

On Some Triply Infinite Sums by Means of N-Fractional Calculus

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Abstract

In this article some triple infinite sums, some related finite sums and mixed sums, which are derived by means of N-fractional calculus, are reported.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\text{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\text{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$f_\nu(z) = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi-z)^{\nu+1}} d\xi \quad (\nu \notin \mathbb{Z}), \quad (1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in \mathbb{Z}^+), \quad (2)$$

where $-\pi \leq \arg(\xi-z) \leq \pi$ for C_- , $0 \leq \arg(\xi-z) \leq 2\pi$ for C_+ ,

$\xi \neq z$, $z \in C$, $\nu \in \mathbb{R}$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\xi}{(\xi-z)^{\nu+1}} \right) \quad (\nu \notin \mathbb{Z}), \quad [\text{Refer to (1)}] \quad (3)$$

with

$$N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in \mathbb{Z}^+), \quad (4)$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta(N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \quad (5)$$

then the set

$$\{N^\nu\} = \{N^\nu \mid \nu \in \mathbb{R}\} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in \mathbb{R}\}$, where $f = f(z)$ and $z \in \mathbb{C}$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$ " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in \mathbb{R}). \quad (7)$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \quad (8)$$

holds. [5]

(III) **Lemma.** We have [1]

$$(i) \quad ((z-c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \quad \left(\left| \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} \right| < \infty \right),$$

$$(ii) \quad (\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty),$$

$$(iii) \quad ((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \quad (|\Gamma(\alpha)| < \infty),$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii). (Γ ; Gamma function),

$$(iv) \quad (u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad \left(\begin{array}{l} u = u(z), \\ v = v(z) \end{array} \right).$$

**§ 1. Triply Infinite, Finite and Mixed Sums which are
Derived by Means of N- Fractional Calculus**

In the following $\alpha, \beta, \gamma, \delta \in \mathbb{R}$,

$$\sum_{k,m,n=0}^{s,p,q} \cdots := \sum_{k=0}^s \sum_{m=0}^p \sum_{n=0}^q \cdots, \quad \sum_{k,m,n=0}^{\infty} \cdots := \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cdots,$$

and

$$\sum_{k,m=0}^{s,p} \cdots := \sum_{k=0}^s \sum_{m=0}^p \cdots, \quad \sum_{k,m=0}^{\infty} \cdots := \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \cdots,$$

for our convenience.

We have then Theorem 1. below by the use of N- fractional calculus of products of some power functions.

Theorem 1. *Let*

$$\begin{aligned} G &= G(\alpha, \beta, \gamma; k, m) \\ &:= \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(m-\beta)\Gamma(k-m-\alpha+\gamma)}{k! \cdot m! \Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(k-\alpha)}, \end{aligned} \quad (1)$$

$$\begin{aligned} H &= H(\alpha, \gamma, \delta; k, m, n) \\ &:= \frac{\Gamma(\delta+1)\Gamma(m+n-\gamma)\Gamma(\gamma+k-\alpha-m+\delta-n)}{n! \Gamma(\delta+1-n)\Gamma(m-\gamma)\Gamma(\gamma+k-\alpha-m)}, \end{aligned} \quad (2)$$

and

$$\begin{aligned} R &= R(\alpha, \beta, \gamma, \delta) \\ &:= -\frac{\sin\pi(\gamma-\alpha-\beta) \cdot \sin\pi(\delta-\alpha)}{\sin\pi(\alpha+\beta) \cdot \sin\pi(\gamma+\delta-\alpha)}, \end{aligned} \quad (3)$$

$$(|R| = M < \infty).$$

(i) When $\alpha, \beta, \gamma, \delta \notin \mathbb{Z}_0^+$, we have the following triply infinite sums ;

$$\begin{aligned} \sum_{k,m,n=0}^{\infty} G \cdot H \cdot \left(\frac{z-c}{z}\right)^{m+n} \left(\frac{c}{z-c}\right)^k \\ = R \cdot \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\delta-\alpha)}{\Gamma(-\alpha-\beta)\Gamma(-\alpha)} \left(\frac{z-c}{z}\right)^{\gamma+\delta-\alpha}, \end{aligned} \quad (4)$$

where

$$|(z-c)/z|, |c/(z-c)| < 1, \quad (5)$$

$$\left| \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \right|, \quad \left| \frac{\Gamma(k - \alpha + \gamma - m)}{\Gamma(k - \alpha)} \right| < \infty \quad (6)$$

and

$$\left| \frac{\Gamma(\gamma + k - \alpha - m + \delta - n)}{\Gamma(\gamma + k - \alpha - m)} \right|, \quad \left| \frac{\Gamma(\delta - \alpha)}{\Gamma(-\alpha)} \right| < \infty. \quad (7)$$

