

A Consideration on Functions Preserving Set Inclusion Relation

富山県立大学・工学部 高木昇 (Noboru Takagi)
Department of Electronics and Informatics,
Toyama Prefectural University

Abstract— This paper discusses functions over the set of non-empty subsets of $\{0, 1, \dots, r-1\}$ that are monotonic in the set inclusion relation. Min, Max and Literal operations play an important role in multiple-valued logic design/circuits because they can realize any function over $\{0, 1, \dots, r-1\}$. Operations over the set of non-empty subsets of $\{0, 1, \dots, r-1\}$ that preserve the set inclusion relation are introduced from Min, Max and Literal operations over $\{0, 1, \dots, r-1\}$. Then, this paper proves some of mathematical properties of functions over the set of non-empty subsets of $\{0, 1, \dots, r-1\}$ that are composed of the operations introduced.

Keywords: Multiple-Valued Logic Design/Circuits, Set Inclusion Relation, Clone Theory

1 Introduction

S. C. Kleene [1] first introduced regularity into ternary operations over the set of truth values $\{0, 1, u\}$ in the following way.

A truth table for a ternary operation is *regular* if it satisfies the condition that “A given column (row) contains 1 in the u row (column), only if the column (row) consists entirely of 1’s; and likewise for 0”.

Kleene’s regularity is one of the ways how binary operations can be expanded into ternary operations. Table 1 is the truth tables of regular ternary operations, which are given from the traditional binary operations AND, OR and NOT.

It is worth to notice that M. Goto [2] independently introduced ternary operations that are identical with the Kleene’s ternary operations in Table 1. He showed that the ternary operations can be a model for analyzing undetermined behavior existing in binary systems, such as hazards in binary logic circuits. After Goto’s work, M. Mukaidono studied mathematical properties of functions over $\{0, 1, u\}$ that can be expressed by a formula composed of the three ternary operations (He called the ternary functions regular ternary logic functions). One of Mukaidono’s main results[3] is that a function f over $\{0, 1, u\}$ is a regular ternary logic function if and only if the function f is monotonic in the partial ordered relation, defined by Figure 1. I. G. Rosenberg [8] indicated that the set of regular ternary logic functions is this clone generated by the Kleene’s ternary logic, i.e., the clone is identical with the clone over the 3-element universe $\{\{0\}, \{1\}, \{0, 1\}\}$ that preserves the set inclusion relation $\subseteq$.

This paper discusses functions over the set of non-empty subsets of $\{0, 1, \dots, r-1\}$ when r is more than 2. In the following, E_r and P_r denote the r -valued set $\{0, 1, \dots, r-1\}$ and the set of non-empty subsets of E_r , respectively.

Table 1: Truth Tables of Regular Ternary Operations NOT, AND and OR

	NOT	AND			OR		
		0	1	u	0	1	u
0	1	0	0	0	0	1	u
1	0	0	1	u	1	1	1
u	u	0	u	u	u	1	u

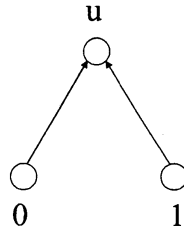


Figure 1: Partial Ordered Relation on $\{0, 1, u\}$

First, this paper shows a definition for expanding operations over E_r into operations over P_r . This definition is identical with the Kleene's regularity when r is equal to 2, and it has already been shown by M. Mukaidono [4] and I. G. Rosenberg [8]. Min, Max, and Literal operations play an important role in multiple-valued logic design/circuits, because they can realize any multiple-valued logic function over E_r . Therefore, Min, Max, and Literal operations are focused on in this paper. This paper then clarifies mathematical properties of functions over P_r , which are expressed by formulas composed of the operations given from Min, Max, and Literal operations over E_r .

This paper is organized below. Section 2 is for preliminaries. This section shows the definition for expanding operations over E_r into operations over P_r , and then gives some of their mathematical properties. Section 3 focuses on Min, Max, and Delta Literal operations over E_r . They are expanded into operations over P_r , and then this section proves a necessary and sufficient condition for a function over P_r to be expressed by a formula composed of these operations. Section 4 shows examples for the results obtained in Section 3. Section 5 discusses mathematical properties of functions over P_r when we selected Min, Max, and Universal Literal operations over E_r . Then, Section 6 gives examples for the results appeared in Section 5. Section 7 concludes the paper.

2 Preliminaries

Let E_r be the r -valued set $\{0, \dots, r-1\}$, and let P_r be the set of all non-empty subsets of E_r , i.e., $P_r = 2^{E_r} - \{\emptyset\}$, where 2^{E_r} is the power set of E_r . If a subset of E_r consists of only one element, then it is called a singleton. The set of all singletons of E_r is denoted by S_r , i.e., $S_r = \{\{0\}, \dots, \{r-1\}\}$. It is evident that the set P_r is a partial ordered set in the set inclusion $\subseteq$. In this paper, elements of the set E_r are denoted by small letters such as a, b, c, x, y , etc., while elements of the set P_r (i.e., non-empty subsets of E_r) are denoted by capital letters such as A, B, C, X, Y etc.

Definition 1 Let o be an n -ary operation on E_r . Then, an n -ary operation $\hat{o}$ on P_r with respect to o is defined by setting

$$\hat{o}(A_1, \dots, A_n) = \{o(a_1, \dots, a_n) \mid a_1 \in A_1, \dots, a_n \in A_n\}$$

for any element $(A_1, \dots, A_n) \in P_r^n$.

(End of Definition)

The following three operations play an important role in multiple-valued logic design because r -valued functions consisting of these operations and the constants $0, \dots, r-1$ are

Table 2: Truth Table of $\wedge$								Table 3: Truth Table of $\sqcup$							
$X \backslash Y$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>	$X \backslash Y$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>1</u>	<u>0</u>	<u>1</u>	<u>1</u>	<u>01</u>	<u>01</u>	<u>1</u>	<u>01</u>	<u>1</u>	<u>1</u>	<u>1</u>	<u>2</u>	<u>1</u>	<u>12</u>	<u>12</u>	<u>12</u>
<u>2</u>	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>
<u>01</u>	<u>0</u>	<u>01</u>	<u>01</u>	<u>01</u>	<u>01</u>	<u>01</u>	<u>01</u>	<u>01</u>	<u>01</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>012</u>	<u>12</u>	<u>012</u>
<u>02</u>	<u>0</u>	<u>01</u>	<u>02</u>	<u>01</u>	<u>02</u>	<u>012</u>	<u>012</u>	<u>02</u>	<u>02</u>	<u>12</u>	<u>2</u>	<u>012</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>12</u>	<u>0</u>	<u>1</u>	<u>12</u>	<u>01</u>	<u>012</u>	<u>12</u>	<u>012</u>	<u>12</u>	<u>12</u>	<u>2</u>	<u>12</u>	<u>12</u>	<u>12</u>	<u>12</u>	<u>12</u>
<u>012</u>	<u>0</u>	<u>01</u>	<u>012</u>	<u>01</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>12</u>	<u>2</u>	<u>012</u>	<u>012</u>	<u>12</u>	<u>012</u>

