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# Gap Number of Groups

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Finite gap number is introduced by J. C. Lennox and J. E. Roseblade in [LR]. We study groups of the small gap number.

## 1 ladder index

Let $T$ be complete theory in an $L$, $\phi(\overline{x},\overline{y})$ $L$-formula ($\overline{x},\overline{y}$ are free variables).

**Definition 1** *An $n$-**ladder** for $\phi$ is a sequence $(\overline{a}_0,\ldots,\overline{a}_{n-1};\overline{b}_0,\ldots,\overline{b}_{n-1})$ of tuples in some model $M$ of $T$, such that*

$$\forall i,j<n, M\models\phi(\overline{a}_i,\overline{b}_j)\Leftrightarrow i\leq j.$$

*We say that $\phi$ is **stable** formula if there exists $n$ such that no $n$-ladder for $\phi$ exists; otherwise it is **unstable**. The least such $n$ is the **ladder index** of $\phi$.*

**Theorem 2** *The theory $T$ is unstable if and only if there exists an unstable formula in $L$ for $T$.*

Henceforth we consider the ladder index for the commutativity formula "$xy=yx$". The ladder index of a group $G$ for the commutativity formula is denoted by $\ell(G)$.

## 2 gap number

Let $G$ be a group.

**Definition 3** *A group G has a finite gap number if for any subgroups* $H_0, H_1, \ldots, H_n, \ldots$ *of G, among the sequence*

$$C_G(H_0) \leq C_G(H_1) \leq \cdots \leq C_G(H_n) \leq \cdots$$

*there exist at most* $m$ *many strict inclusions. The most such* $m$ *is the* ***gap number*** *of G, and denoted by* $g(G)$.

**Lemma 4** *Let* $g(G) = n$. *Suppose that the sequence*

$$C_G(H_0) > C_G(H_1) > \cdots > C_G(H_n)$$

*gives gap number* $n$. *Then there exists* $a_i (0 \leq i \leq n)$ *in G such that* $C_G(H_i) = C_G(\{a_0, a_1, \ldots, a_i\})$ *for each* $i$. *In particular we may do* $a_0 = 1$. *Henceforth we abbreviate as* $C_G(\{a_0, a_1, \ldots, a_i\}) = (a_0, a_1, \ldots, a_i)$.

By Lemma 4, we can prove the following:

**Theorem 5** [ITT] $\ell(G) = g(G) + 2$.

**Lemma 6** *Let* $A, B \subset G$ *with* $A \subset B$. *Then* $(A) \supset (B)$.

**Lemma 7** *Let* $A \subset G$. *Then* $(((A))) = (A)$.

By the above lemma, the following holds:

**Lemma 8** *Let* $g(G) = n$. *Suppose* $(a_0, \ldots, a_n; b_0, \ldots, b_n)$ *is* $(n+1)$*-ladder. Then*

$$\begin{aligned} ((a_0)) &= (b_n, \ldots, b_1, b_0); \\ ((a_0, a_1)) &= (b_n, \ldots, b_1); \\ &\vdots \\ ((a_0, \ldots, a_n)) &= (b_n). \end{aligned}$$

**Lemma 9** *Let* $g(G) = n$. *Suppose that the sequece*

$$G > (a_1) > \cdots > (a_1, a_2, \ldots, a_n)$$

*gives gap nuber* $n$. *Then* $(a_1, a_2, \ldots, a_{n-1})$ *is abelian.*

# 3 Groups of gap number up to four

From now on we do not consider the ladder index but the gap number.

**Theorem 10** [ITT] $g(G) = 0$ *if and only if* $G$ *is abelian.*

**Theorem 11** [ITT] *There exist no groups* $G$ *of* $g(G) = 1$.

**(proof)** Let $g(G) \geq 1$. Then there exists $a \in G$ such that $G > (a)$. Since $(a) \neq G$, there exists $b \notin (a)$. Therefore, we have
$G > (a) > (a, b)$. Thus $g(G) \geq 2$.

**Theorem 12** [ITT] $g(G) = 2$ *if and only if* $G$ *is not abelian, and for any* $a, b \in G \setminus Z(G)$*, if* $(a) \neq (b)$ *then* $(a, b) = Z(G)$.

**Example 13** $g(S_3) = g(D_n) = 2$ *(*$D_n$ *is a dihedral group).*

**Example 14** $g(SL(2, F)) = 2$ *(*$F$ *is a field).*

**Theorem 15** [ITT] *There exist no groups* $G$ *of* $g(G) = 3$.

**(proof)** Let $g(G) \geq 3$. Then there exist $a_1, a_2 \in G$ such that
$G > (a_1) > (a_1, a_2) > Z(G)$.
**Case 1**: $a_1a_2 = a_2a_1$.
Since $(a_1) \neq (a_2)$, we may assume $(a_1) \setminus (a_2) \neq \emptyset$. Let $b \in (a_1) \setminus (a_2)$. As $a_1 \notin G$, there exists a $c \in G \setminus (a_1)$. Therefore, we have

$$G > (a_1) > (a_1, a_2) > (a_1, a_2, b) > (a_1, a_2, b, c).$$

Thus $g(G) \geq 4$.
**Case 2**: $a_1a_2 \neq a_2a_1$.
There exists a $d \in (a_1, a_2) \setminus Z(G)$. Since $d \notin Z(G)$, we can find $e \notin G \setminus (d)$. Then we have

$$G > (d) > (d, a_1) > (d, a_1, a_2) > (d, a_1, a_2, e).$$

Thus $g(G) \geq 4$.

**Example 16** $g(S_4) = g(S_5) = 4$.

# 4 Groups of gap number five

In this section, we investigate whether a group $G$ of gap number 5 exists or not.

