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Elements of prime order in primitive permutation groups.

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In 1873 C. Jordan, and at the beginning of this century W.A. Manning investigated certain properties of an element of prime order in a primitive group which is not alternating or symmetric. They found that the number of fixed points of such an element is bounded by a function of the number of its cycles of prime length, provided that the number of cycles is small. Let us make the following assumptions.

(\*)  $G$  is a primitive permutation group of degree  $n$ , and  $G \not\leq A_n$ .  
 $G$  contains an element of prime order  $p$ , where  $p$  divides  $|G|$ , which has  $q$  cycles of length  $p$  and  $f = n - qp$  fixed points.

In 1873 and 1875 Jordan [2,3] showed (or claimed) that if  $1 \leq q \leq 5$  and  $q < p$  then  $f \leq q + 1$ . The proof for  $q > 1$  was not published however until early this century, when Manning [4] I,II, published a proof of the result and obtained better bounds for  $f$  if  $2 \leq q \leq p - 2$ . In 1928 M. Weiss [11] obtained similar bounds for  $q \in \{6,7\}$ ,  $q < p$ , and very recently Jan Saxl [8,9] improved Manning's bounds in the cases  $q = 4$  and  $q = 5$ .

Now Manning was interested in finding a bound for  $f$  for a larger range of values of  $q$ . His first result in this direction was obtained in 1911 when he showed [5] that if  $q \leq 2p + 3$  and  $G$  is not 2-transitive then  $f \leq \max\{q^2 - q, 2q^2 - p^2\}$ .

This bound is too large to be useful in practice although there are examples in which the value of  $f$  differs from this bound only by a small constant multiple. (If  $G$  is  $A_c$  or  $S_c$  permuting the set of  $n = c(c-1)/2$  unordered pairs of distinct points, where  $c = (5p + 7)/2$ , then  $G$  is a simply transitive primitive group which contains an element of order  $p$  with  $2p + 3$  cycles of length  $p$  and  $f = (9p^2 + 36p + 35)/8$  fixed points, while the bound for  $f$  is  $2q^2 - p^2 = 7p^2 + 24p + 18$ .) Manning's main result [4] III was published in 1918 and gave very useful bounds on  $f$  as well as limiting the  $p$ -part of  $|G|$ . We state it below.

Theorem 1. (Manning) If (\*) is true and  $5 < q \leq (p + 1)/2$  then  $f \leq 4q - 4$  and  $|G|$  is not divisible by  $p^2$ . Moreover if  $G$  is not 2-transitive then  $f \leq 4q - 7$ .

It is easily checked that if  $1 \leq q \leq 4$  and  $q < p$ , then again  $|G|$  is not divisible by  $p^2$ . Thus if  $1 \leq q \leq (p + 1)/2$  we have a good bound both on the degree of  $G$  and on the  $p$ -part of the order of  $G$ . One might ask if these bounds can be extended for larger values of  $q$ , and indeed a partial answer was given a few years ago by Michael O'Nan and myself in [6,7].

Theorem 2. (O'Nan, Praeger) If (\*) is true and  $q < p$  then one of the following is true.

- (a)  $p^2$  does not divide  $|G|$ .
- (b)  $ASL(2, p) \leq G \leq AGL(2, p)$ , and  $G$  permutes the  $n = p^2$  points of an affine plane of order  $p$ .
- (c)  $G = P\Gamma L(2, 8)$  of degree  $n = p^2 = 9$ .

A similar result has been proved for  $q \leq 2p - 2$ , (see [7] II). The problem of extending Manning's bound on the number of fixed points  $f$ , even for the case  $q < p$ , is more difficult since the bound  $f \leq 4q - 4$  does not hold in general. Indeed if  $G = A_c$  or  $S_c$  permuting the  $n = c(c - 1)/2$  unordered pairs of distinct points, where  $p \leq c \leq (3p - 1)/2$ , then  $G$  is a simply transitive primitive group which contains an element of order  $p$  with  $q = c - (p + 1)/2$  cycles of length  $p$ , where  $(p - 1)/2 \leq q < p$ , and  $f = ((p^2 - 1)/4 - q(p - q))/2$  fixed points, (and if  $c = (3p - 1)/2$  then  $f = q(q - 2)/8$ .) I have been able to show very recently that these are the only groups which prevent Manning's bound holding for  $q < p$ , at least for simply transitive groups.