(ii) When $\alpha, \beta, \gamma \notin \mathbb{Z}_0^+$ and $\delta = q \in \mathbb{Z}^+$ we have the following mixed sum ;

$$\begin{aligned} & \sum_{k,m=0}^{\infty} \sum_{n=0}^q G \cdot H(\alpha, \gamma, q; k, m, n) \cdot \left(\frac{z-c}{z} \right)^{m+n} \left(\frac{c}{z-c} \right)^k \\ &= R(\alpha, \beta, \gamma, q) \cdot \frac{\Gamma(\gamma - \alpha - \beta) \Gamma(q - \alpha)}{\Gamma(-\alpha - \beta) \Gamma(-\alpha)} \left(\frac{z-c}{z} \right)^{\gamma+q-\alpha}, \end{aligned} \quad (8)$$

$$(|R(\alpha, \beta, \gamma, q)| = M < \infty)$$

having (5), (6) and

$$\left| \frac{\Gamma(\gamma + k - \alpha - m + q - n)}{\Gamma(\gamma + k - \alpha - m)} \right| < \infty. \quad (9)$$

(iii) When $\alpha, \beta \notin \mathbb{Z}_0^+$ and $\gamma = p, \delta = q$ ($p, q \in \mathbb{Z}^+$) we have the following mixed sum ;

$$\begin{aligned} & \sum_{k,m,n=0}^{\infty, p, q} G(\alpha, \beta, p; k, m) \cdot H(\alpha, p, q; k, m, n) \cdot \left(\frac{z-c}{z} \right)^{m+n} \left(\frac{c}{z-c} \right)^k \\ &= \frac{\Gamma(p - \alpha - \beta) \Gamma(q - \alpha)}{\Gamma(-\alpha - \beta) \Gamma(-\alpha)} \left(\frac{z-c}{z} \right)^{p+q-\alpha}, \end{aligned} \quad (10)$$

where

$$|(z-c)/z| < \infty, \quad |c/(z-c)| < 1. \quad (11)$$

(iv) When $\beta \notin \mathbb{Z}_0^+$ and $\alpha = s, \gamma = p, \delta = q$ ($s, p, q \in \mathbb{Z}^+$) we have the following triply finite sum ;

$$\sum_{k,m,n=0}^{s,p,q} G(s, \beta, p; k, m) \cdot H(s, p, q; k, m, n) \cdot \left(\frac{z-c}{z}\right)^{m+n} \left(\frac{c}{z-c}\right)^k$$

$$= \frac{\Gamma(p-s-\beta)\Gamma(q-s)}{\Gamma(-s-\beta)\Gamma(-s)} \left(\frac{z-c}{z}\right)^{p+q-s}, \quad (12)$$

where

$$|(z-c)/z|, |c/(z-c)| < \infty, \quad (13)$$

and

$$\left| \frac{\Gamma(k-s+p-m)}{\Gamma(k-s)} \right| < \infty. \quad (14)$$

Proof of (i). We have

$$z^\alpha = (z-c)^\alpha \left(1 - \frac{c}{c-z}\right)^\alpha \quad (15)$$

$$= \sum_{k=0}^{\infty} \frac{c^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} (z-c)^{\alpha-k} \quad (|c/(z-c)| < 1) \quad (16)$$

Make (16) $\times z^\beta$, then operate N-fractional calculus operator N^γ to its both sides, we obtain

$$(z^\alpha \cdot z^\beta)_\gamma = \sum_{k=0}^{\infty} \frac{c^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} ((z-c)^{\alpha-k} \cdot z^\beta)_\gamma \quad (17)$$

$$= \sum_{k=0}^{\infty} \frac{c^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma+1)}{m! \Gamma(\gamma+1-m)} ((z-c)^{\alpha-k})_{\gamma-m} (z^\beta)_m, \quad (18)$$

by Lemma (iv).