Table 4: Truth Table of X^S							
X	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
X^0	<u>2</u>	<u>0</u>	<u>0</u>	<u>02</u>	<u>02</u>	<u>0</u>	<u>02</u>
X^1	<u>0</u>	<u>2</u>	<u>0</u>	<u>02</u>	<u>0</u>	<u>02</u>	<u>02</u>
X^2	<u>0</u>	<u>0</u>	<u>2</u>	<u>0</u>	<u>02</u>	<u>02</u>	<u>02</u>
X^{01}	<u>2</u>	<u>2</u>	<u>0</u>	<u>2</u>	<u>02</u>	<u>02</u>	<u>02</u>
X^{02}	<u>2</u>	<u>0</u>	<u>2</u>	<u>02</u>	<u>2</u>	<u>02</u>	<u>02</u>
X^{12}	<u>0</u>	<u>2</u>	<u>2</u>	<u>02</u>	<u>02</u>	<u>2</u>	<u>02</u>
X^{012}	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>

functionally complete on the set E_r [5].

$$\begin{aligned}
 a \cdot b &= \min(a, b), \\
 a + b &= \max(a, b), \\
 x^S &= \begin{cases} r-1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

where $a, b \in E_r$ and $S \subseteq E_r$. The unary operations x^S are often called the universal literals. However, when S is a singleton, x^S is sometimes called a delta literal.

For simplicity, in writing elements of P_r , we will remove brackets and put an underline if no confusion arises. That is, for example, 0, 02 and 012 stand for $\{0\}$, $\{0, 2\}$ and $\{0, 1, 2\}$, respectively. Tables 2, 3 and 4 are truth tables of operations on P_3 with respect to $\cdot$, $+$ and x^S , respectively. Because this paper focuses on the operations on P_r with respect to $\cdot$, $+$ and x^S , they are denoted by $\wedge$, $\sqcup$ and X^S , respectively.¹

This paper does not allow any kinds of compositions of the operations $\wedge$, $\sqcup$ and X^S on P_r . Compositions are restricted by the form of the formulas defined below.

Definition 2 Formulas are defined inductively in the following way.

- (1) Constants $\{0\}, \dots, \{r-1\}$ and literals X_i^S ($i = 1, \dots, n$ and $S \in P_r$) are formulas.
- (2) If G and H are formulas, then $(G \wedge H)$ and $(G \sqcup H)$ are also formulas.
- (3) It is a formula if and only if we get it from (1) and (2) in a finite number of steps.

(End of Definition)

¹The operations $\wedge$ and $\sqcup$ do not satisfy the absorption laws and the distributive laws. Thus, the algebraic system $(P_r, \wedge, \sqcup)$ do not form a lattice.

In writing formulas, we sometimes omit the operation $\wedge$ for simplicity.

It is evident that every formula expresses a function on P_r when each variable X_i takes an element of P_r . Furthermore, it is easy to verify that the formulas can not express all of the functions on P_r , i.e., the functions on P_r expressed by the formulas are not functionally complete on P_r . Thus, one of the main subjects of the paper is to clear what functions on P_r can be expressed by the formulas.

In the following, for any elements $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ of P_r^n , $(A_1, \dots, A_n) \subseteq (B_1, \dots, B_n)$ stands for $A_i \subseteq B_i$ for all i 's. Moreover, $(A_1, \dots, A_n) \cap (B_1, \dots, B_n) = \emptyset$ stands for $A_i \cap B_i = \emptyset$ for some i .

Theorem 1 ² Suppose a function f on P_r can be expressed by a formula. Then, $f(A_1, \dots, A_n) \in S_r$ holds for any element $(A_1, \dots, A_n) \in S_r^n$.

Theorem 2 Suppose a function f on P_r can be expressed by a formula. Then, $f(A_1, \dots, A_n) \subseteq f(B_1, \dots, B_n)$ holds for any elements $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ of P_r^n such that $(A_1, \dots, A_n) \subseteq (B_1, \dots, B_n)$.

3 Functions Expressed by Formulas Composed of $\wedge$, $\sqcup$, and Delta Literals

This section shows a necessary and sufficient condition for functions on P_r that can be expressed by formulas with the operations $\wedge$, $\sqcup$ and delta literals.

Theorem 3 Let $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$ be elements of P_r . If a function f on P_r is expressed by a formula, then the least element of $f(A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_n)$ (which is a subset of E_r) is equal to the least element of $f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n)$ for any elements A and B of $P_r - S_r$, i.e.,

$$\min f(a_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_n) = \min f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n)$$

holds for any elements A and B of $P_r - S_r$.

From Theorems 1, 2 and 3, any function f on P_r expressed by a formula satisfies the following Condition A.

Condition A: Let f be a function on P_r .

- (1) If $(A_1, \dots, A_n) \in S_r^n$, then $f(A_1, \dots, A_n) \in S_r$.
- (2) For any elements $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ of P_r^n , $(A_1, \dots, A_n) \subseteq (B_1, \dots, B_n)$ implies $f(A_1, \dots, A_n) \subseteq f(B_1, \dots, B_n)$.
- (3) Let $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$ be elements of P_r . Then, the least element of $f(A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_n)$ is equal to the least element of $f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n)$ for any elements A and B of $P_r - S_r$, i.e.,

$$\min f(A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_n) = \min f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n)$$

holds for any elements A and B of $P_r - S_r$.

² All of the proofs in this paper are omitted because of the limitation of the space.

In the remainder of this section, it is proven that Condition A is a necessary and sufficient condition for a function on P_r to be expressed by a formula with the operations $\wedge$, $\sqcup$, and delta literals.

