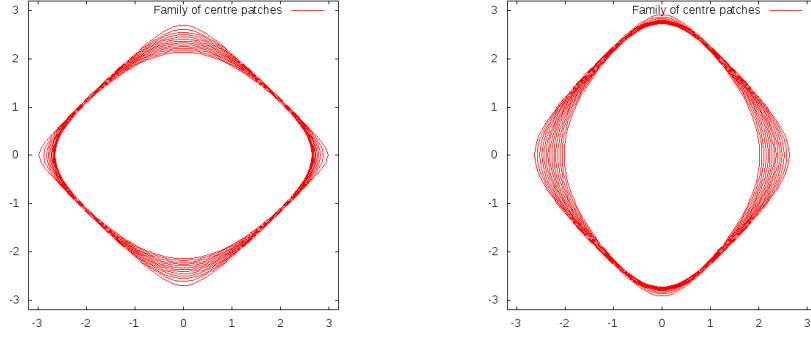


Figure 3.5: The left (resp. right) figure shows the family of the rectangular shapes with various ω for $a = 0.97$ (resp. 1.03). As the area gets smaller, the shape tends to be elliptic; nevertheless it elongates in the direction perpendicular to that of Fig. 3.4.

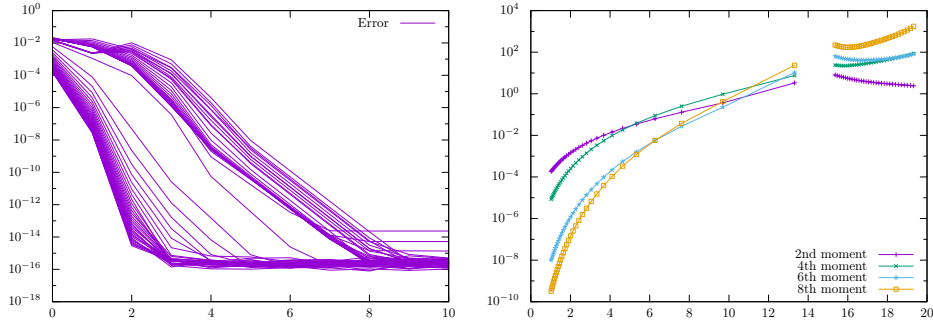


Figure 3.6: (Left) The convergence history of the Newton method is shown. The vertical axis indicates the L^2 norm of $\mathcal{F}_2(f_{r(m)})$. (Right) The absolute values of m -th moments ($m = 2, 4, 6, 8$) of the vorticity region are plotted against 0th moment, namely, its area. The aspect ratio a is 0.975. We can quantitatively observe that the family of the shapes are discontinuous, as is shown in Figs. 3.4 and 3.5.

Chapter 4

Conclusion

A shape derivative formula for singular contour integrals is derived. The formula has a much simpler expression compared to that by Elcrat and Protas [8]. The derivation is elementary and the knowledge of shape calculus is unnecessary, which is an advantage over the existing method [8]. With this new formula, we numerically construct vortex patch lattices in a rectangular array, which is a generalisation of the preceding works on point vortex lattices by Tkachenko [9] and Crowdy [11]. The induced velocity field is described in terms of the K function instead of the Weierstrass zeta function. The shape derivative formula is applicable to this induced velocity, which leads to a numerical method with the Newton method for vortex patch equilibria. We show that the shape of the steady vortex patch is rectangular for the regular lattice and elliptic for the non-regular lattice, respectively. Furthermore, we find a new family of vortex patch equilibria in the nearly regular lattice case which are not obtained by a simple continuation from a point vortex.

When we consider a general flow domain, the integral kernel which comes from the Green function of the flow domain is of different form; nevertheless our shape derivative formula is still easily applicable.

Owing to the simplicity of this formula, we have many applications. First, using our numerical approach, any steady configuration of point vortices will be extended to that of vortex patches. The only requirement in this formula is that the integral kernel has a logarithmic singularity, which is the case in many fluid applications. For example, it is interesting to apply the procedure to point vortex equilibria entrapped around a Kasper wing configuration [20] which is a flow problem in a triply connected exterior domain. Second, the shape derivative formula is applicable in linear stability

analysis as in [8, 5]. Finally, since the shape derivative is evaluated in high accuracy, there could be an alternative and efficient way to solve shape optimisation problems in *Vortex design problem* [21]. These applications will be reported in near future.

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