Theorem 3. If (\*) is true,  $G$  is not 2-transitive, and  $2 < q < p$ , then  $p^2$  does not divide  $|G|$  and either  $f \leq 4q - 7$ , or  $G$  is  $A_c$  or  $S_c$  permuting the  $n = c(c - 1)/2$  unordered pairs of distinct points, where  $c = q + (p + 1)/2$ .

By using an argument shown me by Peter M. Neumann together with the result above a similar bound can be obtained for 2-transitive groups.

Theorem 4. If (\*) is true,  $G$  is 2-transitive, and  $2 < q < p$ , then  $f \leq 4q - 3$ .

It may be possible to reduce the bound on  $f$  in the 2-transitive case. (Notice that I have stated the results for  $q > 2$ , for the bounds of Jordan and Manning for  $q = 3, 4, 5$  are even better than  $4q - 7$ .)

Discussion of the proof of Theorem 3.

Theorem 3 is proved by induction on the degree of  $G$ . The result for  $p \leq 7$  follows from the results of Jordan, Manning and Weiss, so we may assume that  $p \geq 11$ . We may use Jordan's or Manning's results to start the induction, so we assume that  $G$  is a group satisfying (\*) which is not 2-transitive and is such that  $2 < q < p$ , and we assume inductively that the theorem is true for groups of degree less than  $n$ . Let  $A$  be a subgroup of  $G$  of order  $p$ , degree  $qp$ , and with  $f$  fixed points. Without loss of generality we may assume that  $f > 0$ , so suppose that  $\alpha$  is a point fixed by  $A$ . By Theorem 2  $A$  is a Sylow  $p$ -subgroup of  $G$ , and hence of  $G_\alpha$ . If  $\Gamma$  is an orbit of  $G_\alpha$  in  $\Omega - \{\alpha\}$  then by [12] 18.4,  $A$  acts nontrivially on  $\Gamma$ . Thus  $G_\alpha^\Gamma$  is a transitive group of degree  $|\Gamma| < n$  with a subgroup  $A^\Gamma$  of order  $p$  and degree less than  $p^2$ . The next step is to find a primitive representation of degree less than  $n$  associated with this representation which either satisfies (\*) or is alternating or symmetric. Suppose that  $A$  has  $q'$  orbits of length  $p$  and  $f'$  fixed points in  $\Gamma$ .

Lemma Associated with  $G_\alpha^\Gamma$  is a primitive permutation group  $X$  of degree  $x$ , where  $|\Gamma| = xy$ , which contains an element of order  $p$  and degree  $(q'/y)p$  with  $f'/y$  fixed points, for some  $y \geq 1$ .

Proof Let  $M$  be the largest normal subgroup of  $G_\alpha$  such that  $|G_\alpha : M|$  is not divisible by  $p$ , and let  $\Sigma$  be the set of  $M$ -orbits in  $\Gamma$ . Then  $\Sigma$  is a set of blocks of imprimitivity for  $G_\alpha^\Gamma$  and  $|\Sigma|$  is maximal such that  $A^\Sigma = 1$ , (for  $A \subseteq M$ ). Let  $B \in \Sigma$  and let  $H$  be the setwise stabilizer of  $B$  in  $G_\alpha$ , (possibly  $B = \Gamma$ ). Let  $D \subset B$ ,  $D \neq B$ , be a block of imprimitivity for  $H^B$  such that  $|D|$  is maximal, (possibly  $|D| = 1$ ).

Then  $D$  is a block of imprimitivity for  $G_\alpha^\Gamma$ . Let  $\Delta = \{D^g; g \in G_\alpha\}$  and let  $\Delta(B) = \{D^g; g \in H\}$ .

