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Spectral properties of massless Dirac operators with real-valued potentials

By

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Abstract

We prove that a Schnol'-type theorem holds for massless Dirac operators under minimal assumptions on the potential, and apply this result to conclude that the spectrum of a certain class of such operators covers the whole real line. We also discuss embedded eigenvalues of massless Dirac operators with suitable scalar potentials.

§ 1. Introduction

This paper is an announcement of results on spectral properties of Dirac operators with real-valued potentials and will be followed by a complete treatment in which all proofs are given.

The Dirac operators to be considered in this paper are

$$H_2 = -i\,\sigma\cdot\nabla + q(x) \quad \text{in} \;\; L^2(\mathbb{R}^2;\mathbb{C}^2) \tag{1.1}$$

and

$$H_3 = -i\,\alpha\cdot\nabla + q(x) \quad \text{in} \;\; L^2(\mathbb{R}^3;\mathbb{C}^4). \tag{1.2}$$

Here $\sigma = (\sigma_1,\, \sigma_2)$ and $\alpha = (\alpha_1,\, \alpha_2,\, \alpha_3)$ are given as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad (\, j \in \{1,\, 2,\, 3\}) \quad \text{with} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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The dot products are to be read as

$$\sigma \cdot \nabla = \sigma_1 \frac{\partial}{\partial x_1} + \sigma_2 \frac{\partial}{\partial x_2}$$

in (1.1) and

$$\alpha \cdot \nabla = \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3}$$

in (1.2). The potential q is a real-valued function on $\mathbb{R}^d$, where $d = 2$ or $d = 3$, respectively. The operators H_2, H_3 differ from the standard Dirac operator in that they lack a mass term, usually represented by an additional anti-commuting matrix: σ_3 for the two-dimensional case and

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

for the three-dimensional case, where I is a 2×2 identity matrix.

The purpose of the present paper is to show that $\sigma(H_d) = \mathbb{R}$ under minimal assumptions on q. In particular, we shall not require any restriction on the growth or decay of the potential q at infinity.

We have two motivations. First, the spectrum of the one-dimensional massless Dirac operator

$$H_1 = -i\sigma_2 \frac{d}{dx} + q(x) \quad \text{in} \quad L^2(\mathbb{R}; \mathbb{C}^2)$$

covers the whole real axis and is purely absolutely continuous whenever $q \in L^1_{loc}(\mathbb{R}, \mathbb{R})$. This surprising fact was first pointed out by one of the authors in [8]. By separation in spherical polar coordinates, this result also implies that $\sigma(H_d) = \mathbb{R}$ if q is rotationally symmetric; see [9]. Second, it is believed that the energy spectrum of graphene, in which electron transport is governed by Dirac equations in two dimensions without a mass term, has no bandgap (zero bandgap); see [2], [4], [7]. For these reasons, it is natural to make an attempt to show that $\sigma(H_d) = \mathbb{R}$ under minimal assumptions on q.

§ 2. Embedded eigenvalues

It is difficult to imagine that the spectra of H_2 and H_3 are always purely absolutely continuous regardless of q. Actually, in the three-dimensional case, we have an example of q which gives rise to a zero mode of H_3, *i.e.* an example of q for which H_3 has the embedded eigenvalue 0.

Example 1. Let $q(x) = -3/\langle x \rangle^2$, where $\langle x \rangle = \sqrt{1 + |x|^2}$. Then there exists a unique self-adjoint realization of H_3 in $L^2(\mathbb{R}^3; \mathbb{C}^4)$ with $\mathrm{Dom}(H_3) = H^1(\mathbb{R}^3; \mathbb{C}^4)$, the

Sobolev space of order 1. If one puts

$$f(x) = \langle x \rangle^{-3}(I_4 + i\alpha \cdot x)\phi_0$$

with ϕ_0 a unit vector in $\mathbb{C}^4$, then a direct calculation shows that $H_3 f = 0$. This implies that $0 \in \sigma_{\mathrm{p}}(H_3)$, because $f \in \mathrm{Dom}(H_3)$. Thus H_3 has a zero mode. Since $\sigma(H_3) = \mathbb{R}$, the energy 0 is an embedded eigenvalue of H_3.

The potential q and the zero mode f in Example 1 were motivated by [6].

Example 2 below indicates that there is a difference in spectral property between H_3 and H_2. In fact, quite a similar construction in Example 2 below gives a zero resonance of H_2, not a zero mode of H_2. On the other hand, we do not know if the potential q in Example 2 gives rise to a zero mode of H_2.

