

$L_p - L_q$ ESTIMATES OF THE OSEEN SEMIGROUP IN EXTERIOR DOMAINS

YUKO ENOMOTO AND YOSHIHIRO SHIBATA

We consider the following Oseen equation:

$$(1) \quad \begin{cases} u_t - \Delta u + (u_\infty \cdot \nabla)u + \nabla \pi = f & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0, \infty) \times \Omega, \\ u|_{\partial\Omega} = 0, \quad u|_{t=0} = a, \end{cases}$$

where Ω is an exterior domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$ ($n \geq 3$). When $u_\infty = 0$, the equation is the Stokes one. Our treatment below is including the Stokes equation.

First of all, we introduce the notation throughout the paper. For two Banach spaces X and Y , $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y . We put

$$\begin{aligned} B_b^n &= \{x \in \mathbb{R}^n \mid |x| < b\} \quad (b > 0), \\ \Omega_b &= \Omega \cap B_b^n, \\ C_{0,\sigma}^\infty(\Omega)^n &= \{u \in C_0^\infty(\Omega)^n \mid \nabla \cdot u = 0\}, \\ J_p(\Omega) &= \overline{C_{0,\sigma}^\infty(\Omega)^n}^{\|\cdot\|_{L_p}}, \\ J_{p,b}(\Omega) &= \{u \in J_p(\Omega) \mid u(x) = 0 \text{ for } |x| > b\}, \\ G_p(\Omega) &= \{\nabla \pi \in L_p(\Omega)^n \mid \pi \in L_{p,loc}(\Omega)\} \end{aligned}$$

and $\varphi_b(x)$ is a function in $C^\infty(\mathbb{R}^n)$ such that $\varphi_b(x) = 0$ for $|x| \leq b-1$ and $\varphi_b(x) = 1$ for $|x| \geq b$.

The Banach space $L_p(\Omega)^n$ admits the Helmholtz decomposition:

$$L_p(\Omega)^n = J_p(\Omega) \oplus G_p(\Omega).$$

Let $\mathbb{P}$ be a continuous projection from $L_p(\Omega)^n$ onto $J_p(\Omega)$. Applying $\mathbb{P}$ to the Oseen equation, we have

$$\begin{cases} u_t + \mathbb{P}(-\Delta + (u_\infty \cdot \nabla))u = \mathbb{P}f, \\ u|_{\partial\Omega} = 0, \quad u|_{t=0} = a. \end{cases}$$

Let us define the operator $\mathbb{O}_{u_\infty}$ by $\mathbb{O}_{u_\infty} = \mathbb{P}(-\Delta + (u_\infty \cdot \nabla))$ with the domain:

$$\mathcal{D}_p(\mathbb{O}_{u_\infty}) = \{u \in J_p(\Omega) \cap W_p^2(\Omega) \mid u|_{\partial\Omega} = 0\}$$

By Miyakawa [3], we know that $\mathbb{O}_{u_\infty}$ generates an analytic semigroup $\{T_{u_\infty}(t)\}_{t \geq 0}$.

Our main theorem is the following one.

Department of Mathematical Sciences Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo, 169-8555, Japan.

Theorem 1. Let σ_0 be a positive number and $1 \leq p \leq q \leq \infty$. Assume that $|u_\infty| \leq \sigma_0$. For any $t > 0$,

$$(2) \quad \|T_{u_\infty}(t)a\|_{L_q(\Omega)} \leq C_{p,q,\sigma_0} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|a\|_{L_p(\Omega)} \quad (p,q) \neq (1,1), (\infty, \infty),$$

$$(3) \quad \|\nabla T_{u_\infty}(t)a\|_{L_q(\Omega)} \leq C_{p,q,\sigma_0} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|a\|_{L_p(\Omega)} \quad 1 \leq p \leq q \leq n, (p,q) \neq (1,1).$$

Moreover, if $|u_\infty| \neq 0$ and $t > 1$ then we have

$$(4) \quad \|\partial_t T_{u_\infty}(t)a\|_{L_q(\Omega)} \leq C_{p,q,\sigma_0} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|a\|_{L_p(\Omega)} \quad (p,q) \neq (1,1), (\infty, \infty),$$

$$(5) \quad \|\partial_t \nabla T_{u_\infty}(t)a\|_{L_q(\Omega)} \leq C_{p,q,\sigma_0} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-1} \|a\|_{L_p(\Omega)} \quad 1 \leq p \leq q \leq n, (p,q) \neq (1,1).$$

A crucial step of our approach is to show the following local energy decay of the Oseen semigroup.

