

Geometric properties of certain meromorphic functions

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Abstract

In this paper, we aim at investigating several geometric properties of the solutions of the following differential equations:

$$w''(z) + a(z)w'(z) + b(z)w(z) = 0,$$

where the functions $a(z)$ and $b(z)$ are meromorphic in the punctured disk $\mathbb{D} = \{z : 0 < |z| < 1\}$.

1 Introduction

Let Σ be the class of functions of the form

$$(1.1) \quad f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$$

which are meromorphic in the punctured disk $\mathbb{D} = \{z : 0 < |z| < 1\}$.

A function $f(z) \in \Sigma$ is said to meromorphic starlike of order α in $\mathbb{D}$ if it satisfies

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < -\alpha \quad (z \in \mathbb{D})$$

for some α ($0 \leq \alpha < 1$). We denoted by $\Sigma S_0^*(\alpha)$ the subclass of Σ consisting of all such functions.

2 A class of bounded functions

We begin with the definition and lemma.

Definition 1 Let $\mathcal{H}_J$ be the class of complex functions $h(s, t)$ satisfying:

- (i) $h(s, t)$ is continuous in a domain $\mathbb{D} \subset \mathbb{C} \times \mathbb{C}$,
- (ii) $(0, 0) \in \mathbb{D}$ and $|h(0, 0)| < J$ ($J > 0$),
- (iii) $|h(Je^{i\theta}, Ke^{i\theta})| \geq J$ when $(Je^{i\theta}, Ke^{i\theta}) \in \mathbb{D}$, θ is real and $K \geq J$.

Definition 2 Let $h \in \mathcal{H}_J$ with corresponding domain $\mathbb{D}$. We denote by $\mathcal{B}_J(h)$ the class of functions $u(z) = u_1z + u_2z^2 + \dots$ which are analytic in the unit disk $\Delta = \{z : |z| < 1\}$ and satisfy

- (i) $(u(z), zu'(z)) \in \mathbb{D}$,
- (ii) $|h(u(z), zu'(z))| < J \quad (z \in \Delta)$.

Lemma 1 ([3]) Let $h \in \mathcal{H}_J$ and $b(z)$ be an analytic function in Δ with $|b(z)| < J$. If the differential equation

$$(2.1) \quad h(u(z), zu'(z)) = b(z)$$

has a solution $u(z)$ analytic in Δ , then $|u(z)| < J$.

Lemma 2 ([1]) If $f(z) \in \Sigma$ satisfies $f(z)f'(z) \neq 0$ in $\mathbb{D}$ and

$$(2.2) \quad -\operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} < 4 - \beta \quad (z \in \Delta),$$

then

$$(2.3) \quad -\operatorname{Re} \{ z^2 f'(z) \} > \frac{1}{5 - 2\beta} \quad (z \in \Delta),$$

that is, $f(z)$ is meromorphic close-to-convex of order $\frac{1}{5 - 2\beta}$, where $\frac{3}{2} \leq \beta < 2$.

3 Main results

First, we prove

Theorem 1 Let $w(z), a(z) \in \Sigma$ and $b(z)$ are meromorphic in $\mathbb{D}$ with

$$(3.1) \quad \left| z^2 \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}(a(z))^2 \right) \right| < \frac{1}{2} \quad (z \in \mathbb{D})$$

and

$$\operatorname{Re} \{ za(z) \} \geq 2 + 2\alpha \quad (0 \leq \alpha < 1).$$

Also, let $w(z)$ be the solution of the following second order linear differential equation

$$(3.2) \quad w''(z) + a(z)w'(z) + b(z)w(z) = 0.$$

Then $w(z)$ is meromorphic starlike of order α .

Proof. Put $w(z) = e^{-\frac{1}{2} \int a(\xi) d\xi} v(z)$. Then (3.2) leads to the normal form

$$(3.3) \quad v''(z) + \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}(a(z))^2 \right) v(z) = 0.$$

If we put

$$(3.4) \quad u(z) = \frac{zv'(z)}{v(z)} - \frac{1}{2} \quad (z \in \mathbb{D}),$$

then $u(z)$ is analytic in Δ and (3.3) becomes

$$(3.5) \quad (u(z))^2 + zu'(z) - \frac{1}{4} = -z^2 \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}(a(z))^2 \right),$$

or equivalently

$$(3.6) \quad h(u(z), zu'(z)) = -z^2 \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}(a(z))^2 \right),$$

where $h(s, t) = s^2 + t - \frac{1}{4}$. It is easy to check $h(s, t) \in \mathcal{H}_{\frac{1}{2}}$, that is

- (i) $h(s, t)$ is continuous in $\mathbb{C} \times \mathbb{C}$,
- (ii) $|h(0, 0)| = \frac{1}{4} < \frac{1}{2}$,
- (iii) $\left| h\left(\frac{1}{2}e^{i\theta}, Ke^{i\theta}\right) \right| \geq \frac{1}{2} \quad \left(K \geq \frac{1}{2} \right)$.