Example 2. Let $q(x) = -2/\langle x \rangle^2$. Then there exists a unique self-adjoint realization of H_2 in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ with $\mathrm{Dom}(H_2) = H^1(\mathbb{R}^2;\mathbb{C}^2)$, and $\sigma(H_2) = \mathbb{R}$. If

$$\psi(x) = \langle x \rangle^{-2}(I_2 + i\sigma \cdot x)\phi_0,$$

ϕ_0 a unit vector in $\mathbb{C}^2$, then one sees that $H_2\psi = 0$. However, it is clear that $\psi \notin L^2(\mathbb{R}^2;\mathbb{C}^2)$. Therefore, ψ is not a zero mode of H_2. On the other hand, one finds that $\psi \in L^{2,-s}(\mathbb{R}^2;\mathbb{C}^2)$ for $\forall s > 0$, where

$$L^{2,-s}(\mathbb{R}^2;\mathbb{C}^2) = \left\{ \varphi \;\middle|\; \|\langle x \rangle^{-s}\varphi\|_{L^2} < \infty \right\}.$$

This means that ψ is a zero resonance of H_2.

It is not an easy task to clarify whether H_d has embedded eigenvalues for general potentials. However, we have a good control of the embedded eigenvalues of H_d if $q(x)$ is rotationally symmetric. To formulate a result, we need to introduce the definition of the limit range $\mathcal{R}_\infty(q)$ of q:

$$\mathcal{R}_\infty(q) = \bigcap_{r>0} \overline{\left\{ q(x) \;\middle|\; |x| \geq r \right\}},$$

where $\overline{A}$ denotes the closure of a subset $A \subset \mathbb{R}$.

Theorem 2.1 (Schmidt[9]). *Let $q(x) = \eta(|x|)$ and let $\eta \in L^1_{loc}(0, \infty)$. Suppose that there exists a real number $E \in \mathbb{R} \setminus \mathcal{R}_\infty(q)$ such that*

$$\frac{1}{r(E - \eta(r)) - 1} \in BV(r_0, \infty)$$

for some $r_0 > 0$, where $BV(r_0,\infty)$ denotes the set of functions of bounded variations on the interval (r_0,∞). Then $\sigma_p(H_d) \subset \mathcal{R}_\infty(q)$ for $d \in \{2, 3\}$.

Theorem 2.1 is a direct consequence of [9, Corollary 1]. If q is not rotationally symmetric, we can prove the following.

Theorem 2.2. *Let $q \in C^1(\mathbb{R}^d;\mathbb{R})$, $d \in \{2, 3\}$, and suppose that both q and $(x \cdot \nabla)q$ are bounded functions. Then $\sigma_p(H_d) \subset [m_q, M_q]$, where*

$$m_q = \inf_x \{q(x) + (x \cdot \nabla)q(x)\}, \quad M_q = \sup_x \{q(x) + (x \cdot \nabla)q(x)\}.$$

The proof of Theorem 2.2 is based on a virial theorem in an abstract setting; see [1, Lemma 2.1].

§ 3. Schnol's theorem for Dirac operators

We now prepare a Schnol' theorem for Dirac operators. As for Schnol's theorem, we refer the reader [3, p.21, Theorem 2.9] which is a characterization of the spectra of Schrödinger operators in terms of polynomially bounded eigensolutions. In the three-dimensional Dirac operators, the theorem can be stated as follows:

Theorem 3.1. *Let $q \in L^2_{loc}(\mathbb{R}^3;\mathbb{R})$, and let E be a real number. Suppose f is a polynomially bounded measurable function on $\mathbb{R}^3$, not identically 0, and satisfies the equation*

$$(-i\alpha \cdot \nabla + q)f = Ef \tag{3.1}$$

in the distribution sense. Then $E \in \sigma(H_3)$ for any self-adjoint realization H_3 such that $\mathrm{Dom}(H_3) \supset H^1(\mathbb{R}^3;\mathbb{C}^4) \cap \mathrm{Dom}(q)$.

Outline of the proof of Theorem 3.1. We follow the line of the proof of [3, p. 21, Theorem 2.9].

We may suppose $f \notin L^2(\mathbb{R}^3;\mathbb{C}^4)$ without loss of generality. Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $\varphi(x) = 1$ $(|x| \leq 1)$ and $\varphi(x) = 0$ $(|x| \geq 2)$, and put $\varphi_n(x) = \varphi(x/n)$ for $n \in \mathbb{N}$. Then introducing a monotonically increasing sequence $\big(M(n)\big)_{n\in\mathbb{N}}$ by

$$M(n) = \int_{|x|\leq n} |f(x)|^2 \, dx,$$

and defining $f_n := \varphi_n f / \|\varphi_n f\|_{L^2}$, we can show that there exists a positive constant C such that

$$\|(H_3 - E)f_n\|^2_{L^2} \leq C \frac{M(2n) - M(n)}{n^2 M(n)}. \tag{3.2}$$

for all n.