Definition 3 Let f be a function on P_r , and let $A = (A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n)$ be an element of S_r^{n-1} . Then, we define one-variable functions $\check{f}_A^i(X)$ and $\hat{f}_A^i(X)$ ($i = 1, \dots, n$) expressed by the following formulas.

$$\check{f}_A^i(X) = \bigsqcup_{s \in E_r} \left\{ \{s\} \wedge \bigsqcup_{B \in P_A^i(s)} X^B \right\}, \quad (1)$$

where $\check{P}_A^i(s)$ is the set of all maximal elements of the set

$$P_A^i(s) = \{B \in P_r \mid \min f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n) = s\}, \quad (2)$$

and

$$\hat{f}_A^i(X) = \bigsqcup_{S \in P_r - S_r} \left(\bigsqcup_{t \in S} \left[\{t\} \wedge \left\{ \bigsqcup_{B \in \hat{Q}_A^i(S)} \left(\bigwedge_{e \in B} X^{\{e\}} \right) \right\} \right] \right), \quad (3)$$

where $\hat{Q}_A^i(S)$ is the set of all minimal elements of

$$Q_A^i(S) = \{B \in P_r - S_r \mid f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n) = S\}. \quad (4)$$

In the formulas (1) and (3), if $\check{P}_A^i(s)$ and $\hat{Q}_A^i(S)$ are the empty set, then

$$\bigsqcup_{B \in \check{P}_A^i(s)} X^B \text{ and } \bigsqcup_{B \in \hat{Q}_A^i(S)} \left(\bigwedge_{k \in B} X^{\{k\}} \right)$$

are defined as $\{0\}$, respectively. Moreover, in the formula (1), when $B = E_r$, then X^B is the constant $\{r-1\}$. (End of Definition)

In the formula (1), if f is a function satisfying Condition A, then any subset $\check{P}_A^i(s)$ is a subset of S_r , or it is equal to $\{E_r\}$. Now, let us show this property. Suppose an element B of $P_r - S_r$ is a member of $P_A^i(s)$. Then, by Condition A(3), it follows that E_r is also a member of $P_A^i(s)$. Therefore, when an element of $P_r - S_r$ is a member of $P_A^i(s)$, then E_r is also a member of $P_A^i(s)$. This fact implies that $\check{P}_A^i(s)$ is a subset of S_r , or it is equal to $\{E_r\}$. So, the formula (1) is well-defined when f is a function satisfying Condition A.

Lemma 1 Let $A = (A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n)$ be an element of S_r^{n-1} . Then, for a function f satisfying Condition A,

$$\check{f}_A^i(B) = \begin{cases} f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n) & \text{if } f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n) \in S_r \\ K & \text{otherwise} \end{cases}$$

holds for any element B of P_r , where K is an element of P_r such that

$$\{f_0\} \subseteq K \subseteq f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n)$$

and f_0 is the least element of $f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n)$. (End of Lemma)

Lemma 2 Let $A = (A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n)$ be an element of S_r^{n-1} . Then, for a function f satisfying Condition A,

$$\hat{f}_A^i(B) = \begin{cases} \{0\} & \text{if } f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n) \in S_r \\ \{0\} \cup f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n) & \text{otherwise} \end{cases}$$

holds for any element $B \in P_r$.

(End of Lemma)

Lemma 3 Let $A = (A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n)$ be an element of S_r^{n-1} . Then, for a function f satisfying Condition A,

$$\check{f}_A^i(B) \sqcup \hat{f}_A^i(B) = f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n)$$

holds for any element $B \in P_r$.

In the following, this section proves that any function satisfying Condition A can be expressed by a formula, and also shows a method how a formula can be formulated by a function satisfying Condition A.

Definition 4 Let f be a function on P_r . Then, f_1 is defined as a function on P_r expressed by the following formula.

$$f_1(X_1, \dots, X_n) = \bigsqcup_{i=1}^n f^i(X_1, \dots, X_n), \quad (5)$$

where

$$f^i(X_1, \dots, X_n) = \bigsqcup_{A=(A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n) \in S_r^{n-1}} \left(\bigwedge_{j=1(j \neq i)}^n X_j^{A_j} \wedge (\check{f}_A^i(X_i) \sqcup \hat{f}_A^i(X_i)) \right).$$

(End of Definition)

Here, let us introduce a subset of P_r^n , which will be denoted by $I(r, n)$, below.

$$I(r, n) = \bigcup_{i=1}^n \{(A_1, \dots, A_n) \in P_r^n \mid A_i \in P_r - S_r \text{ and } A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n \in S_r\}$$

That is, each element $(A_1, \dots, A_n)$ of $I(r, n)$ consists of elements of S_r , but except for one.

Lemma 4 Let F be a function satisfying Condition A. Then,

$$f_1(A_1, \dots, A_n) = \begin{cases} f(A_1, \dots, A_n) & \text{if } f(A_1, \dots, A_n) \in S_r \cup I(r, n) \\ K & \text{otherwise} \end{cases}$$

holds for any element $(A_1, \dots, A_n) \in P_r^n$, where K is an element of P_r such that $\{0\} \subseteq K \subseteq \{0\} \cup f(A_1, \dots, A_n)$.

(End of Lemma)

Definition 5 Let f be a function on P_r , let S be an element of $P_r - S_r$, and let $\hat{T}(f, S)$ is the set of all minimal elements of the following subset of P_r^n .

$$T(f, S) = \{(A_1, \dots, A_n) \in P_r^n \mid f(A_1, \dots, A_n) = S \text{ and } (A_1, \dots, A_n) \notin S_r^n \cup I(r, n)\} \quad (6)$$

Then, f_2 is defined as a function on P_r expressed by the following formula.

$$f_2(X_1, \dots, X_n) = \{s_0\} \sqcup \left[\bigsqcup_{S \in P_r - S_r} \left\{ \bigsqcup_{t \in S} \{t\} \wedge f_S(X_1, \dots, X_n) \right\} \right], \quad (7)$$

where

$$f_S(X_1, \dots, X_n) = \begin{cases} \bigsqcup_{(A_1, \dots, A_n) \in \hat{T}(f, S)} \left\{ \bigwedge_{b \in A_1} X_1^{\{b\}} \wedge \dots \wedge \bigwedge_{b \in A_n} X_n^{\{b\}} \right\} & \text{if } \hat{T}(f, S) \neq \emptyset \\ \{0\} & \text{otherwise} \end{cases} \quad (8)$$

and s_0 is the least element of $\bigcup_{(A_1, \dots, A_n) \in P_r^n} f(A_1, \dots, A_n)$. (End of Definition)

Lemma 5 Let f be a function on P_r satisfying Condition A. Then,

$$f_2(A_1, \dots, A_n) = \begin{cases} \{s_0\} & \text{if } (A_1, \dots, A_n) \in S_r^n \cup I(r, n) \\ f(A_1, \dots, A_n) & \text{otherwise,} \end{cases}$$

holds for any element $(A_1, \dots, A_n) \in P_r^n$, where s_0 is the least element of the union

$$\bigcup_{(A_1, \dots, A_n) \in P_r^n} f(A_1, \dots, A_n). \quad (\text{End of Lemma})$$

Theorem 4 Let f be a function on P_r satisfying Condition A. Then,

$$f(A_1, \dots, A_n) = f_1(A_1, \dots, A_n) \sqcup f_2(A_1, \dots, A_n)$$

holds for any element $(A_1, \dots, A_n) \in P_r^n$, where f_1 and f_2 are the formulas (5) and (7), respectively. (End of Theorem)

4 Examples of Functions Satisfying Condition A

Consider the function f on P_3 whose truth table is given in Table 5. It is not difficult to verify that f satisfies Condition A. Then, this section illustrates how we can form the formula that expresses the function f .

