

Exceptional sets of algebraic varieties of hyperbolic type

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1. Let V be a projective n -manifold (i.e., non-singular projective algebraic variety of dimension n) of hyperbolic type. Assume that the m -canonical system $|mK(V)|$ has no base point for large m . We then have a morphism $f = \Phi_m : V \rightarrow W \subset \mathbb{P}^{N_m}$ where $N_m = P_m(V) - 1$. Choose m so large that f is birational and that W is normal. Let $\Sigma = \{w \in W; \dim \bar{f}^{-1}(w) > 0\}$, which is a Zariski closed subset of W . Σ is a union of irreducible components $\Sigma_1, \dots, \Sigma_r$. Our purpose here is to prove that each Σ_j is an algebraic variety of elliptic type. We call Σ the total exceptional set and Σ_j an exceptional subvariety of V .

2. We prove this by induction on $\dim f(\Sigma_j)$. If $\dim f(\Sigma_j) > 0$, then consider a general hyperplane section W_1 of $W \subset \mathbb{P}^{N_m}$. $V_1 = \bar{f}^{-1}W_1$ is an $(n-1)$ -submanifold of V , which is linearly equivalent to $mK(V)$ as a divisor i.e., $V_1 \sim mK(V)$. By the symbol $\mathcal{O}(E)$ we indicate the sheaf of germs of holomorphic sections of the vector bundle or the divisor E . Considering the exact sequence

$$0 \rightarrow \mathcal{O}(eK(V)) \rightarrow \mathcal{O}((m+e)K(V)) \rightarrow \mathcal{O}((m+e)K(V)|V_1) \rightarrow 0$$

with the fact that $K(V_1) \sim (K(V) + V_1)|V_1 \sim (1+m)K(V)|V_1$, we obtain the exact sequence:

$$H^0(V, \mathcal{O}(me_1)K(V)) \rightarrow H^0(V_1, \mathcal{O}(me_1(V_1))) \rightarrow H^1(V, \mathcal{O}(e_1(m-1)K(V)))$$

where $e = m(e_1 - 1)$. By a generalization of Kodaira's vanishing theorem, $H^p(-mK(V)) = 0$ for any $m > 0$ and $p < n$. Hence,

the Serre duality implies $H^1((m+1)K(V)) = 0$. Therefore,

$$\text{Tr}_{V_1} | e_1 m K(V) | \text{ is complete, so } \Phi_{e_1 m K(V)} | V_1 = \Phi_{e_1 K(V)}.$$

Hence, $\sum_j \cap V_1$ is exceptional for $\Phi_{e_1 K(V_1)}$. By induction

hypothesis, it follows that $\kappa(\sum_j \cap V_1) = -\infty$, which induces

$\kappa(\sum_j) = -\infty$. Thus we can assume that $f(\sum_j)$ is a point p

and write $E = \sum_j$ which is a subvariety of codimension r .

3. If $r = 1$, by the generalized adjunction formula, we have

$$\kappa(E) \leq \kappa(\{K(V) + E\} | E, E) = \kappa(E | E, E).$$

A general hyperplane section W_1 does not contain p and $|W_1|_p$ has a member W' . Then

$$f^*W' = r_1 E + G \sim f^*(W_1),$$

where each component of G differs from E and $r_1 > 0$. Hence

$$f^*(W_1) | E = r_1 E | E + G | E_1 \sim f^*(W_1 | p) = 0,$$

which implies $\kappa(E | E, E) = -\infty$. Thus $\kappa(E) = -\infty$.