Theorem 2. Let $1 < p < \infty$ and $\sigma_0 > 0$. Assume that $|u_\infty| \leq \sigma_0$. Then, for any $b > b_0$ and any nonnegative integer k , there exists a positive constant $C_{k,p,b,\sigma_0,n}$ such that

$$\|\partial_t^k T_{u_\infty}(t)a\|_{W_p^2(\Omega_b)} \leq C_{k,p,b,\sigma_0,n} t^{-\frac{n+k}{2}} \|a\|_{L_p(\Omega)} \quad \text{for } \forall t \geq 1, \forall a \in J_{p,b}(\Omega),$$

where b_0 is a fixed positive number such that $\Omega^c \subset B_{b_0-3}^n$.

To use a cut-off technique later on under the Helmholtz decomposition, we use the following Bogovskii lemma.

Lemma 3 ([1], [2]). Let $1 < p < \infty$ and let m be a nonnegative integer. Then, there exists a bounded linear operator $\mathbb{B} : \dot{W}_{p,a}^m(D) \rightarrow \dot{W}_p^{m+1}(D)$ such that

$$\nabla \cdot \mathbb{B}[f] = f \text{ in } D \quad \text{and} \quad \|\mathbb{B}[f]\|_{W_p^{m+1}(D)} \leq C \|f\|_{W_p^m(D)}$$

where D is a bounded domain with Lipschitz boundary in $\mathbb{R}^n$, $\dot{W}_p^m(D) = \overline{C_0^\infty(D)}^{\|\cdot\|_{W_p^m}}$ and $\dot{W}_{p,a}^m(D) = \{u \in \dot{W}_p^m(D) \mid \int_D u dx = 0\}$.

Sketch of proof of Theorem 1. We define a solution operator in $\mathbb{R}^n$. Let $c(x)$ be a function in $L_p(\mathbb{R}^n)$ satisfying $\nabla \cdot c = 0$ in $\mathbb{R}^n$. We define $S_{u_\infty}(t)c(x)$ by the formula:

$$S_{u_\infty}(t)c(x) = \left(\frac{1}{4\pi t}\right)^n \int_{\mathbb{R}^n} e^{-\frac{|x-y-tu_\infty|^2}{4t}} c(y) dy.$$

Put $v(t, x) = S_{u_\infty}(t)c(x)$, then v satisfies the equation:

$$\begin{cases} v_t - \Delta v + (u_\infty \cdot \nabla)v = 0 & \text{in } (0, \infty) \times \mathbb{R}^n, \\ \nabla \cdot v = 0 & \text{in } (0, \infty) \times \mathbb{R}^n, \\ v|_{t=0} = c. \end{cases}$$

Moreover, when $1 \leq p \leq q \leq \infty$, by the Young inequality we can show that

$$(6) \quad \|\partial_t^j \partial_x^\alpha v(t)\|_{L_q(\mathbb{R}^n)} \leq C t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{j+|\alpha|}{2}} \|c\|_{L_p(\Omega)} \quad t \geq 1,$$

$$(7) \quad \|\partial_t^j \partial_x^\alpha v(t)\|_{L_q(\mathbb{R}^n)} \leq C t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-(j+\frac{|\alpha|}{2})} \|c\|_{L_p(\Omega)} \quad 0 < t \leq 1.$$

For $t \geq 1$, we will prove the following $L_p - L_q$ estimates:

$$(8) \quad \|T_{u_\infty}(t)a\|_{L_q(\Omega)} \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|a\|_{L_p(\Omega)} \quad 1 < p \leq q \leq \infty,$$

$$(9) \quad \|\nabla T_{u_\infty}(t)a\|_{L_q(\Omega)} \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|a\|_{L_p(\Omega)} \quad 1 < p \leq q \leq n.$$

To do this put $\tilde{a}(x) = T_{u_\infty}(1)a(x)$ and $u(t, x) = T_{u_\infty}(t)\tilde{a}(x)$. By the analytic semigroup theory, for any nonnegative integer N

$$\tilde{a} \in \mathcal{D}_p(\mathcal{O}_{u_\infty}^N) \quad \text{and} \quad \|\tilde{a}\|_{W_p^{2N}(\Omega)} \leq C\|a\|_{L_p(\Omega)}.$$

1st step.

Let m be a nonnegative integer. For any $t \geq 0$, we shall prove the following estimates:

$$(10) \quad \|u(t)\|_{W_p^{2m}(\Omega_b)} \leq C(1+t)^{-\frac{n}{2p}} \|a\|_{L_p(\Omega)},$$

$$(11) \quad \|u_t(t)\|_{W_p^{2m}(\Omega_b)} \leq C(1+t)^{-\frac{n}{2p}-\frac{1}{2}} \|a\|_{L_p(\Omega)},$$

$$(12) \quad \|u(t)\|_{W_\infty^{2m}(\Omega_b)} \leq C(1+t)^{-\frac{n}{2p}} \|a\|_{L_p(\Omega)},$$

$$(13) \quad \|u_t(t)\|_{W_\infty^{2m}(\Omega_b)} \leq C(1+t)^{-\frac{n}{2p}-\frac{1}{2}} \|a\|_{L_p(\Omega)},$$

where $1 < p < \infty$.