Set  $X = H^{\Delta(B)}$ . Then by the maximality of  $|D|$ ,  $X$  is a primitive group of degree  $|\Delta(B)| = x$  say where  $|\Gamma| = x|D| \cdot |\Sigma|$ . By the maximality of  $|\Sigma|$ ,  $A$  acts nontrivially on  $\Delta$ , and as  $q' < p$  it follows that  $A$  permutes  $q'p/|D|$  elements of  $\Delta$  and fixes  $f'/|D|$  elements of  $\Delta$  pointwise. Moreover as  $A$  is a Sylow  $p$ -subgroup of  $M$ ,  $A$  permutes the same number of points in each element of  $\Sigma$ , and it follows that  $A$  permutes  $q'p/|D| \cdot |\Sigma|$  elements of  $\Delta(B)$  and fixes  $f'/|D| \cdot |\Sigma|$  elements of  $\Delta(B)$ . Since  $|\Gamma| = x|D| \cdot |\Sigma|$ , the lemma is proved.

Before proceeding we note that  $M^{\Delta(B)}$  is a transitive normal subgroup of  $X$ .

If  $q'/y = 1$  then by Jordan's result either  $f'/y \leq 2$  or  $X \supseteq A_x$ , and in the latter case clearly  $M^{\Delta(B)} \supseteq A_x$  so that  $A_x$  is a composition factor of  $G_\alpha$ . If  $2 \leq q'/y \leq (p+1)/2$  then by Jordan's and Manning's results,  $f'/y \leq 4(q'/y) - 4$ , (for  $X \not\supseteq A_x$  since  $p^2$  does not divide  $|X|$ ). Finally if  $q'/y \geq (p+3)/2$ , then since  $q' \leq q < p$ ,  $y = 1$  and so  $X = G_\alpha^\Gamma$  is primitive. Since  $q' > q/2$  this situation can arise for at most one orbit of  $G_\alpha$ . Moreover since the number of nontrivial orbits of  $A$  in an orbit  $\Gamma'$  of  $G_\alpha$  other than  $\Gamma$  is at most  $q - q' \leq (p-1) - (p+3)/2 \leq q' - 2$ , a small calculation shows that for every possibility  $|\Gamma'| < 2|\Gamma|$ . It follows from [1] that  $G_\alpha^\Gamma$  is not 2-transitive. Thus by induction either  $f' \leq 4q' - 7$  or  $G_\alpha^\Gamma$  is  $A_c$  or  $S_c$  on the set of  $|\Gamma| = c(c-1)/2$  unordered pairs of distinct points for some  $c \geq p$ . In the latter case  $A_c$  is a composition factor of  $G_\alpha$ .

If  $G_\alpha$  has no composition factors isomorphic to  $A_c$  for some  $c \geq p$ , then essentially by adding the bounds for the number of fixed points of  $A$  in each orbit of  $G_\alpha$  we obtain the required bound on  $f$ . If  $G_\alpha$  has a composition factor isomorphic to  $A_c$  for some  $c \geq p$  then the result follows from the following proposition.

Proposition Let  $G$  be a primitive permutation group of degree  $n$  and suppose that  $G$  contains an element of prime order  $p \geq 11$  and degree less than  $p^2$ . Then,

- (a) if  $G$  has a composition factor  $A_c$  for some  $c \geq p$ , either  $n = c$  and  $G \supseteq A_n$ , or  $n = c(c-1)/2$  and  $G$  is  $A_c$  or  $S_c$  permuting the set of unordered pairs of distinct points, and
- (b) if a one-point stabilizer in  $G$  has a composition factor  $A_c$  for some  $c \geq p$ , either  $n = c+1$  and  $G \supseteq A_n$ , or  $n = (c+2)(c+1)/2$  and  $G$  is  $A_{c+2}$  or  $S_{c+2}$  permuting the set of unordered pairs of distinct points.

The proof of this proposition is very complicated. One half of the proof involves a combinatorial argument using ideas from graph theory. The other half involves an investigation of subgroups of prime power order (for some prime less than  $p$ ), and exploits the methods of O'Nan's paper [6].

Conclusion I would like to improve these results in two ways. First, using an idea of Peter Neumann it may be possible to reduce Manning's bound to, perhaps,  $f \leq 2q$ . Second, I would like to obtain bounds on  $f$  for  $q \leq 2p-2$ . To do this I need a generalization of the Proposition.

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