Example 1 Let us first consider the formulas (1) and (3). It follows by Eq. (2) that we have the following three subsets of P_3 .

$$\begin{aligned} P_0^1(0) &= \{B \in P_3 \mid \min f(B, \underline{0}) = 0\} = \{\underline{0}, \underline{2}, \underline{01}, \underline{02}, \underline{12}, \underline{012}\} \\ P_0^1(1) &= \{B \in P_3 \mid \min f(B, \underline{0}) = 1\} = \{\underline{1}\} \\ P_0^1(2) &= \{B \in P_3 \mid \min f(B, \underline{0}) = 2\} = \emptyset \end{aligned}$$

Table 5: Example of Function f Satisfying Condition A

$X \backslash Y$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>02</u>	<u>02</u>
<u>1</u>	<u>1</u>	<u>1</u>	<u>2</u>	<u>1</u>	<u>12</u>	<u>12</u>	<u>12</u>
<u>2</u>	<u>0</u>	<u>2</u>	<u>0</u>	<u>02</u>	<u>0</u>	<u>02</u>	<u>02</u>
<u>01</u>	<u>01</u>	<u>01</u>	<u>02</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>02</u>	<u>0</u>	<u>02</u>	<u>0</u>	<u>02</u>	<u>0</u>	<u>02</u>	<u>02</u>
<u>12</u>	<u>012</u>	<u>012</u>	<u>02</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>012</u>	<u>012</u>	<u>012</u>	<u>02</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>

Thus, since $\check{P}_0^1(0) = \{\underline{012}\}$, $\check{P}_0^1(1) = \{\underline{1}\}$, and $\check{P}_0^1(2) = \emptyset$, we have the formula $\check{f}_0^1(X)$ by Eq. (1).

$$\begin{aligned}\check{f}_0^1(X) &= (\underline{0} \wedge \underline{012}) \sqcup (\underline{1} \wedge X^1) \sqcup (\underline{2} \wedge \underline{0}) \\ &= \underline{1} \wedge X^1\end{aligned}\tag{9}$$

In a similar way, we have the formulas $\check{f}_1^1(X)$, $\check{f}_2^1(X)$, $\check{f}_0^2(Y)$, $\check{f}_1^2(Y)$, and $\check{f}_2^2(Y)$, below.

$$\left. \begin{aligned}\check{f}_1^1(X) &= \underline{1}X^1 \sqcup X^2 \\ \check{f}_2^1(X) &= X^1 \\ \check{f}_0^2(Y) &= \underline{0} \\ \check{f}_1^2(Y) &= \underline{1} \sqcup Y^2 \\ \check{f}_2^2(Y) &= Y^1\end{aligned}\right\}\tag{10}$$

Moreover, it follows by Eq. (4) that we have

$$\begin{aligned}Q_0^1(\underline{01}) &= \{B \in P_3 - S_3 \mid f(B, \underline{0}) = \underline{01}\} = \{\underline{01}\}, \\ Q_0^1(\underline{02}) &= \{B \in P_3 - S_3 \mid f(B, \underline{0}) = \underline{02}\} = \emptyset, \\ Q_0^1(\underline{12}) &= \{B \in P_3 - S_3 \mid f(B, \underline{0}) = \underline{12}\} = \emptyset, \text{ and} \\ Q_0^1(\underline{012}) &= \{B \in P_1 - S_3 \mid f(B, \underline{0}) = \underline{012}\} = \{\underline{12}, \underline{012}\}.\end{aligned}$$

Thus, since $\hat{Q}_0^1(\underline{01}) = \{\underline{01}\}$, $\hat{Q}_0^1(\underline{02}) = \hat{Q}_0^1(\underline{12}) = \emptyset$ and $\hat{Q}_0^1(\underline{012}) = \{\underline{12}\}$, we have the formula $\hat{f}_0^1(X)$ by Eq. (3).

$$\begin{aligned}\hat{f}_0^1(X) &= (\underline{0}X^0X^1 \sqcup \underline{1}X^0X^1) \sqcup \underline{0} \sqcup \underline{0} \sqcup (\underline{0}X^2X^2 \sqcup \underline{1}X^2X^2 \sqcup \underline{2}X^2X^2) \\ &= \underline{1}X^0X^1 \sqcup \underline{1}X^1X^2 \sqcup X^1X^2.\end{aligned}\tag{11}$$

In a similar way, we have the formulas $\hat{f}_1^1(X)$, $\hat{f}_2^1(X)$, $\hat{f}_0^2(Y)$, $\hat{f}_1^2(Y)$, and $\hat{f}_2^2(Y)$, below.

$$\left. \begin{aligned}\hat{f}_1^1(X) &= \underline{1}X^0X^1 \sqcup X^0Y^2 \sqcup \underline{1}X^1X^2 \sqcup X^1X^2, \\ \hat{f}_2^1(X) &= X^0X^1 \sqcup X^1X^2, \\ \hat{f}_0^2(Y) &= Y^1Y^2, \\ \hat{f}_1^2(Y) &= \underline{1}Y^0Y^2 \sqcup X^0Y^2 \sqcup \underline{1}Y^1Y^2 \sqcup Y^1Y^2, \\ \hat{f}_2^2(Y) &= Y^0Y^1 \sqcup Y^1Y^2.\end{aligned}\right\}\tag{12}$$

Table 6: Truth Tables of $\tilde{f}_A^1(X)$ and $\hat{f}_A^1(X)$

X	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
$\tilde{f}_0^1(X)$	<u>0</u>	<u>1</u>	<u>0</u>	<u>01</u>	<u>0</u>	<u>01</u>	<u>01</u>
$\tilde{f}_1^1(X)$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>012</u>	<u>012</u>
$\tilde{f}_2^1(X)$	<u>0</u>	<u>2</u>	<u>0</u>	<u>02</u>	<u>0</u>	<u>02</u>	<u>02</u>
$\hat{f}_0^1(X)$	<u>0</u>	<u>0</u>	<u>0</u>	<u>01</u>	<u>0</u>	<u>012</u>	<u>012</u>
$\hat{f}_1^1(X)$	<u>0</u>	<u>0</u>	<u>0</u>	<u>01</u>	<u>02</u>	<u>012</u>	<u>012</u>
$\hat{f}_2^1(X)$	<u>0</u>	<u>0</u>	<u>0</u>	<u>02</u>	<u>0</u>	<u>02</u>	<u>02</u>