Let N be a natural number such that $N \geq \frac{1}{2} \left(\frac{n}{p} + 2m + 6 \right)$ and let $1 < p < \infty$. There exists a $c(x) \in W_p^{2N}(\mathbb{R}^n)$ such that $c(x) = \tilde{a}(x)$ on Ω and $\nabla \cdot c = 0$ in $\mathbb{R}^n$. Moreover,

$$(14) \quad \|c\|_{W_p^{2N}(\mathbb{R}^n)} \leq C\|\tilde{a}\|_{W_p^{2N}(\Omega)} \leq C\|a\|_{L_p(\Omega)}.$$

By (6), (7) and (14), for any $t \geq 0$ we put $v(t, x) = S_{u_\infty}(t)c(x)$ then we have

$$\|\partial_t^j v(t)\|_{W_\infty^{2m+1}(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{2p}-\frac{j}{2}} \|a\|_{L_p(\Omega)},$$

where $j = 0, 1, 2$. Let us define w by the following formula:

$$w = u - \varphi_{b+1}v - \mathbb{B}[(\nabla\varphi_{b+1}) \cdot v].$$

Then, $w = u$ in Ω_b and w satisfies the equation:

$$\begin{cases} w_t - \Delta w + (u_\infty \cdot \nabla)w + \nabla\pi = g & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot w = 0 & \text{in } (0, \infty) \times \Omega, \\ w|_{\partial\Omega} = 0, \quad w|_{t=0} = d, \end{cases}$$

where

$$\begin{aligned} g &= -2(\nabla\varphi_{b+1})(\nabla v) - (\Delta\varphi_{b+1})v + [(u_\infty \cdot \nabla)\varphi_{b+1}]v \\ &\quad - (\partial_t - \Delta + (u_\infty \cdot \nabla))\mathbb{B}[(\nabla\varphi_{b+1}) \cdot v], \\ d &= \varphi_{b+1}c - \mathbb{B}[(\nabla\varphi_{b+1}) \cdot c]. \end{aligned}$$

It is easy to show that g and d satisfy the properties:

$$\begin{aligned} \partial_t^j g(t) &\in \mathcal{D}_p(\mathcal{O}_{u_\infty}^m) \cap J_{p,b+1}(\Omega), \quad \|\partial_t^j g(t)\|_{W_p^{2m}(\Omega)} \leq C(1+t)^{-\frac{n}{2p}-\frac{j}{2}} \|a\|_{L_p(\Omega)}, \\ d &\in \mathcal{D}_p(\mathcal{O}_{u_\infty}^N) \cap J_{p,b+1}(\Omega), \quad \|d\|_{W_p^{2N}(\Omega)} \leq C\|a\|_{L_p(\Omega)}. \end{aligned}$$

By Duhamel's principle, w is represented by

$$w(t, x) = T_{u_\infty}(t)d(x) + \int_0^t T_{u_\infty}(t-s)g(s)ds.$$

Since g and d have compact supports, we can use the local energy decay theorem. Then, by the local energy decay estimate, we have

$$\|w(t)\|_{W_p^{2m}(\Omega_b)} \leq C(1+t)^{-\frac{n}{2p}} \|a\|_{L_p(\Omega)}.$$

Moreover, we have

$$\|w_t(t)\|_{W_p^{2m}(\Omega_b)} \leq C(1+t)^{-\frac{n}{2p}-\frac{1}{2}} \|a\|_{L_p(\Omega)}.$$

Therefore we obtain (10) and (11). Since m is arbitrary, by Sobolev's embedding theorem, we obtain (12) and (13).

2nd step.

We estimate the pressure π , that is, for $t \geq 0$ we prove

$$(15) \quad \|\pi(t)\|_{W_p^{2m}(\Omega_b)} \leq C(1+t)^{-\frac{n}{2p}} \|a\|_{L_p(\Omega)},$$

$$(16) \quad \|\pi(t)\|_{W_\infty^{2m}(\Omega_b)} \leq C(1+t)^{-\frac{n}{2p}} \|a\|_{L_p(\Omega)},$$

where $1 < p < \infty$.

We may assume without loss of generality that $\int_{\Omega_b} \pi(x)dx = 0$. By Poincaré's inequality, we have

$$\begin{aligned} \|\pi(t)\|_{W_p^{2m}(\Omega_b)} &\leq C\|\nabla\pi(t)\|_{W_p^{2m-1}(\Omega_b)} \\ &\leq C\|u_t - \Delta u + (u_\infty \cdot \nabla)u\|_{W_p^{2m-1}(\Omega_b)}, \end{aligned}$$

which implies that (15) holds. Using the Sobolev's embedding theorem again, we obtain (16).

3rd step.