4. In the case of $r > 1$, we first resolve the singularity of E following the method of Hironaka. Let $\mu_1 : V_1 \rightarrow V$ be a monoidal transformation with non-singular center $C \subset \text{sing}(E)$, the singular locus of E . Let E_1 be the strict

transform and L_1 the exceptional divisor for μ_1 . Then

$$K(V_1) = \mu^*K(V) + (v_1 - 1)L_1 \text{ and } K(V_1)|_{E_1} = \mu^*(K(V)|_E) + (v_1 - 1)L_1|_E$$

where $v_1 = \text{codim. of } C \text{ in } V$. If C is a divisor on E ,

then $v_1 - 1 = r$ and if not, $L_1|_{E_1}$ is exceptional for

$\mu_1|_{E_1} : E_1 \rightarrow E$. If E_1 is still singular, we have to perform

another monoidal transformation $\mu_2 : V_2 \rightarrow V_1$. Repeating this

process, we have a sequence of monoidal transformations $V_\ell \xrightarrow{\mu_\ell}$

$V_{\ell-1} \rightarrow \dots \rightarrow V_1 \xrightarrow{\mu_1} V$ and a non-singular E_ℓ , being the

strict transform of $E_{\ell-1}$. Let $\varphi_j = \mu_j \circ \dots \circ \mu_\ell : V_\ell \rightarrow V_{j-1}$.

Then we have $K(V_\ell) = \varphi_1^*K(V) + \sum (v_j - 1) \varphi_j^*L_{j-1}$. Moreover by an

exact sequence

$$0 \rightarrow \Omega_{E_\ell}^1 \rightarrow \Omega_{V_\ell}^1|_{E_\ell} \rightarrow \mathcal{O}(N_{V_\ell/E_\ell}^\vee) \rightarrow 0$$

where $N_{V_\ell/E_\ell}^\vee$ is the dual bundle of the normal bundle, we have

$$K(E_\ell) = K(V_\ell)|_{E_\ell} + [\det N_{V_\ell/E_\ell}].$$

In order to compute the divisor $[\det N_{V_\ell/E_\ell}]$ we blow V_ℓ up

with center E_ℓ . Then we have a monoidal transformation $\psi :$

$V^* \rightarrow V_\ell$, whose exceptional divisor L^* is isomorphic to

$\mathbb{P}(N_{V_\ell/E_\ell})$. $\mathcal{O}(L^*)|_{L^*}$ is the dual of the fundamental sheaf

$\mathcal{O}_{\mathbb{P}}(1)$ of $\mathbb{P} = \mathbb{P}(N_{V_\ell/E_\ell})$. Consider a general hyperplane

section D^* of V^* and write $D = f \varphi_1(D^*)$, which is a prime

divisor passing through p . Write $f^*D = D_0 + \dots$, D_0 being

the proper transform of D . Then since $\text{codim } E = r > 1$,

D_0 contains E . Letting ε_1 be the order of D_0 at the generic point of C , we have $\mu_1^* D_0 = D_1 + \varepsilon_1 L_1$, where D_1 is the strict transform of D . Considering in the same way as above, we have

$$\varphi_1^* D_0 = D_\ell + \varepsilon_1 \varphi_1^* L_1 + \varepsilon_2 \varphi_2^* L_2 + \cdots + \varepsilon_\ell L_\ell,$$

$$\psi_1^* D = \psi^* \varphi_1^* D_0 = D^* + \varepsilon_1 \psi_1^* L_1 + \varepsilon_2 \psi_2^* L_2 + \cdots + \varepsilon_\ell \varphi^* L_\ell + \varepsilon L^*$$

where $\psi_j = \varphi_j \circ \psi$, and ε is the order of D at the general point of E_ℓ . Hence $\varepsilon_j \geq \varepsilon$. Since $\mathcal{O}(\varepsilon L^*)|L^* \cong \mathcal{O}_{\mathbb{P}}(-\varepsilon)$, we have $\pi_*(\mathcal{O}(-\varepsilon L^*)|L^*) \cong S^\varepsilon(\mathcal{O}_{N_{V_\ell/E_\ell}}^\vee)$, where