Table 7: Truth Tables of $\tilde{f}_A^2(Y)$ and $\hat{f}_A^2(Y)$

Y	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
$\tilde{f}_0^2(Y)$	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
$\tilde{f}_1^2(Y)$	<u>1</u>	<u>1</u>	<u>2</u>	<u>1</u>	<u>12</u>	<u>12</u>	<u>12</u>
$\tilde{f}_2^2(Y)$	<u>0</u>	<u>2</u>	<u>0</u>	<u>02</u>	<u>0</u>	<u>02</u>	<u>02</u>
$\hat{f}_0^2(Y)$	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>02</u>	<u>02</u>
$\hat{f}_1^2(Y)$	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>012</u>	<u>012</u>	<u>012</u>
$\hat{f}_2^2(Y)$	<u>0</u>	<u>0</u>	<u>0</u>	<u>02</u>	<u>0</u>	<u>02</u>	<u>02</u>

Table 8: Truth Table of $f_A^i = \tilde{f}_A^i \sqcup \hat{f}_A^i$

X or Y	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
$\tilde{f}_0^1(X)$	<u>0</u>	<u>1</u>	<u>0</u>	<u>01</u>	<u>0</u>	<u>012</u>	<u>012</u>
$\tilde{f}_1^1(X)$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>012</u>	<u>012</u>
$\tilde{f}_2^1(X)$	<u>0</u>	<u>2</u>	<u>0</u>	<u>02</u>	<u>0</u>	<u>02</u>	<u>02</u>
$\tilde{f}_0^2(Y)$	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>02</u>	<u>02</u>
$\tilde{f}_1^2(Y)$	<u>1</u>	<u>1</u>	<u>2</u>	<u>1</u>	<u>12</u>	<u>12</u>	<u>12</u>
$\tilde{f}_2^2(Y)$	<u>0</u>	<u>2</u>	<u>0</u>	<u>02</u>	<u>0</u>	<u>02</u>	<u>02</u>

Tables 6 and 7 show the truth tables of $\tilde{f}_A^i$ and $\hat{f}_A^i$ for which $i = 1, 2$ and $A \in \{\underline{0}, \underline{1}, \underline{2}\}$.

(End of Example)

It follows by Lemma 3 that

$$f(X, B) = \tilde{f}_A^1(X) \sqcup \hat{f}_A^1(X) \text{ and } f(B, Y) = \tilde{f}_A^2(Y) \sqcup \hat{f}_A^2(Y)$$

hold for every $A \in \{\underline{0}, \underline{1}, \underline{2}\}$ and every $B \in P_3$, where $\tilde{f}_A^1(X)$, $\hat{f}_A^1(X)$, $\tilde{f}_A^2(Y)$ and $\hat{f}_A^2(Y)$ have been obtained in Eqs. (9), (10), (11) and (12). Table 8 shows the truth tables of $\tilde{f}_A^i \sqcup \hat{f}_A^i$, where $i = 1, 2$ and $A \in \{\underline{0}, \underline{1}, \underline{2}\}$.

Example 2 Let us next consider the formula (5) in Definition 4. It follows by Eqs. (9) and (11) that we have the formula $\tilde{f}_0^1(X) \sqcup \hat{f}_0^1(X)$ below.

$$\begin{aligned} \tilde{f}_0^1(X) \sqcup \hat{f}_0^1(X) &= \underline{1}X^1 \sqcup \underline{1}X^0X^1 \sqcup \underline{1}X^1X^2 \sqcup X^1X^2 \\ &= \underline{1}X^1 \sqcup X^1X^2 \end{aligned}$$

In a similar way, by Eqs. (10), (11) and (12), we have the formulas

$$\begin{aligned} \tilde{f}_1^1(X) \sqcup \hat{f}_1^1(X) &= \underline{1}X^1 \sqcup X^2, \\ \tilde{f}_2^1(X) \sqcup \hat{f}_2^1(X) &= X^1, \\ \tilde{f}_0^2(Y) \sqcup \hat{f}_0^2(Y) &= Y^1Y^2, \\ \tilde{f}_1^2(Y) \sqcup \hat{f}_1^2(Y) &= \underline{1} \sqcup Y^2, \\ \tilde{f}_2^2(Y) \sqcup \hat{f}_2^2(Y) &= Y^1. \end{aligned}$$

Therefore, the formula $f_1(X, Y)$ of (5) in Definition 4 is given as

$$f_1(X, Y) = f^1(X, Y) \sqcup f^2(X, Y), \quad (13)$$

Table 9: Truth Table of $f_1(X, Y)$

$X \backslash Y$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>02</u>	<u>02</u>
<u>1</u>	<u>1</u>	<u>1</u>	<u>2</u>	<u>1</u>	<u>12</u>	<u>12</u>	<u>12</u>
<u>2</u>	<u>0</u>	<u>2</u>	<u>0</u>	<u>02</u>	<u>0</u>	<u>02</u>	<u>02</u>
<u>01</u>	<u>01</u>	<u>01</u>	<u>02</u>	<u>01</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>02</u>	<u>0</u>	<u>02</u>	<u>0</u>	<u>02</u>	<u>0</u>	<u>02</u>	<u>02</u>
<u>12</u>	<u>012</u>	<u>012</u>	<u>02</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>012</u>	<u>012</u>	<u>012</u>	<u>02</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>

Table 10: Truth Table of $f_2(X, Y)$

$X \backslash Y$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
<u>1</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
<u>2</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
<u>01</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>02</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>02</u>	<u>0</u>	<u>02</u>	<u>02</u>
<u>12</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>012</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>

where

$$f^1(X, Y) = Y^0 \left(\hat{f}_0^1(X) \sqcup \hat{f}_0^1(X) \right) \sqcup Y^1 \left(\hat{f}_1^1(X) \sqcup \hat{f}_1^1(X) \right) \sqcup Y^2 \left(\hat{f}_2^1(X) \sqcup \hat{f}_2^1(X) \right) \text{ and}$$

$$f^2(X, Y) = X^0 \left(\hat{f}_0^2(Y) \sqcup \hat{f}_0^2(Y) \right) \sqcup X^1 \left(\hat{f}_1^2(Y) \sqcup \hat{f}_1^2(Y) \right) \sqcup X^2 \left(\hat{f}_2^2(Y) \sqcup \hat{f}_2^2(Y) \right).$$

Table 9 is the truth table of $f_1(X, Y)$.