We shall prove the following L_p - L_q estimates:

$$(17) \quad \|u(t)\|_{L_q(\Omega)} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|a\|_{L_p(\Omega)} \quad 1 < p \leq q \leq \infty,$$

$$(18) \quad \|\nabla u(t)\|_{L_q(\Omega)} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|a\|_{L_p(\Omega)} \quad 1 < p \leq q \leq n.$$

Since we have already had the estimate of u in Ω_b , in order to obtain (17) and (18), it is sufficient to estimate u outside of Ω_b . Let us define z by the formula:

$$z = (1 - \varphi_b)u + \mathbb{B}[(\nabla\varphi_b) \cdot u].$$

Then, $z = u$ in Ω_b^c and z satisfies the equation:

$$\begin{cases} z_t - \Delta z + (u_\infty \cdot \nabla)z + \nabla[(1 - \varphi_b)\pi] = h & \text{in } (0, \infty) \times \mathbb{R}^n, \\ \nabla \cdot z = 0 & \text{in } (0, \infty) \times \mathbb{R}^n, \\ z|_{t=0} = e, \end{cases}$$

where

$$\begin{aligned} h &= 2(\nabla\varphi_b)(\nabla u) + (\Delta\varphi_b)u + [(u_\infty \cdot \nabla)\varphi_b]u - (\nabla\varphi_b)\pi \\ &\quad + (\partial_t - \Delta + (u_\infty \cdot \nabla))\mathbb{B}[(\nabla\varphi_b) \cdot u], \\ e &= (1 - \varphi_b)\tilde{a} + \mathbb{B}[(\nabla\varphi_b) \cdot \tilde{a}]. \end{aligned}$$

It is easy to show that h and e satisfy the inequalities:

$$\begin{aligned} \|h(t)\|_{W_p^{2m-1}(\mathbb{R}^n)} &\leq C(1+t)^{-\frac{n}{2p}}\|a\|_{L_p(\Omega)}, \\ \|e\|_{W_p^{2m}(\mathbb{R}^n)} &\leq C\|a\|_{L_p(\Omega)}, \end{aligned}$$

where $1 < p < \infty$. By Duhamel's principle, z is represented by

$$z(t, x) = S_{u_\infty}(t)e(x) + \int_0^t S_{u_\infty}(t-s)\mathbb{P}h(s)ds.$$

By L_p - L_q estimate, we have

$$\begin{aligned} \|S_{u_\infty}(t)e\|_{L_q(\mathbb{R}^n)} &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|a\|_{L_p(\Omega)}, \\ \|\nabla S_{u_\infty}(t)e\|_{L_q(\mathbb{R}^n)} &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|a\|_{L_p(\Omega)}, \end{aligned}$$

where $1 \leq p \leq q \leq \infty$. Let ρ be a number such that $1 < \rho < \min(\frac{n}{2}, p)$. Since

$$\|\mathbb{P}h(t)\|_{L_\rho(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{2\rho}}\|a\|_{L_p(\Omega)},$$

we have

$$\begin{aligned} \left\| \int_0^t S_{u_\infty}(t-s)\mathbb{P}h(s)ds \right\|_{L_q(\mathbb{R}^n)} &\leq CI_\rho(t)\|a\|_{L_p(\Omega)}, \\ \left\| \nabla \int_0^t S_{u_\infty}(t-s)\mathbb{P}h(s)ds \right\|_{L_q(\mathbb{R}^n)} &\leq CJ_\rho(t)\|a\|_{L_p(\Omega)}, \end{aligned}$$

where

$$\begin{aligned} I_\rho(t) &= \int_0^t (1+t-s)^{-\frac{n}{2}(\frac{1}{\rho}-\frac{1}{q})}(1+s)^{-\frac{n}{2\rho}}ds, \\ J_\rho(t) &= \int_0^t (1+t-s)^{-\frac{n}{2}(\frac{1}{\rho}-\frac{1}{q})-\frac{1}{2}}(1+s)^{-\frac{n}{2\rho}}ds \end{aligned}$$

and $1 < p \leq q \leq \infty$. Therefore, we obtain

$$\begin{aligned} \|z(t)\|_{L_q(\mathbb{R}^n)} &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|a\|_{L_p(\Omega)} \quad 1 < p \leq q \leq \infty, \\ \|\nabla z(t)\|_{L_q(\mathbb{R}^n)} &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|a\|_{L_p(\Omega)} \quad 1 < p \leq q \leq \infty. \end{aligned}$$