Now we have

$$(z^\alpha \cdot z^\beta)_\gamma = e^{-i\pi\gamma} P(\alpha, \beta, \gamma) \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} z^{\alpha+\beta-\gamma} \quad (19)$$

$$\left(\left| \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \right| < \infty \right),$$

where

$$P(\alpha, \beta, \gamma) = \frac{\sin \pi \alpha \cdot \sin \pi (\gamma - \alpha - \beta)}{\sin \pi (\alpha + \beta) \cdot \sin \pi (\gamma - \alpha)} \quad (20)$$

$$\left(\begin{array}{l} |P(\alpha, \beta, \gamma)| = M < \infty, \\ \operatorname{Re}(\alpha + \beta + 1) > 0, \quad (1 + \alpha - \gamma) \notin \mathbb{Z}_0^- \end{array} \right)$$

(Refer to J. Frac. Calc. Vol.27, pp.83 - 88) [19].

Next we have

$$\left((z - c)^{\alpha - k} \right)_{\gamma - m} = e^{-i\pi(\gamma - m)} \frac{\Gamma(k - \alpha + \gamma - m)}{\Gamma(k - \alpha)} (z - c)^{\alpha - k - \gamma + m}, \quad (21)$$

$$\left(\left| \frac{\Gamma(k - \alpha + \gamma - m)}{\Gamma(k - \alpha)} \right| < \infty \right)$$

and

$$(z^\beta)_m = e^{-i\pi m} \frac{\Gamma(m - \beta)}{\Gamma(-\beta)} z^{\beta - m} \quad (22)$$

by Lemma (i), respectively.

We have then

$$\begin{aligned} \sum_{k, m=0}^{\infty} G(\alpha, \beta, \gamma; k, m) c^k (z - c)^{\alpha - k - \gamma + m} z^{-m} \\ = P(\alpha, \beta, \gamma) \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} z^{\alpha - \gamma} \end{aligned} \quad (23)$$

$$\left(\left| \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \right| < \infty \right)$$

from (18), (19), (21) and (22).

Make (23) $\times z^\gamma$, then operate N^δ to its both sides, we obtain

$$\sum_{k, m=0}^{\infty} G \cdot c^k \left((z - c)^{\alpha - k - \gamma + m} \cdot z^{\gamma - m} \right)_\delta = P(\alpha, \beta, \gamma) \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z^{\alpha - \gamma} \cdot z^\gamma)_\delta, \quad (24)$$

hence

$$\begin{aligned} \sum_{k, m=0}^{\infty} G \cdot c^k \sum_{n=0}^{\infty} \frac{\Gamma(\delta + 1)}{n! \Gamma(\delta + 1 - n)} \left((z - c)^{\alpha - k - \gamma + m} \right)_{\delta - n} (z^{\gamma - m})_n \\ = P(\alpha, \beta, \gamma) \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} (z^{\alpha - \gamma} \cdot z^\gamma)_\delta. \end{aligned} \quad (25)$$

Now we have

$$(z^{\alpha-\gamma} \cdot z^\gamma)_\delta = P(\alpha-\gamma, \gamma, \delta) (z^\alpha)_\delta \quad (26)$$

$$= e^{-i\pi\delta} \frac{\sin\pi(\alpha-\gamma) \cdot \sin\pi(\delta-\alpha)}{\sin\pi\alpha \cdot \sin\pi(\delta+\gamma-\alpha)} \cdot \frac{\Gamma(\delta-\alpha)}{\Gamma(-\alpha)} z^{\alpha-\delta} \quad (27)$$

$$\left(\left| \frac{\Gamma(\delta-\alpha)}{\Gamma(-\alpha)} \right| < \infty \right)$$

$$\left(\begin{array}{l} |P(\alpha-\gamma, \gamma, \delta)| = M < \infty, \\ \text{Re}(\alpha+1) > 0, \quad (1+\alpha-\gamma-\delta) \notin \mathbb{Z}_0^- \end{array} \right)$$

(Refer to J. Frac. Calc. Vol.27, pp.83 - 88) [19].

Next we have

$$((z-c)^{\alpha-k-\gamma+m})_{\delta-n} = e^{-i\pi(\delta-n)} \frac{\Gamma(k+\gamma-\alpha-m+\delta-n)}{\Gamma(k+\gamma-\alpha-m)} (z-c)^{m+\alpha-\gamma-k-\delta+n}, \quad (28)$$

$$\left(\left| \frac{\Gamma(k+\gamma-\alpha-m+\delta-n)}{\Gamma(k+\gamma-\alpha-m)} \right| < \infty \right)$$

and

$$(z^{\gamma-m})_n = e^{-i\pi n} \frac{\Gamma(m-\gamma+n)}{\Gamma(m-\gamma)} z^{\gamma-m-n} \quad (29)$$

by Lemma (i), respectively.