(End of Example)

Example 3 In this example, let us consider the formula (7) in Definition 5. It follows by Eq. (6) that we have the following subsets of P_3^2 .

$$\begin{aligned} T(f, \underline{01}) &= \emptyset \\ T(f, \underline{02}) &= \{(\underline{02}, \underline{01}), (\underline{02}, \underline{12}), (\underline{02}, \underline{012})\} \\ T(f, \underline{12}) &= \emptyset \\ T(f, \underline{012}) &= \{(A, B) \mid A \in \{\underline{01}, \underline{12}, \underline{012}\} \text{ and } B \in P_3 - S_3\} \end{aligned}$$

Therefore, since we have

$$\begin{aligned} \hat{T}(f, \underline{01}) &= \emptyset, \\ \hat{T}(f, \underline{02}) &= \{(\underline{02}, \underline{01}), (\underline{02}, \underline{12})\}, \\ \hat{T}(f, \underline{12}) &= \emptyset, \text{ and} \\ \hat{T}(f, \underline{012}) &= \{(\underline{01}, \underline{01}), (\underline{01}, \underline{02}), (\underline{01}, \underline{12}), (\underline{12}, \underline{01}), (\underline{12}, \underline{02}), (\underline{12}, \underline{12})\}, \end{aligned}$$

it follows by Eq. (8) that we have the following formulas.

$$\begin{aligned} f_{\underline{01}}(X, Y) &= \underline{0} \\ f_{\underline{02}}(X, Y) &= X^0 X^2 Y^0 Y^1 \sqcup X^0 X^2 Y^1 Y^2 \\ f_{\underline{12}}(X, Y) &= \underline{0} \\ f_{\underline{012}}(X, Y) &= X^0 X^1 Y^0 Y^1 \sqcup X^0 X^1 Y^0 Y^2 \sqcup X^0 X^1 Y^1 Y^2 \sqcup \\ &\quad X^1 X^2 Y^0 Y^1 \sqcup X^1 X^2 Y^0 Y^2 \sqcup X^1 X^2 Y^1 Y^2 \end{aligned}$$

Thus, the formula $f_2(X, Y)$ of (7) in Definition 5 is obtained as the formula below.

$$\begin{aligned} f_2(X, Y) &= \underline{0} \sqcup \{0f_{\underline{02}}(X, Y) \sqcup 2f_{\underline{02}}(X, Y)\} \sqcup \{0f_{\underline{012}}(X, Y) \sqcup 1f_{\underline{012}}(X, Y) \sqcup 2f_{\underline{012}}(X, Y)\} \\ &= f_{\underline{02}}(X, Y) \sqcup 1f_{\underline{012}}(X, Y) \sqcup f_{\underline{012}}(X, Y) \end{aligned} \quad (14)$$

Table 10 is the truth table of $f_2(X, Y)$.

(End of Example)

It follows by Theorem 4 that the function f of Table 5 can be expressed by the formula $f_1(X, Y) \sqcup f_2(X, Y)$, where $f_1(X, Y)$ and $f_2(X, Y)$ are the formulas given in (13) and (14), respectively.

5 Functions Expressed by Formulas Composed of $\wedge$, $\sqcup$ and Universal Literals

This section discusses functions on P_r expressed by formulas, which are composed of the operations $\wedge$, $\sqcup$ and universal literals. Then, a necessary and sufficient condition for a function on P_r to be expressed by a formula when r is equal to 3.

Theorem 5 Let f be a function on P_r . If f can be expressed by a formula, then

$$\bigcap_{A \in P_r - S_r} f(A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_n) \neq \emptyset$$

holds for any elements $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$ of P_r .

(End of Theorem)

By Theorems 1, 2 and 5, any function f on P_r expressed by a formula satisfies the following Condition B.

Condition B: Let f be a function on P_r .

- (1) If $(A_1, \dots, A_n) \in S_r^n$, then $f(A_1, \dots, A_n) \in S_r$.
- (2) For any elements $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ of P_r^n , $(A_1, \dots, A_n) \subseteq (B_1, \dots, B_n)$ implies $f(A_1, \dots, A_n) \subseteq f(B_1, \dots, B_n)$.
- (3) $\bigcap_{A \in P_r - S_r} f(A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_n) \neq \emptyset$ holds for any elements $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$ of P_r .

In the following, this section proves that Condition B is a necessary and sufficient condition for a function on P_3 to be expressed by a formula with the operations $\wedge$, $\sqcup$, and universal literals.

Definition 6 Let $(A_1, \dots, A_n)$ be any element of P_r^n . Then, $\alpha = X_1^{A_1} \wedge \dots \wedge X_n^{A_n}$ is said to be the type-1 term corresponding to $(A_1, \dots, A_n)$. Next, let $(B_1, \dots, B_n)$ be any element of $P_r^n - S_r^n$. Then, $\beta = \bigwedge_{e \in B_1} X_1^{\{e\}} \wedge \dots \wedge \bigwedge_{e \in B_n} X_n^{\{e\}}$ is said to be the type-2 term corresponding to $(B_1, \dots, B_n)$.

(End of Definition)

Let S be an element of P_r , and let T be an element of $P_r - S_r$. Then, it is easy to verify that the following two equations are valid.

$$X^S = \begin{cases} \{r-1\} & \text{if } X \subseteq S \\ \{0\} & \text{if } X \cap S = \emptyset \\ \{0, r-1\} & \text{otherwise} \end{cases} \quad (15)$$

$$\bigwedge_{e \in T} X^{\{e\}} = \begin{cases} \{0, r-1\} & \text{if } T \subseteq X \\ \{0\} & \text{otherwise} \end{cases} \quad (16)$$

Therefore, for any type-1 term α and any type-2 term β , $\alpha(A_1, \dots, A_n) = \{r-1\}$, $\{0, r-1\}$, or $\{0\}$, and $\beta(A_1, \dots, A_n) = \{0, r-1\}$ or $\{0\}$ hold for any element $(A_1, \dots, A_n) \in P_r^n$.