$\pi: L^* = \mathbb{P}(N_{V_\ell/E_\ell}) \rightarrow E_\ell$ is the natural projection. Moreover,

$$\psi_1^* D|L^* = D^*|L^* + \pi^* \bar{\varphi}_1^*(\varepsilon_1 L_1|E_1) + \cdots$$

$$+ \pi^* \bar{\varphi}_{\ell-1}^*(\varepsilon_{\ell-1} L_{\ell-1}|E_{\ell-1}) + \pi^*(\varepsilon_\ell L_\ell|E_\ell) + \varepsilon L^*|L^*,$$

where $\bar{\varphi}_j = \varphi_j|E_j$. On the other hand, in view of $\psi_1^*(D)|L^* = \psi_1^*(D|p) \sim 0$, it follows that

$$-\varepsilon L^*|L^* \sim D^*|L^* + \pi^* \bar{\varphi}_1^*(\varepsilon_1 L_1|E_1) + \cdots + \pi^*(\varepsilon_\ell L_\ell|E_\ell).$$

Hence, $S^\varepsilon(\mathcal{O}_{N_{V_\ell/E_\ell}}^\vee) \cong \mathcal{O}(D^*|L^*) \otimes \pi^* \mathcal{O}(\bar{\varphi}_1^*(\varepsilon_1 L_1|E_1)) \otimes \cdots$

$\otimes \pi^* \mathcal{O}(\varepsilon_\ell L_\ell|E_\ell)$. Moreover,

$$\det S^\varepsilon(\mathcal{O}_{N_{V_\ell/E_\ell}}^\vee) = \mathcal{O}\left(\begin{pmatrix} \varepsilon-1+r \\ \varepsilon-1 \end{pmatrix} [\det N_{V_\ell/E_\ell}]\right)$$

and

$$\det\{\pi_* \mathcal{O}(D^*|L^*) \otimes \mathcal{L}\} \cong \det \pi_* \mathcal{O}(D^*|L^*) \otimes \mathcal{L}^\alpha,$$

where $\mathcal{L}$ is the invertible sheaf and $\alpha = \begin{pmatrix} \varepsilon-1+r \\ \varepsilon \end{pmatrix}$.

Hence,

$$-\begin{pmatrix} \varepsilon-1+r \\ \varepsilon-1 \end{pmatrix} [\det(N_{V_\ell/E_\ell})] \sim [\det \pi_* \mathcal{O}(D^*|L^*)] + \alpha \sum_{j=1}^{\ell} \bar{\varphi}_j^* \varepsilon_j L_j|E_j.$$

Thus,

$$\begin{aligned}
\binom{\varepsilon-1+r}{\varepsilon-1} K(E_\ell) &= \binom{\varepsilon-1+r}{\varepsilon-1} K(V_\ell) |_{E_\ell} - [\det \pi_* \mathcal{O}(D^* | L^*)] \\
&\quad - \alpha \cdot \sum \varepsilon_j \bar{\varphi}_j^* (\varepsilon_j L_j | E_j) \\
&= -[\det \pi_* \mathcal{O}(D^* | L^*)] + \binom{\varepsilon+r-1}{\varepsilon} \left\{ \frac{\varepsilon}{r} \sum (v_j-1) \bar{\varphi}_j^* (L_j | E_j) - \sum \varepsilon_j \bar{\varphi}_j^* L_j | E_j \right\} \\
&\leq -[\det \pi_* \mathcal{O}(D^* | L^*)] + \binom{\varepsilon+r-1}{\varepsilon} \left\{ \sum \varepsilon_j \left(\frac{v_j-1-r}{r} \right) \bar{\varphi}_j^* (L_j | E_j) \right\},
\end{aligned}$$

because $L_j | E_j$ is effective and $\varepsilon_j \geq \varepsilon$. If $v_j-1-r > 0$, then $L_j | E_j$ is exceptional. Hence

$$\kappa \left(\sum \varepsilon_j \left(\frac{v_j-1-r}{r} \right) \bar{\varphi}_j^* L_j | E_j, E_\ell \right) \leq 0.$$

On the other hand, the left hand side is not less than

$$\kappa \left([\det \pi_* (\mathcal{O} D^*) | L^*] + \binom{\varepsilon-1+r}{\varepsilon-1} K(E_\ell), E_\ell \right).$$

From the following lemma, $\kappa([\det(\pi_*(\mathcal{O} D^*) | L^*)], E_\ell) = n-r$, which implies $\kappa(E_\ell) = -\infty$.