Now, for $0 < t \leq 1$, we shall prove the following L_p - L_q estimates:

$$(19) \quad \|T_{u_\infty}(t)a\|_{L_q(\Omega)} \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|a\|_{L_p(\Omega)},$$

$$(20) \quad \|\nabla T_{u_\infty}(t)a\|_{L_q(\Omega)} \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|a\|_{L_p(\Omega)},$$

where $1 < p \leq q < \infty$. In the similar manner, we have

$$(21) \quad \|\partial_t T_{u_\infty}(t)a\|_{L_q(\Omega)} \leq C t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|a\|_{L_p(\Omega)},$$

$$(22) \quad \|\partial_t \nabla T_{u_\infty}(t)a\|_{L_q(\Omega)} \leq C t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-1} \|a\|_{L_p(\Omega)}.$$

If u together with some π satisfies the equation:

$$\begin{cases} \lambda u - \Delta u + (u_\infty \cdot \nabla)u + \nabla \pi = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then there exists an $R > 0$ such that for $\lambda \in \Sigma_\varepsilon = \{\lambda \in \mathbb{C} \mid |u_\infty|^2 \operatorname{Re} \lambda + |\operatorname{Im} \lambda|^2 > 0\}$ with $|\lambda| \geq R$,

$$(23) \quad |\lambda| \|u\|_{L_p(\Omega)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{L_p(\Omega)} + \|\nabla^2 u\|_{L_p(\Omega)} \leq C \|f\|_{L_p(\Omega)},$$

where $1 < p < \infty$. The analytic semigroup $T_{u_\infty}(t)a$ is represented by

$$T_{u_\infty}(t)a = \int_\Gamma e^{\lambda t} (\lambda + \mathbb{O}_{u_\infty})^{-1} a \, d\lambda$$

with suitable contour Γ in some sector. By the resolvent estimate (23), for $0 < t \leq 1$

$$(24) \quad \|T_{u_\infty}(t)a\|_{L_p(\Omega)} + t^{\frac{1}{2}} \|\nabla T_{u_\infty}(t)a\|_{L_p(\Omega)} + t \|\nabla^2 T_{u_\infty}(t)a\|_{L_p(\Omega)} \leq C \|f\|_{L_p(\Omega)}.$$

In view of the complex interpolation: $W_p^{n(\frac{1}{p}-\frac{1}{q})} = [L_p, W_p^2]_{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$, interpolating (24) and Sobolev's embedding theorem, we obtain the $L_p - L_q$ estimates (19) and (20).

Next, for $0 < t \leq 1$ we shall prove

$$(25) \quad \|T_{u_\infty}(t)a\|_{L_\infty(\Omega)} \leq C t^{-\frac{n}{2p}} \|a\|_{L_p(\Omega)} \quad 1 < p < \infty.$$

A Besov space $B_{p,1}^{\frac{n}{p}}$ is continuously included in L_∞ and it is obtained by the real interpolation: $B_{p,1}^{\frac{n}{p}} = [L_p, W_p^2]_{\frac{n}{2p},1}$. Interpolating two formulas: $\|T_{u_\infty}(t)a\|_{L_p(\Omega)} \leq C \|a\|_{L_p(\Omega)}$ and $\|T_{u_\infty}(t)a\|_{W_p^2(\Omega)} \leq C t^{-1} \|a\|_{L_p(\Omega)}$, we have (25).

Finally, for $t > 0$ we shall prove

$$(26) \quad \|T_{u_\infty}(t)a\|_{L_q(\Omega)} \leq C t^{-\frac{n}{2}(1-\frac{1}{q})} \|a\|_{L_1(\Omega)} \quad 1 < q \leq \infty.$$

For $a \in J_1(\Omega)$, we define $T_{u_\infty}(t)a$ by the duality

$$(T_{u_\infty}(t)a, b) = (a, T_{-u_\infty}(t)b) \quad \text{for } \forall b \in C_{0,\sigma}^\infty(\Omega).$$

Then, we have

$$\begin{aligned} |(T_{u_\infty}(t)a, b)| &\leq C \|a\|_{L_1(\Omega)} \|T_{-u_\infty}(t)b\|_{L_\infty(\Omega)} \\ &\leq C \|a\|_{L_1(\Omega)} t^{-\frac{n}{2q'}} \|b\|_{L_{q'}(\Omega)}, \end{aligned}$$

where $1 < q < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$ which implies that (26) holds. This completes the proof of Theorem 1. $\square$

Now, we shall prove our local energy decay theorem. Before going to a sketch of the proof of Theorem 2, we introduce the following definition concerning some regularity of the resolvent operator.