Lemma 6 For any type-1 term α corresponding to $(A_1, \dots, A_n) \in P_r^n$,

- (1) $(B_1, \dots, B_n) \subseteq (A_1, \dots, A_n)$ iff $\alpha(B_1, \dots, B_n) = \{r-1\}$,
- (2) $(A_1, \dots, A_n) \cap (B_1, \dots, B_n) = \emptyset$ iff $\alpha(B_1, \dots, B_n) = \{0\}$,
- (3) $(B_1, \dots, B_n) \not\subseteq (A_1, \dots, A_n)$ and $(A_1, \dots, A_n) \cap (B_1, \dots, B_n) = \emptyset$ iff $\alpha(B_1, \dots, B_n) = \{0, r-1\}$

hold for any $(B_1, \dots, B_n) \in P_r^n$. (End of Lemma)

Lemma 7 For any type-2 term α corresponding to $(A_1, \dots, A_n) \in P_r^n - S_r^n$,

- (1) $(A_1, \dots, A_n) \subseteq (B_1, \dots, B_n)$ iff $\alpha(B_1, \dots, B_n) = \{0, r-1\}$,
- (2) $(A_1, \dots, A_n) \not\subseteq (B_1, \dots, B_n)$ iff $\alpha(B_1, \dots, B_n) = \{0\}$

hold for any $(B_1, \dots, B_n) \in P_r^n$. (End of Lemma)

Let f be a function satisfying Condition B, and let S be an element of P_r . Then, define two subsets of P_r^n , denoted by $L(f, S)$ and $U(f, S)$, below.

$$\begin{aligned} L(f, S) &= \{(A_1, \dots, A_n) \in P_r^n \mid f(A_1, \dots, A_n) \subseteq S\} \text{ and} \\ U(f, S) &= \{(A_1, \dots, A_n) \in P_r^n \mid f(A_1, \dots, A_n) \cap S \neq \emptyset\}. \end{aligned}$$

Let $\tilde{L}(f, S)$ and $U'(f, S)$ be the sets of all maximal elements of $L(f, S)$ and of all minimal elements of $U(f, S)$, respectively. Further, let $\hat{U}(f, S) = U'(f, S) - S_r^n$.

Lemma 8 Let f be a function satisfying Condition B, and let S be an element of P_r . Then, $(f)^S$ can be expressed by the following formula.

$$(f)^S = \begin{cases} \bigsqcup_{A \in \tilde{L}(f, S)} \alpha_A \sqcup \bigsqcup_{A \in \hat{U}(f, S)} \beta_A & \text{if } \tilde{L}(f, S) \neq \emptyset \text{ or } \hat{U}(f, S) \neq \emptyset \\ \{0\} & \text{otherwise} \end{cases} \quad (17)$$

where α_A and β_A are the type-1 and type-2 terms corresponding to A , respectively.

(End of Lemma)

Now, let us consider formulas of one-variable functions satisfying Condition B. Any one-variable function f satisfying Condition B is in at least one of the following three cases³.

(B-1) $f(A) \neq \underline{01}$ holds for any element $A \in P_3 - S_3$.

(B-2) $f(A) \neq \underline{02}$ holds for any element $A \in P_3 - S_3$.

(B-3) $f(A) \neq \underline{12}$ holds for any element $A \in P_3 - S_3$.

³If f is in neither one of the cases (B-1), (B-2), (B-3), then it implies that we have three distinct elements A, B, C in $P_3 - S_3$ such that $f(A) = \underline{01}$, $f(B) = \underline{02}$, and $f(C) = \underline{12}$. However, this contradicts to the fact that f satisfies Condition B(3).

Table 11: Example of (B-4)

$x \backslash y$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>0</u>	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>012</u>	<u>012</u>
<u>1</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>
<u>2</u>	<u>0</u>	<u>2</u>	<u>1</u>	<u>02</u>	<u>012</u>	<u>12</u>	<u>012</u>
<u>01</u>	<u>02</u>	<u>12</u>	<u>2</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>02</u>	<u>0</u>	<u>12</u>	<u>12</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>12</u>	<u>02</u>	<u>2</u>	<u>12</u>	<u>02</u>	<u>012</u>	<u>12</u>	<u>012</u>
<u>012</u>	<u>02</u>	<u>12</u>	<u>12</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>

Table 12: Example of (B-5)

$x \backslash y$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
<u>1</u>	<u>0</u>	<u>1</u>	<u>0</u>	<u>01</u>	<u>0</u>	<u>01</u>	<u>01</u>
<u>2</u>	<u>0</u>	<u>2</u>	<u>1</u>	<u>02</u>	<u>012</u>	<u>12</u>	<u>012</u>
<u>01</u>	<u>0</u>	<u>01</u>	<u>0</u>	<u>01</u>	<u>0</u>	<u>01</u>	<u>01</u>
<u>02</u>	<u>0</u>	<u>02</u>	<u>01</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>12</u>	<u>0</u>	<u>012</u>	<u>01</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>012</u>	<u>0</u>	<u>012</u>	<u>01</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>

Table 13: Example of (B-6)

$x \backslash y$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
<u>1</u>	<u>0</u>	<u>1</u>	<u>1</u>	<u>01</u>	<u>01</u>	<u>1</u>	<u>01</u>
<u>2</u>	<u>0</u>	<u>2</u>	<u>2</u>	<u>02</u>	<u>02</u>	<u>2</u>	<u>02</u>
<u>01</u>	<u>0</u>	<u>01</u>	<u>012</u>	<u>01</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>02</u>	<u>0</u>	<u>012</u>	<u>02</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>12</u>	<u>0</u>	<u>12</u>	<u>12</u>	<u>012</u>	<u>012</u>	<u>12</u>	<u>012</u>
<u>012</u>	<u>0</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>

Property 1 Any one-variable function f satisfying Condition B can be expressed by the following formula.

$$f(X) = \begin{cases} f^{12}(X) \wedge (\underline{1} \sqcup f^{02}(X)) & \text{if } f \text{ is in the case (B-1)} \\ (\underline{1} \wedge f^1(X)) \sqcup f^2(X) \sqcup (\underline{1} \wedge f^{12}(X)) & \text{if } f \text{ is in the case (B-2)} \\ (\underline{1} \wedge f^1(X)) \sqcup f^2(X) & \text{if } f \text{ is in the case (B-3)} \end{cases} \quad (18)$$

(End of Property)

By Property 1, every one-variable function satisfying Condition B can be expressed by a formula.

Next, let us consider the case where functions satisfying Condition B depend more than one variable. Then, any function f satisfying Condition B is in at least one of the three cases below.

(B-4) $f(A_1, \dots, A_n) \neq \underline{01}$ holds for any element $(A_1, \dots, A_n) \in (P_3 - S_3)^n$.

(B-5) $f(A_1, \dots, A_n) \neq \underline{12}$ holds for any element $(A_1, \dots, A_n) \in (P_3 - S_3)^n$.

(B-6) $\bigcap_{(A_1, \dots, A_n) \in (P_3 - S_3)^n} f(A_1, \dots, A_n) = \underline{1}$.

Tables 11, 12 and 13 are examples of two-variable functions being in the cases (B-4), (B-5), and (B-6), respectively.

Then, we can prove Properties 1 ~ 6, which show a way for constructing formulas of n -variable functions satisfying Condition B.

Let A be an element $(A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n)$ of P_r^{n-1} . Then, denote the one-variable function $f(A_1, \dots, A_{i-1}, X, A_{i+1}, \dots, A_n)$ by $f_A^i(X)$.