Lemma. Let M be a projective n -manifold and $\mathcal{E}$ a locally free sheaf of rank r . The projective bundle $\pi : \mathbb{P}(\mathcal{E}) \rightarrow M$ has the tautological line bundle E . Assume that $\varepsilon E + \pi^*(D)$ is very ample, where $\varepsilon > 0$ and D is a divisor on M . Then $\pi_* \mathcal{O}(\varepsilon E + \pi^* D)$ is the ample sheaf. Hence $[\det \pi_* (\mathcal{O} \varepsilon E + \pi^* D)]$ is ample.

Since $\mathbb{P}(S^\varepsilon(\mathcal{E})) \cong \mathbb{P}(S^\varepsilon(\mathcal{E}) \otimes \mathcal{O}(D)) \hookrightarrow \mathbb{P}(\varepsilon E + \pi^* D)$, whose isomorphism i transforms E_ε to $\varepsilon E + \pi_\varepsilon^* D$, we have the imbeddings $\mathbb{P}(\mathcal{E}) \subset \mathbb{P}(S^\varepsilon(\mathcal{E})) \hookrightarrow \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}(\varepsilon E + \pi^* D)))$,

E_ε being the tautological bundle of $\mathbb{P}(S^\varepsilon(\mathcal{E}))$. The hyperplane section H of $\mathbb{P}(H^0(\mathbb{P}, \mathcal{O}(\varepsilon E + \pi^*D)))$ induces $E_\varepsilon + \pi_\varepsilon^*D$ on $\mathbb{P}(S^\varepsilon(\mathcal{E}))$ and so $E_\varepsilon + \pi_\varepsilon^*D$ is ample. Hence $\pi_{\varepsilon*} \mathcal{O}(E_\varepsilon + \pi_\varepsilon^*D) = \pi_* \mathcal{O}(\varepsilon E + \pi^*D) = S^\varepsilon(\mathcal{E}) \otimes \mathcal{O}(D)$ is also ample by Hartshorne's theorem.

Consequently we obtain

Theorem. Let V be a projective n -manifold of hyperbolic type, whose $|mK(V)|$ has no base point for $m \gg 0$. Then exceptional subvarieties for $\mathbb{P}_{mK(V)}$ are algebraic varieties of elliptic type.

Corollary. Let V be a submanifold of an abelian variety. Then the minimal canonical fibered manifold $f : V \rightarrow W$ is defined. K_V is a pull back of an ample divisor on W .

Problem. In the above situation, we assume that V is of hyperbolic type. Then is $|3K_V|$ very ample?

Remark. Let V be a 3-manifold of hyperbolic type whose $|mK(V)|$ has no base point for $m \gg 0$. We first assume that a non-singular surface S is the total exceptional set. Then we have two cases: (1) $f(S) = p \in W$. S^3 = the multiplicity of W at p is 2, 3, 4, ..., 9 since $-K(S)$ is ample. S is a rational surface called Del Pezzo surface, including $\mathbb{P}^2$, quadrics, cubics, (2) $f(S) = \Gamma$ a curve. $S \rightarrow \Gamma$ is a (2,1)-fibered surface whose general fiber is $\mathbb{P}^1$. The multiplicity of W at a general point of Γ is 2. Second we assume that

a non-singular curve C is the total exceptional set of V . Then $C \cong \mathbb{P}^1$ and $N_{V/C} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Moreover, any non-singular curve can never be the total exceptional set of n -manifold ($n \geq 4$) of hyperbolic type under our assumption (i.e., $B_s \mid mK(V) \mid = \emptyset$).

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