Therefore, we obtain

$$\begin{aligned} \sum_{k,m,n=0}^{\infty} G(\alpha, \beta, \gamma; k, m) H(\alpha, \gamma, \delta; k, m, n) c^k (z-c)^{\alpha-\gamma-\delta+n+m-k} z^{\gamma-m-n} \\ = R(\alpha, \beta, \gamma, \delta) \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\delta-\alpha)}{\Gamma(-\alpha-\beta)\Gamma(-\alpha)} z^{\alpha-\delta} \end{aligned} \quad (30)$$

from (25) ~ (29), since

$$P(\alpha, \beta, \gamma) P(\alpha-\gamma, \gamma, \delta) = R(\alpha, \beta, \gamma, \delta). \quad (31)$$

We have then (4) from (30), under the conditions, using the notations (1), (2) and (3).

Note 1. When we use

$$(z^\alpha z^\beta)_\gamma = (z^{\alpha+\beta})_\gamma = e^{-i\pi\gamma} \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} z^{\alpha+\beta-\gamma} \quad (32)$$

instead of $(z^\alpha \cdot z^\beta)_\gamma$ (see Lemma (i v)), we obtain

$$\sum_{k,m=0}^{\infty} G(\alpha, \beta, \gamma; k, m) c^k (z-c)^{\alpha-k-\gamma+m} z^{-m} = \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} z^{\alpha-\gamma}, \quad (33)$$

instead of (23), from (18).

Therefore, we have the following doubly infinite sum ;

$$\sum_{k,m=0}^{\infty} G \cdot \left(\frac{z-c}{z} \right)^m \left(\frac{c}{z-c} \right)^k = \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \left(\frac{z-c}{z} \right)^{\gamma-\alpha} \quad (34)$$

from (33).

This result is reported in a previous paper of the author (cf. JFC Vol. 24, (2003), pp.68 - 70.). [11]

When

$$P(\alpha, \beta, \gamma) = 1, \quad (35)$$

(23) is reduced to (34).

Note 2. When we use

$$(z^{\alpha-\gamma} z^\gamma)_\delta = (z^\alpha)_\delta = e^{-i\pi\delta} \frac{\Gamma(\delta - \alpha)}{\Gamma(-\alpha)} z^{\alpha-\delta} \quad \left(\left| \frac{\Gamma(\delta - \alpha)}{\Gamma(-\alpha)} \right| < \infty \right) \quad (36)$$

instead of $(z^{\alpha-\gamma} \cdot z^\gamma)_\delta$, we obtain

$$\begin{aligned} \sum_{k,m,n=0}^{\infty} G \cdot H \cdot c^k (z-c)^{m+n-k+\alpha-\gamma-\delta} z^{\gamma-m-n} \\ = P(\alpha, \beta, \gamma) \frac{\Gamma(\gamma - \alpha - \beta) \Gamma(\delta - \alpha)}{\Gamma(-\alpha - \beta) \Gamma(-\alpha)} z^{\alpha-\delta} \end{aligned} \quad (37)$$

instead of (30), from (25).

Moreover, for the case of (35), we have the following triply infinite sum ;

$$\sum_{k,m,n=0}^{\infty} G \cdot H \cdot \left(\frac{z-c}{z}\right)^{m+n} \left(\frac{c}{z-c}\right)^k$$

$$= \frac{\Gamma(\gamma - \alpha - \beta)\Gamma(\delta - \alpha)}{\Gamma(-\alpha - \beta)\Gamma(-\alpha)} \left(\frac{z-c}{z}\right)^{\gamma+\delta-\alpha} \quad (38)$$

from (37).

And this result is a special case of (4), in which

$$R(\alpha, \beta, \gamma, \delta) = 1. \quad (39)$$

In a previous paper of the author, this result (38) is reported as Theorem 3. in JFC Vol. 24, (2003), p.71 . [11]

Note 3. The identity (4) is same as the one shown in a paper by S. - D. Lin, H. M. Srivastava and S. - T. Tu (cf. JFC Vol. 27, p. 48.) [21].