Definition 4. Let B be a Banach space and $\|\cdot\|_B$ its norm. Let $T_{u_\infty}(t)$ be a function in $C^\infty(\mathbb{R}^n \setminus \{0\})$ with its value in B , which depends on $u_\infty \in \mathbb{R}^n$. Let σ_0 be a positive number. Assume that $\|T_{u_\infty}(t)\|_{L_\infty(\mathbb{R}, B)} \leq C$ when $|u_\infty| \leq \sigma_0$ for some constant C independent of σ_0 . We say that $T_{u_\infty}(t)$ is *uniformly n -regular* in B if whenever $|u_\infty| \leq \sigma_0$, $T_{u_\infty}(t)$ satisfies the following properties :

When n is even, for any nonnegative integers m , M and N with $N \geq m$ there hold the following seven inequalities :

$$\begin{aligned} \|\Delta_h^2 [s^N \partial_s^{\frac{n}{2}-1+m} T_{u_\infty}(s)]\|_{L_1(\mathbb{R}, B)} &\leq C|h|; \\ \|\Delta_h [s^N \partial_s^{\frac{n}{2}-1+m} T_{u_\infty}(s)]\|_{L_q(\mathbb{R}, B)} &\leq C|h|^{\frac{1}{2}} \quad 1 \leq q < 2; \\ \|\Delta_h [s^{N+1} \partial_s^{\frac{n}{2}+m} T_{u_\infty}(s)]\|_{L_1(\mathbb{R}, B)} &\leq C|h|^{\frac{1}{2}}; \\ \|s^N \partial_s^{\frac{n}{2}+m} T_{u_\infty}(s)\|_{L_q(\mathbb{R}, B)} &\leq C \quad 1 \leq q < \infty; \\ \|s^{N+1} \partial_s^{\frac{n}{2}+m} T_{u_\infty}(s)\|_{L_q(\mathbb{R}, B)} &\leq C \quad 1 \leq q < 2; \\ \|\Delta_h [s^M \partial_s^m T_{u_\infty}(s)]\|_{L_q(\mathbb{R}, B)} &\leq C|h|^r \\ &\quad 1 \leq q < \infty, \quad 0 \leq m \leq \frac{n}{2} - 2, \quad r = 1 \text{ and } \frac{1}{2}; \\ \|s^M \partial_s^m T_{u_\infty}(s)\|_{L_\infty(\mathbb{R}, B)} &\leq C \quad 1 \leq m \leq \frac{n}{2} - 2. \end{aligned}$$

When n is odd, for any nonnegative integer m , M and N with $N \geq 2m$ there hold the following seven inequalities :

$$\begin{aligned} \|\Delta_h^2 [s^{N+1} \partial_s^{\lfloor \frac{n}{2} \rfloor + m} T_{u_\infty}(s)]\|_{L_1(\mathbb{R}, B)} &\leq C|h|; \\ \|\Delta_h [s^N \partial_s^{\lfloor \frac{n}{2} \rfloor + m} T_{u_\infty}(s)]\|_{L_1(\mathbb{R}, B)} &\leq C|h|^{\frac{1}{2}}; \\ \|\Delta_h [s^{N+1} \partial_s^{\lfloor \frac{n}{2} \rfloor + m} T_{u_\infty}(s)]\|_{L_\infty(\mathbb{R}, B)} &\leq C; \\ \|s^N \partial_s^{\lfloor \frac{n}{2} \rfloor + m} T_{u_\infty}(s)\|_{L_q(\mathbb{R}, B)} &\leq C \quad 1 \leq q < 2; \\ \|\Delta [s^M \partial_s^m T_{u_\infty}(s)]\|_{L_q(\mathbb{R}, B)} &\leq C|h| \quad 1 \leq q < 2, \quad 0 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor - 1; \\ \|\Delta_h [s^M \partial_s^m T_{u_\infty}(s)]\|_{L_q(\mathbb{R}, B)} &\leq C|h|^{\frac{1}{2}} \quad 1 \leq q < \infty, \quad 0 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor - 1; \\ \|s^M \partial_s^m T_{u_\infty}(s)\|_{L_\infty(\mathbb{R}, B)} &\leq C \quad 1 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \end{aligned}$$

where $\left[\frac{n}{2}\right] = \frac{n-1}{2}$. Here, the constant C depends on n, m, r, M, N and σ_0 , but is independent of h and u_∞ ; and for any B -valued function $g(s)$ and $h \in \mathbb{R} \setminus \{0\}$ we have put

$$\begin{aligned} \|g\|_{L_q(\mathbb{R}, B)} &= \left\{ \int_{-\infty}^{\infty} \|g(s)\|_B^q ds \right\}^{\frac{1}{q}} \quad 1 \leq q < \infty; \\ \|g\|_{L_\infty(\mathbb{R}, B)} &= \operatorname{ess\,sup}_{s \in \mathbb{R} \setminus \{0\}} \|g(s)\|_B; \\ \Delta_h^2 g(s) &= g(s+h) - 2g(s) + g(s-h); \\ \Delta_h g(s) &= g(s+h) - g(s). \end{aligned}$$

Theorem 5. Let $X = \mathcal{L}(L_{p,b}(\mathbb{R}^n), W_p^2(B_b^n))$, and $\sigma_0 > 0$. Assume $|u_\infty| \leq \sigma_0$. If we put

$$U_{u_\infty}(s) = \left(\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \frac{e^{ix \cdot \xi}}{|\xi|^2 + is + i(u_\infty \cdot \xi)} \frac{\xi_j \xi_k}{|\xi|^2} d\xi \right)$$

and $E_{u_\infty}(s)f = U_{u_\infty}(s) * f$, then $E_{u_\infty}(s)$ is uniformly n -regular in X .