Let $g(G) = 5$ and let $(1, a_1, \dots, a_5; b_0, \dots, b_4, 1)$ be 6-ladder.

**Case 1**: $a_1a_2 = a_2a_1, a_1a_3 = a_3a_1$ and $a_2a_3 = a_3a_2$.
Then we have

$$G > (a_1) > (a_1, a_2) > (a_1, a_2, a_3) > (a_1, a_2, a_3, b_2) > (a_1, a_2, a_3, b_2, b_1) > Z(G).$$

Thus, $g(G) \geq 6$.

**Case 2**: $a_1a_2 = a_2a_1, a_1a_3 = a_3a_1, a_2a_3 \neq a_3a_2$ and $a_1a_4 \neq a_4a_1$.
Then we have

$$G > (b_4) > (b_4, a_1) > (b_4, a_1, a_2) > (b_4, a_1, a_2, a_3) > (b_4, a_1, a_2, a_3, a_4) > Z(G).$$

Thus, $g(G) \geq 6$.

**Case 3**: $a_1a_2 = a_2a_1, a_1a_3 = a_3a_1, a_2a_3 \neq a_3a_2$ and $a_1a_4 = a_4a_1$.
Then we have

$$G > (a_1) > (a_1, b_3) > (a_1, b_3, a_2) > (a_1, b_3, a_2, a_3) > (a_1, b_3, a_2, a_3, a_4) > Z(G).$$

Thus, $g(G) \geq 6$.

**Case 4**: $a_1a_2 = a_2a_1, a_1a_3 \neq a_3a_1$ and $a_2a_3 = a_3a_2$.
Then we have

$$G > (a_2) > (a_2, a_1) > (a_2, a_1, a_3) > (a_2, a_1, a_3, a_4) > Z(G).$$

Moreover $a_2a_1 = a_1a_2, a_2a_3 = a_3a_2$ and $a_1a_3 \neq a_3a_1$. By case 2, 3, $g(G) \geq 6$.

**Case 5**: $a_1a_2 = a_2a_1$ and $a_1a_3 \neq a_3a_1$.
Then we have

$$G > (b_4) > (b_4, b_3) > (b_4, b_3, a_1) > (b_4, b_3, a_1, b_1) > Z(G).$$

Moreover $b_4a_1 = a_1b_4$ and $b_3a_1 = a_1b_3$. By case 1, 4, $g(G) \geq 6$.

Therefore, by case 1 through 5, we hold $a_1a_2 \neq a_2a_1$.

**Case 6**: $a_1a_2 \neq a_2a_1$ and $a_1a_3 = a_3a_1$.
Then we have

$$G > (a_1) > (a_1, a_3) > (a_1, a_3, a_2) > (a_1, a_3, a_2, a_4) > Z(G).$$

Moreover $a_1a_3 = a_3a_1$. Thus, $g(G) \geq 6$.

**Case 7**: $a_1a_2 \neq a_2a_1, a_1a_3 \neq a_3a_1$ and $a_2a_3 = a_3a_2$.
Then we have

$$G > (a_2) > (a_2, a_1) > (a_2, a_1, a_3) > (a_2, a_1, a_3, a_4) > Z(G).$$

Moreover $a_2a_3 = a_3a_2$. By case 6, $g(G) \geq 6$.

Therefore, by case 1 through 7, we hold $a_1a_2 \neq a_2a_1, a_1a_3 \neq a_3a_1$ and $a_2a_3 \neq a_3a_2$.

**Case 8**: all of $a_1, a_2, a_3, a_4$ are noncommutative except $a_1a_4 = a_4a_1, a_2a_4 = a_4a_2$.
Then we have

$$G > (a_1) > (a_1, a_2) > (a_1, a_2, a_4) > (a_1, a_2, a_4, a_3) > Z(G).$$

Moreover $a_1a_4 = a_4a_1$. By case 6, $g(G) \geq 6$.

**Case 9**: all of $a_1, a_2, a_3, a_4$ are noncommutative except $a_1a_4 = a_4a_1, a_3a_4 = a_4a_3$.
Then we have

$$G > (a_1) > (a_1, a_3) > (a_1, a_3, a_4) > (a_1, a_3, a_4, a_2) > Z(G).$$

Moreover $a_1a_4 = a_4a_1$. By case 6, $g(G) \geq 6$.

**Case 10**: all of $a_1, a_2, a_3, a_4$ are noncommutative except $a_2a_4 = a_4a_2, a_3a_4 = a_4a_3$.
Then we have

$$G > (a_2) > (a_2, a_3) > (a_2, a_3, a_4) > (a_2, a_3, a_4, a_1) > Z(G).$$

Moreover $a_2a_4 = a_4a_2$. By case 6, $g(G) \geq 6$.

In the cases of remaining we understand the following:

**Lemma 17** *Let* $G > (a_1) > (a_1, a_2) > \cdots > (a_1, a_2, a_3, a_4, a_5) = Z(G)$. *Then we can do as follows: all of* $a_1, a_2, a_3, a_4$ *are noncommutative except* $a_1a_4 = a_4a_1, a_2a_4 = a_4a_2, a_3a_4 = a_4a_3, a_1a_5 = a_5a_1, a_2a_5 = a_5a_2, a_3a_5 = a_5a_3$.

**(proof)** We have

$$G > (a_1) > (a_1, a_2) > (a_1, a_2, a_3) > (a_1, a_2, a_3, b_4) > (a_1, a_2, a_3, b_4, b_3).$$

Moreover all of $a_1, a_2, a_3$ are noncommutative, and all of $b_4, b_3, b_2$ are noncommutative, as desired.

**Question 18** *Does there exist a group* $G$ *of* $g(G) = 5$*?*

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