Proof of (ii). Set $\delta = q \in \mathbb{Z}^+$ in (4).

Proof of (iii). Set $\gamma = p, \delta = q (p, q \in \mathbb{Z}^+)$ in (4).

Proof of (iv). Set $\alpha = s, \gamma = p, \delta = q (s, p, q \in \mathbb{Z}^+)$ in (4).

§ 2. Direct Calculation of Triply Infinite Sum

In the following G, H and R are the ones shown in § 1, respectively.

Now we have

$$\sum_{k,m,n=0}^{\infty} G \cdot H \cdot \left(\frac{z-c}{z}\right)^{m+n} \left(\frac{c}{z-c}\right)^k$$

$$= \sum_{k,m,n=0}^{\infty} \frac{[-\alpha]_k [-\gamma]_m [-\delta]_n [-\beta]_m [m-\gamma]_n}{k! \cdot m! \cdot n! \cdot (-1)^{-k-m-n}}$$

$$\times \frac{\Gamma(k - \alpha + \gamma + \delta - m - n)}{\Gamma(k - \alpha)} \left(\frac{z-c}{z}\right)^{m+n} \left(\frac{c}{z-c}\right)^k \quad (1)$$

using the relationship

$$\Gamma(\lambda + 1 - k) = (-1)^{-k} \frac{\Gamma(\lambda + 1)\Gamma(-\lambda)}{\Gamma(k - \lambda)}, \quad (2)$$

where

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda), \quad [\lambda]_0 = 1. \quad (3)$$

(Notation of Pochhammer).

Next we have

$$\frac{\Gamma(k-\alpha+\gamma+\delta-m-n)}{\Gamma(k-\alpha)} = \frac{[\gamma+\delta-\alpha-m-n]_k}{[-\alpha]_k} \cdot \frac{\Gamma(\gamma+\delta-\alpha-m-n)}{\Gamma(-\alpha)} \quad (4)$$

$$= \frac{[\gamma+\delta-\alpha-m-n]_k}{[-\alpha]_k} \cdot (-1)^{-(m+n)} \frac{\Gamma(\gamma+\delta-\alpha)}{\Gamma(-\alpha)[\alpha-\gamma-\delta+1]_{m+n}} \quad (5)$$

$$= \frac{\Gamma(\gamma+\delta-\alpha)[\gamma+\delta-\alpha-m-n]_k}{\Gamma(-\alpha)[- \alpha]_k} \times (-1)^{-(m+n)} \frac{1}{[\alpha-\gamma-\delta+1]_m [m+\alpha-\gamma-\delta+1]_n} \quad (6)$$

since

$$[\alpha-\gamma-\delta+1]_{m+n} = [\alpha-\gamma-\delta+1]_m [m+\alpha-\gamma-\delta+1]_n. \quad (7)$$

Therefore, we obtain

$$\begin{aligned} & \sum_{k,m,n=0}^{\infty} G \cdot H \cdot \left(\frac{z-c}{z} \right)^{m+n} \left(\frac{c}{z-c} \right)^k \\ &= \sum_{k,m,n=0}^{\infty} \frac{\Gamma(\gamma+\delta-\alpha)}{\Gamma(-\alpha)} \cdot \frac{[-\gamma]_m [-\delta]_n [-\beta]_m [m-\gamma]_n}{k! \cdot m! \cdot n! \cdot (-1)^{-k}} \\ & \quad \times \frac{[\gamma+\delta-\alpha-m-n]_k}{[\alpha-\gamma-\delta+1]_m [m+\alpha-\gamma-\delta+1]_n} \left(\frac{z-c}{z} \right)^{m+n} \left(\frac{c}{z-c} \right)^k \end{aligned} \quad (8)$$

from (1) and (6).