To prove Theorem 2, we construct a parametrix. For $f(x) \in L_{p,b}(\Omega)$, we put $f_0(x) = f(x)$ for $x \in \Omega$ and $f_0(x) = 0$ for $x \notin \Omega$. Let us put

$$\begin{aligned} \Phi_{u_\infty}(\lambda)f &= \varphi_{b-1} E_{u_\infty}(\lambda)f_0 + (1 - \varphi_{b-1})F_{u_\infty}(\lambda)f + G_{u_\infty}(\lambda)f, \\ P_{u_\infty}(\lambda)f &= \varphi_{b-1} \Pi f_0 + (1 - \varphi_{b-1})\Pi_{u_\infty}(\lambda)f, \end{aligned}$$

where $G_{u_\infty}(\lambda)f = \mathbb{B}[(\nabla \varphi_{b-1}) \cdot (E_{u_\infty}(\lambda)f_0 - F_{u_\infty}(\lambda)f)]$ and $(v, \pi) = (F_{u_\infty}(\lambda)f, \Pi_{u_\infty}(\lambda)f)$ is a solution to the Oseen equation:

$$\begin{cases} (\lambda - \Delta + (u_\infty \cdot \nabla))v + \nabla \pi = f & \text{in } \Omega_b, \\ \nabla \cdot v = 0 & \text{in } \Omega_b, \\ v = 0 & \text{on } \partial\Omega_b. \end{cases}$$

Then, $\Phi_{u_\infty}(\lambda)f$ and $P_{u_\infty}(\lambda)f$ satisfy the equation:

$$\begin{cases} (\lambda - \Delta + (u_\infty \cdot \nabla))\Phi_{u_\infty}(\lambda)f + \nabla P_{u_\infty}(\lambda)f = (I + \Psi_{u_\infty}(\lambda))f & \text{in } \Omega, \\ \nabla \cdot \Phi_{u_\infty}(\lambda)f = 0 & \text{in } \Omega, \\ \Phi_{u_\infty}(\lambda)f = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $\Phi_{u_\infty}(\lambda)f$ is uniformly n -regular in $C^\infty(\mathbb{R} \setminus \{0\}; \mathcal{L}(L_{p,b}(\Omega), W_p^2(\Omega_b)))$. For $I + \Psi_{u_\infty}(\lambda)$, we obtain the following lemma.

Lemma 6. Let $1 < p < \infty$ and $\lambda \in \Sigma_{u_\infty} \cup \{0\}$. Then, $I + \Psi_{u_\infty}(\lambda) : L_{p,b}(\Omega) \longrightarrow L_{p,b}(\Omega)$ has the bounded inverse $(I + \Psi_{u_\infty}(\lambda))^{-1}$. Moreover, $(I + \Psi_{u_\infty}(\lambda))^{-1}$ is uniformly n -regular in $\mathcal{L}(L_{p,b}(\Omega), L_{p,b}(\Omega))$.

Note that, the resolvent operator of the Oseen equation is represented by

$$(\lambda + \mathbb{O}_{u_\infty})^{-1}f = \Phi_{u_\infty}(\lambda)(I + \Psi_{u_\infty}(\lambda))^{-1}f \quad \text{for } \forall f \in J_{p,b}(\Omega).$$

Sketch of proof of Theorem 2. Let $X = \mathcal{L}(J_{p,b}(\Omega), W_p^2(\Omega_b))$. Using a cut off function $\varphi_R(s)$, we have

$$\begin{aligned} T_{u_\infty}(t) &= \int_{-\infty}^{\infty} e^{-its} \varphi_R(s) \Phi_{u_\infty}(is) (I + \Psi_{u_\infty}(is))^{-1} ds \\ &\quad + \int_{-\infty}^{\infty} e^{-its} (1 - \varphi_R(s)) (is + \mathbb{O}_{u_\infty})^{-1} ds \\ &= I_1(t) + I_2(t) \in X. \end{aligned}$$

In order to estimate $I_2(t)$, we use the following theorems about the resolvent.

Theorem 7. Let $1 < p < \infty$. Then, $\rho(\mathbb{O}_{u_\infty}) \supset -\Sigma_{u_\infty}$. Moreover, for any $\sigma_0 > 0$ and $\lambda_0 > 0$ there exists a $C_{p,\sigma_0,\lambda_0} > 0$ such that

$$\|(\lambda + \mathbb{O}_{u_\infty})^{-1} f\|_{W_p^2(\Omega)} + |\lambda| \|(\lambda + \mathbb{O}_{u_\infty})^{-1} f\|_{L_p(\Omega)} \leq C_{p,\sigma_0,\lambda_0} \|f\|_{L_p(\Omega)}, \quad \forall f \in \mathbb{J}_p(\Omega),$$

provided that $\operatorname{Re} \lambda \geq 0$, $|\lambda| \geq \lambda_0$ and $|u_\infty| \leq \sigma_0$.