On the other hand, we can deduce that

$$\liminf_{n\to\infty} \frac{M(2n)-M(n)}{n^2 M(n)} = 0,$$

which implies that there exists a subsequence $\big(M(n_k)\big)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{M(2n_k)-M(n_k)}{n_k^2\, M(n_k)} = 0. \tag{3.3}$$

Combining (3.2) with (3.3), we can conclude that $E \in \sigma(H_3)$. ■

We give an application of the Schnol' theorem, and show that the spectra of massless Dirac operators with real-valued potentials always coincide with the whole real axis, provided that the potentials are of the form specified in Theorem 3.2 below.

Theorem 3.2. *Let $\eta \in C^1(\mathbb{R};\mathbb{R})$ and define $q(x) := \eta(x\cdot k)$ on $\mathbb{R}^d$, $d \in \{2, 3\}$, where $k \in \mathbb{R}^d$ is a unit vector. Then $\sigma(H_d) = \mathbb{R}$.*

Outline of the proof of Theorem 3.2. We only give the proof for $d = 3$. Put

$$\xi(t) = \int_0^t \eta(\tau)\, d\tau.$$

Choose a unit vector $\phi_0 \in \mathbb{C}^4$, $\neq 0$, so that $(\alpha\cdot k)\phi_0 = \phi_0$. For a given $E \in \mathbb{R}$, define

$$f(x) = e^{-i(\alpha\cdot k)\,\xi(x\cdot k)} e^{iEx\cdot k}\phi_0.$$

Then f satisfies the equation (3.1). Moreover, $f \in C^1(\mathbb{R}^3;\mathbb{C}^4)$, and $|f(x)|_{\mathbb{C}^4} = 1$ for all $x \in \mathbb{R}^3$. It follows from Theorem 3.1 that $E \in \sigma(H_3)$. Since E is an arbitrary real number, we can conclude that $\sigma(H_3) = \mathbb{R}$. ■

§ 4. The main result

We now state the main theorem, which greatly generalizes Theorem 3.2.

Theorem 4.1. *Let $q \in L^2_{loc}(\mathbb{R}^3;\mathbb{R})$. Suppose that there is a sequence $(k_n)_{n\in\mathbb{N}}$ of unit vectors in $\mathbb{R}^3$, a sequence $(B_{r_n}(a_n))_{n\in\mathbb{N}}$ of balls with centre $a_n \in \mathbb{R}^3$ and radius $r_n \to \infty$ $(n \to \infty)$, and a sequence of square-integrable functions $q_n : (-r_n, r_n) \to \mathbb{R}$ $(n \in \mathbb{N})$ such that*

$$r_n^{-3} \int_{B(a_n,\, r_n)} \big|q(x) - q_n\big((x-a_n)\cdot k_n\big)\big|^2\, dx \to 0$$

as $n \to \infty$. Then $\sigma(H_3) = \mathbb{R}$ for any self-adjoint extension H_3 of

$$(-i\alpha\cdot\nabla + q)\big|_{C_0^\infty(\mathbb{R}^3)^4}.$$

The two dimensional analogue of the statements above holds true.

Outline of the proof of Theorem 4.1. For a given energy $E \in \mathbb{R}$, we shall construct a singular sequence $(h_n)_{n\in\mathbb{N}}$, thus showing that $E \in \sigma(H_3)$. To this end, we first choose a function $\widetilde{\eta}_n \in C^\infty(-r_n, r_n)$ so that

$$\frac{1}{2r_n}\int_{-r_n}^{r_n} |q_n(t) - \widetilde{\eta}_n(t)|^2\, dt \to 0$$

as $n \to \infty$. Then following the idea in the proof of Theorem 3.2, we put

$$\xi_n(t) = \int_0^t \widetilde{\eta}_n(\tau)\, d\tau,$$

and choose a sequence of unit vectors $\phi_n \in \mathbb{C}^4$, $\neq 0$, so that $(\alpha \cdot k_n)\phi_n = \phi_n$, and define a sequence of functions $(f_n)_{n\in\mathbb{N}}$ by

$$f_n(x) = e^{-i(\alpha\cdot k_n)\,\xi_n((x-a_n)\cdot k_n)} e^{iEx\cdot k_n}\phi_n \;:\; B_{r_n}(a_n) \to \mathbb{C}^4.$$

We now choose a function $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\mathrm{supp}\chi \subset B_1(0)$ and that $\|\chi\|_{L^2} = 1$. Putting

$$h_n(x) = r_n^{-3/2}\chi\Big(\frac{x-a_n}{r_n}\Big) f_n(x),$$

we can obtain the desired singular sequence. ■

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