Next we have the identity

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} z^k = (1-z)^{-\lambda}, \quad (9)$$

hence

$$\sum_{k=0}^{\infty} \frac{[\gamma + \delta - \alpha - m - n]_k}{k!} (-1)^k \left(\frac{c}{z-c} \right)^k = \left(\frac{z}{z-c} \right)^{m+n+\alpha-\gamma-\delta}. \quad (10)$$

Then applying (10) to (8) we obtain

$$\begin{aligned} & \sum_{k,m,n=0}^{\infty} G \cdot H \cdot \left(\frac{z-c}{z} \right)^{m+n} \left(\frac{c}{z-c} \right)^k \\ &= \frac{\Gamma(\gamma + \delta - \alpha)}{\Gamma(-\alpha)} \left(\frac{z}{z-c} \right)^{\alpha-\gamma-\delta} \sum_{m=0}^{\infty} \frac{[-\gamma]_m [-\beta]_m}{m! [\alpha - \gamma - \delta + 1]_m} \\ & \quad \times \sum_{n=0}^{\infty} \frac{[-\delta]_n [m - \gamma]_n}{n! [m + \alpha - \gamma - \delta + 1]_n} \end{aligned} \quad (11)$$

$$= \frac{\Gamma(\gamma + \delta - \alpha) \Gamma(\alpha - \gamma - \delta + 1) \Gamma(\alpha + \beta + 1)}{\Gamma(-\alpha) \Gamma(\alpha - \delta + 1) \Gamma(\alpha + \beta - \gamma + 1)} \left(\frac{z-c}{z} \right)^{\gamma+\delta-\alpha}, \quad (12)$$

because (see Note 4.)

$$\sum_{n=0}^{\infty} \frac{[-\delta]_n [m - \gamma]_n}{n! [m + \alpha - \gamma - \delta + 1]_n} = {}_2F_1(-\delta, m - \gamma; m + \alpha - \gamma - \delta + 1; 1) \quad (13)$$

$$= \frac{\Gamma(m + \alpha - \gamma - \delta + 1) \Gamma(\alpha + 1)}{\Gamma(m + \alpha - \gamma + 1) \Gamma(\alpha + 1 - \delta)} \quad \left(\begin{array}{l} \operatorname{Re}(\alpha + 1) > 0, \\ (m + \alpha - \gamma - \delta + 1) \notin \mathbb{Z}_0^- \end{array} \right) \quad (14)$$

$$= \frac{[\alpha - \gamma - \delta + 1]_m}{[\alpha - \gamma + 1]_m} \cdot \frac{\Gamma(\alpha - \gamma - \delta + 1) \Gamma(\alpha + 1)}{\Gamma(\alpha - \gamma + 1) \Gamma(\alpha + 1 - \delta)}, \quad (15)$$

and

$$\sum_{m=0}^{\infty} \frac{[-\gamma]_m [-\beta]_m}{m! [\alpha - \gamma + 1]_m} = {}_2F_1(-\gamma, -\beta; \alpha - \gamma + 1; 1) \quad (16)$$

$$= \frac{\Gamma(\alpha - \gamma + 1) \Gamma(\alpha + \beta + 1)}{\Gamma(\alpha + 1) \Gamma(\alpha + \beta - \gamma + 1)} \quad \left(\begin{array}{l} \operatorname{Re}(\alpha + \beta + 1) > 0, \\ (\alpha - \gamma + 1) \notin \mathbb{Z}_0^- \end{array} \right). \quad (17)$$

Therefore, we obtain

$$\begin{aligned}
& \sum_{k,m,n=0}^{\infty} G \cdot H \cdot \left(\frac{z-c}{z} \right)^{m+n} \left(\frac{c}{z-c} \right)^k \\
&= - \frac{\sin \pi(\delta - \alpha) \cdot \sin \pi(\gamma - \alpha - \beta)}{\sin \pi(\alpha + \beta) \cdot \sin \pi(\gamma + \delta - \alpha)} \cdot \frac{\Gamma(\gamma - \alpha - \beta) \Gamma(\delta - \alpha)}{\Gamma(-\alpha - \beta) \Gamma(-\alpha)} \left(\frac{z-c}{z} \right)^{\gamma + \delta - \alpha}, \\
& \hspace{25em} (\S 1. (4))
\end{aligned}$$

from (12), using the relationship

$$\Gamma(\lambda) \Gamma(1-\lambda) = \frac{\pi}{\sin \pi \lambda} \quad (\lambda \notin \mathbb{Z}). \quad (18)$$

Note 4. We have the following identity ;

$$\sum_{k=0}^{\infty} \frac{[a]_k [b]_k}{k! [c]_k} = {}_2F_1(a, b; c; 1) \quad (19)$$

$$= \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \quad \left(\begin{array}{l} \operatorname{Re}(c-a-b) > 0, \\ c \notin \mathbb{Z}_0^- \end{array} \right). \quad (20)$$

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