Theorem 8. Let $1 < p < \infty$, $\sigma_0 > 0$ and $|u_\infty| \leq \sigma_0$. Then there exist $0 < \delta_0 < \frac{\pi}{2}$ and $R_0 = R_0(p, \sigma_0) > 0$ independent of u_∞ such that

$$|\lambda| \|(\lambda + \mathbb{O}_{u_\infty})^{-1} f\|_{L_p(\Omega)} + \|(\lambda + \mathbb{O}_{u_\infty})^{-1} f\|_{W_p^2(\Omega)} \leq C_p \|f\|_{L_p(\Omega)} \quad \forall f \in \mathbb{J}_p(\Omega)$$

provided that $|\lambda| \geq R_0$ and $|\arg \lambda| \leq \pi - \delta_0$.

By Theorems 7 and 8, we have

$$\|\partial_t^k I_2(t)\|_X \leq C_{k,l} t^{-l} \quad \forall k, \forall l \in \mathbb{N}.$$

Next, we estimate $I_1(t)$. Observe that

$$\partial_t^k I_1(t) = \int_{-\infty}^{\infty} (-is)^k e^{-its} \varphi_R(s) \Phi_{u_\infty}(is) (I + \Psi_{u_\infty}(is))^{-1} ds.$$

To estimate $I_1(t)$, we introduce the following space.

Definition 9 ([4]). Let X be a Banach space with norm $|\cdot|_X$. Let N be a positive integer and $\alpha = N + \sigma$ with $0 < \sigma \leq 1$. Put

$$C^\alpha(\mathbb{R}; X) = \{f \in C^{N-1}(\mathbb{R}; X) \cap C^\infty(\mathbb{R} \setminus \{0\}; X) \mid \langle f \rangle_{\alpha, X} < \infty\},$$

where

$$\begin{aligned} \langle f \rangle_{\alpha, X} &= \sum_{j=0}^N \int_{-\infty}^{\infty} \left| \left(\frac{d}{d\tau} \right)^j f(\tau) \right|_X d\tau + \sup_{h \neq 0} |h|^{-\sigma} \int_{-\infty}^{\infty} \left| \Delta_h \left(\frac{d}{d\tau} \right)^N f(\tau) \right|_X d\tau \\ &\quad \text{if } 0 < \sigma < 1, \\ \langle f \rangle_{\alpha, X} &= \sum_{j=0}^N \int_{-\infty}^{\infty} \left| \left(\frac{d}{d\tau} \right)^j f(\tau) \right|_X d\tau + \sup_{h \neq 0} |h|^{-1} \int_{-\infty}^{\infty} \left| \Delta_h^2 \left(\frac{d}{d\tau} \right)^N f(\tau) \right|_X d\tau \\ &\quad \text{if } \sigma = 1. \end{aligned}$$

Theorem 10. *If $f \in C^\alpha(\mathbb{R}; X)$ then*

$$\|\hat{f}(\tau)\|_X \leq C(1 + |\tau|)^{-\alpha} \|f\|_{\alpha, X},$$

where

$$\hat{f}(\tau) = \int_{-\infty}^{\infty} e^{-i\tau t} f(t) dt.$$

Since $(-is)^k \varphi_R(s) \Phi_{u_\infty}(is) (I + \Psi_{u_\infty}(is))^{-1} \in \mathcal{C}^{\frac{n+k}{2}}(\mathbb{R}, X)$, by Theorem 10 we have

$$\|\partial_t^k I_1(t)\|_X \leq C_{n,k} (1+t)^{-\frac{n+k}{2}} \quad \forall k \in \mathbb{N}.$$

This completes the proof of Theorem 2. □

REFERENCES

- [1] M. E. Bogovskii, *Solution of the first boundary value problem for the equation of an incompressible medium*, Sov. Math. Dokl., **20**, pp. 1094–1098 (1979).
- [2] ———, *Solution for some vector analysis problems connected with operators div and grad functional*, Trudy sem. S. L. Sobolev, No. 1, Novosibirsk: Acad. Nauk SSSR, Sibirsk. Otdel., Inst. Mat., pp. 5–40 (1980).
- [3] T. Miyakawa, *On nonstationary solutions of the Navier-Stokes equations in an exterior domain*, Hiroshima Math. J., **12**, pp. 115–140 (1982).
- [4] Y. Shibata, *On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain*, Tokuha J. Math., **7**, No. 1, pp. 1–68 (1983).