From assumption, we have

$$\left| -z^2 \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}(a(z))^2 \right) \right| < \frac{1}{2} \quad (z \in \mathbb{D}).$$

By using Lemma 1, we have $|u(z)| < \frac{1}{2} \quad (z \in \Delta)$. Therefore, we obtain

$$\left| \frac{zv'(z)}{v(z)} - \frac{1}{2} \right| < \frac{1}{2} \quad (z \in \Delta).$$

This implies

$$0 < \operatorname{Re} \left\{ \frac{zv'(z)}{v(z)} \right\} < 1 \quad (z \in \Delta).$$

From $w(z) = e^{-\frac{1}{2} \int a(\xi) d\xi} v(z)$, we have

$$(3.7) \quad \exp \left(\frac{1}{2} \int a(\xi) d\xi \right) w(z) = v(z).$$

Logarithmically differentiating of (3.7) leads to

$$(3.8) \quad \frac{zw'(z)}{w(z)} = \frac{zv'(z)}{v(z)} - \frac{1}{2}za(z).$$

Combining (3.8) and $\operatorname{Re}\{za(z)\} \geq 2 + 2\alpha \quad (0 \leq \alpha < 1)$, we obtain

$$\operatorname{Re} \left\{ \frac{zw'(z)}{w(z)} \right\} = \operatorname{Re} \left\{ \frac{zv'(z)}{v(z)} \right\} - \frac{1}{2}\operatorname{Re}\{za(z)\} < 1 - \frac{1}{2}(2 + 2\alpha) = -\alpha \quad (z \in \mathbb{D}),$$

that is, $w(z)$ is meromorphic starlike of order α . □

Example 1 In Theorem 1, let $a(z) = \frac{2}{z}$ and $b(z) = \frac{1}{2}$. The solution of

$$(3.9) \quad w''(z) + \frac{2}{z}w'(z) + \frac{1}{2}w(z) = 0$$

is given by $w(z) = \frac{\cos \frac{z}{\sqrt{2}}}{z}$. This solution $w(z)$ is meromorphic starlike function.

Next, we prove

Theorem 2 Let $w(z), Q(z) \in \Sigma$. We consider the following second order differential equation.

$$(3.10) \quad w''(z) + Q(z)w(z) = 0 \quad (z \in \mathbb{D}).$$

If

$$\operatorname{Re} \left\{ Q(z) \frac{zw(z)}{w'(z)} \right\} < 4 - \beta \quad (z \in \mathbb{D}),$$

then we have

$$-\operatorname{Re}\{z^2w'(z)\} > \frac{1}{5-2\beta} \quad \left(\frac{3}{2} \leq \beta < 2 \right).$$

Proof. From (3.10), we have

$$(3.11) \quad Q(z) \frac{zw(z)}{w'(z)} = -\frac{zw''(z)}{w'(z)}.$$

Applying Lemma 2 to (3.11), we can prove Theorem 2. □

Example 2 In Theorem 2, let $Q(z) = -\frac{2}{z^2}$. A solution of

$$w''(z) - \frac{2}{z^2}w(z) = 0$$

is give by $w(z) = \frac{1}{z} + \frac{3}{50}z^2$. Then

$$\operatorname{Re} \left\{ Q(z) \frac{zw(z)}{w'(z)} \right\} < 2.404 \dots < \frac{5}{2}$$

and

$$-\operatorname{Re}\{z^2w'(z)\} > 0.88 > \frac{1}{2}.$$

Therefore, $w(z)$ is meromorphic close-to-convex function.

Remark 1 Let $\mathcal{MC}(\alpha)$ be the subclass of Σ consisting of functions $f(z)$ which satisfy

$$(3.12) \quad -\operatorname{Re}\{z^2f'(z)\} > \alpha \quad (z \in \Delta)$$

for some α ($0 \leq \alpha < 1$). A function $f(z) \in \mathcal{MC}(\alpha)$ is meromorphic close-to-convex of order α in $\mathbb{D}$.

References

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