

Resolution of dihedral orbifolds

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Let $G \subset \mathrm{GL}(2, \mathbb{C})$ be the following **small binary dihedral group**:

$$G = \left\langle \alpha = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix}, \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \varepsilon^{2n} = 1, (2n, a) = 1, a^2 \equiv 1 \pmod{2n} \right\rangle$$

where $A = \langle \alpha \rangle = \frac{1}{2n}(1, a)$ is a maximal normal of index 2, and we consider the minimal resolution $Y \rightarrow \mathbb{C}^2/G$.

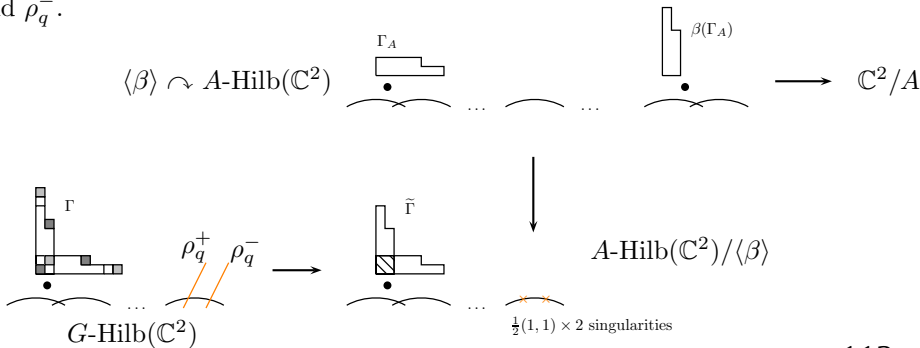
$Y = G\text{-Hilb}(\mathbb{C}^2)$: Moduli space parametrising G -clusters ([Ishii])
 $= \mathcal{M}_\theta(Q, R)$: Moduli of θ -stable representations of the bound McKay quiver

Definition: Let $G \subset \mathrm{GL}(2, \mathbb{C})$ be a finite subgroup. A G -graph is a subset $\Gamma \subset \mathbb{C}[x, y]$ such that it contains $\dim \rho$ elements in each irreducible representation ρ .

Motivation: For any G -cluster $\mathcal{Z} \in G\text{-Hilb}(\mathbb{C}^2)$, the basis of $\mathcal{O}_{\mathcal{Z}}$ as a vector space is a G -graph. Given a G -graph Γ , all the G -clusters with Γ as basis for $\mathcal{O}_{\mathcal{Z}}$ form an open set $U_\Gamma \subset G\text{-Hilb}(\mathbb{C}^2)$, and the collection of distinguished $\{U_{\Gamma_i}\}_{i \in I}$ covers $G\text{-Hilb}(\mathbb{C}^2)$.

General construction

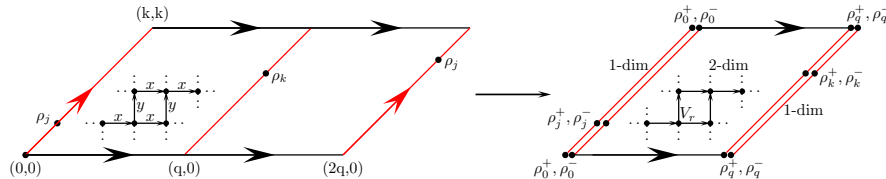
- **Fact:** $G\text{-Hilb}(\mathbb{C}^2) = G/A\text{-Hilb}(A\text{-Hilb}(\mathbb{C}^2))$.
- The symmetry of the continued fraction $\frac{2n}{a}$ implies that (i) the coordinates along the exceptional divisor $E \subset A\text{-Hilb}(\mathbb{C}^2)$ are also symmetric, and (ii) β is an involution on the middle curve $E_m \cong \mathbb{P}^1$ on the exceptional divisor on $A\text{-Hilb}(\mathbb{C}^2)$.
- Every G -graph Γ is either the unique extension of the union of two symmetric A -graphs, or it comes from a choice on the special irreducible representations ρ_q^+ and ρ_q^- .



Orbifold McKay quiver

Let $\mathrm{Irr} A = \{\rho_0, \dots, \rho_{2n-1}\}$ the irreducible representations of A . The McKay quiver of $A = \frac{1}{2n}(1, a)$ can be written on a torus, and the quotient $G/A \cong \mathbb{Z}/2 = \langle \beta \rangle$ acts on $\mathrm{Irr} A$ by conjugation. Then

McKay quiver of $G = \mathbb{Z}/2$ -orbifold of the McKay quiver of A



Fixed points $\rho_j \in \mathrm{Irr} A$ by β become two 1-dimensional representations ρ_j^+ and ρ_j^- . Free orbits $\{\rho_r, \rho_{ar}\}$ by β become one 2-dimensional representation V_r .

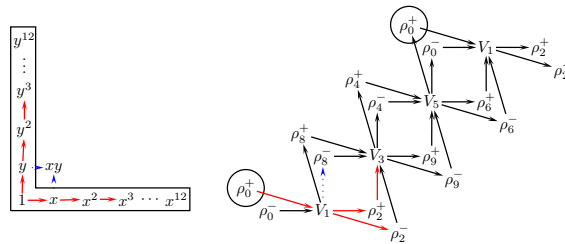
Explicit description of an open cover of Y

Let (Q, R) the bound McKay quiver, $\mathbf{d} = (\dim \rho_i)_{i \in Q_0}$ the dimension vector and the generic stability condition $\theta = (-\sum_{\rho_i \in \mathrm{Irr} G} \dim \rho_i, 1, \dots, 1)$. This choice of θ implies that *there exist $\dim \rho_j$ nonzero paths from the distinguished source ρ_0^+ to every other irreducible representation ρ_j .*

Any G -graph Γ produces the choices for nonzero maps in the representation space of (Q, R) . Therefore, given any G -graph Γ we can associate an open set $M_\Gamma \subset \mathcal{M}_\theta(Q, R)$, and the $\{M_{\Gamma_i}\}_{i \in I}$ covers $\mathcal{M}_\theta(Q, R)$.

Using the relations R of Q the equations of M_Γ are explicitly obtained.

Example: Let $G = \langle \frac{1}{12}(1, 7), \beta \rangle$. The minimal resolution Y consists of 5 open sets given by the G -graphs $\Gamma_2, \dots, \Gamma_5$. For instance, for the G -graph Γ_0 we have:



Remaining open sets for $\mathcal{M}_\theta(Q, R)$ as hypersurfaces in $\mathbb{C}^3$:

$$\begin{aligned} M_{\Gamma_2} &: b_2^+ E = (b_2^+ + 1) D^+ \\ M_{\Gamma_3} &: b_2^- G = (b_2^- + 1) D^- \\ M_{\Gamma_4} &: ef = (e^2 f - 1) D_+ \\ M_{\Gamma_5} &: gh = (g^2 h - 1) D_- \end{aligned}$$

and M_{Γ_1} is given by $(cd = (1 + cd^2)E) \subset \mathbb{C}^3$