Property 2 Suppose a function f satisfying Condition B is in the case (B-5). Let f' be a function expressed by the formula

$$f' = p^1 \sqcup \dots \sqcup p^n, \quad (19)$$

where

$$p^i(X_1, \dots, X_n) = \bigsqcup_{A=(A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n) \in S_3^{n-1}} (f_A^i(X_i) \wedge X_1^{A_1} \wedge \dots \wedge X_{i-1}^{A_{i-1}} \wedge X_{i+1}^{A_{i+1}} \wedge \dots \wedge X_n^{A_n}). \quad (20)$$

Then, for any element $(A_1, \dots, A_n) \in P_3^n$,

$$f'(A_1, \dots, A_n) = \begin{cases} f(A_1, \dots, A_n) & \text{if } (A_1, \dots, A_n) \notin (P_3 - S_3)^n \\ K & \text{otherwise} \end{cases}$$

where K is an element of P_r such that $\{0\} \subseteq K \subseteq \{0\} \cup F(A_1, \dots, A_n)$. (End of Property)

Property 3 Suppose a function f satisfying Condition B is in the case (B-5). Let f'' be a function expressed by the formula

$$f''(X_1, \dots, X_n) = \bigsqcup_{S \in P_3 - S_3} \left\{ \bigsqcup_{t \in S} (\{t\} \wedge f_{\hat{T}(S)}(X_1, \dots, X_n)) \right\}, \quad (21)$$

where $\hat{T}(S)$ is the set of all minimal elements of the set

$$T(S) = \{(A_1, \dots, A_n) \in (P_3 - S_3)^n \mid f(A_1, \dots, A_n) = S\}$$

and

$$f_{\hat{T}(S)}(X_1, \dots, X_n) = \bigsqcup_{(A_1, \dots, A_n) \in \hat{T}(S)} \left\{ \bigwedge_{e \in A_1} X^{\{e\}} \wedge \dots \wedge \bigwedge_{e \in A_n} X^{\{e\}} \right\}. \quad (22)$$

Then, for any element $(A_1, \dots, A_n) \in P_3^n$,

$$f''(A_1, \dots, A_n) = \begin{cases} \{0\} & \text{if } (A_1, \dots, A_n) \notin (P_3 - S_3)^n \\ f(A_1, \dots, A_n) & \text{otherwise} \end{cases}$$

(End of Property)

Property 4 Any function f satisfying Condition B can be expressed by $f = f' \sqcup f''$, if f is in the case (B-5). (End of Property)

Property 5 Suppose a function f satisfying Condition B is in the case (B-4). Let $A = (A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n)$ be an element of S_3^{n-1} . Then, define g_A^i and h as functions on P_r expressed by the following formulas.

$$g_A^i(X_1, \dots, X_n) = f_A^i(X_i) \sqcup (X_1^{A'_1} \wedge \dots \wedge X_{i-1}^{A'_{i-1}} \wedge X_{i+1}^{A'_{i+1}} \wedge \dots \wedge X_n^{A'_n}), \quad (23)$$

where $A'_j = E_3 - A_j$ ($j = 1, \dots, i-1, i+1, \dots, n$).

$$h(X_1, \dots, X_n) = \left\{ \left(\bigsqcup_{i=1}^n \bigsqcup_{A \in S_3} X_i^A \right) \sqcup f^{12}(X_1, \dots, X_n) \right\} \wedge \left\{ \left(\bigsqcup_{i=1}^n \bigsqcup_{A \in S_3} X_i^A \right) \sqcup f^{02}(X_1, \dots, X_n) \sqcup \underline{1} \right\} \quad (24)$$

Then, f can be expressed by the following formula.

$$f(X_1, \dots, X_n) = G(X_1, \dots, X_n) \wedge h(X_1, \dots, X_n), \quad (25)$$

where G is $\wedge$ -ing of all the g_A^i 's of Eq. (23), i.e.,

$$G(X_1, \dots, X_n) = \bigwedge_{i=1}^n \left(\bigwedge_{A \in S_3^{n-1}} g_A^i(X_1, \dots, X_n) \right) \quad (26)$$

(End of Property)

Property 6 Suppose a function f satisfying Condition B is in the case (B-6). Then, f is in either one of the following two cases.

- (1) $f(A) \neq \underline{02}$ holds for any element $A \in P_3^n$, or
- (2) $f(A) = \underline{02}$ holds for some element $A \in P_3^n$.

If f is in the case (1), then f can be expressed by

$$f(X_1, \dots, X_n) = (\underline{1} \wedge f^1(X_1, \dots, X_n)) \sqcup f^2(X_1, \dots, X_n) \sqcup (\underline{1} \wedge f^{12}(X_1, \dots, X_n)). \quad (27)$$

Let w be a function expressed by the following formula.

$$w(X_1, \dots, X_n) = \bigsqcup_{(A_1, \dots, A_n) \in Q_{02}} \xi_1(A_1) \sqcup \dots \sqcup \xi_n(A_n), \quad (28)$$

where $Q_{02} = \{(A_1, \dots, A_n) \in P_3^n \mid f(A_1, \dots, A_n) = \underline{02}\}$ ⁴, and

$$\xi_i(A) = \begin{cases} X_i^A & \text{if } A \in S_3 \\ \underline{0} & \text{otherwise.} \end{cases}$$

Then, f can be expressed by the following formula, if f is in the case (2).

$$f(X_1, \dots, X_n) = G(X_1, \dots, X_n) \wedge w(X_1, \dots, X_n), \quad (29)$$

where $G(X_1, \dots, X_n)$ is given by Eq. (26).

(End of Property)

6 Examples of Function Satisfying Condition B

This section shows examples of 2-variable functions satisfying Condition B, and illustrates how they can be expressed by formulas.

Example 4 Consider the function f defined by Table 12, which is in the case (B-5). The formula expressing f is given by Properties 2, 3, and 4. First, consider the formulas $f_A^1(X)$ and $f_A^2(Y)$, which appear in Eq. (20). Table 14 shows the truth tables of the six one-variable functions $f_0^1(X)$, $f_{\underline{1}}^1(X)$, $f_{\underline{2}}^1(X)$, $f_0^2(Y)$, $f_{\underline{1}}^2(Y)$ and $f_{\underline{2}}^2(Y)$. Since $f_{\underline{2}}^1(X)$ and $f_{\underline{1}}^2(Y)$ are in (B-2) (or (B-3)), $f_{\underline{1}}^1(X)$ is in (B-3), and $f_{\underline{2}}^2(Y)$ is in (B-1), it follows by Eq. (17) and (18) that these one-variable functions are expressed by the following formulas.

⁴ $Q_{02} \cap (P_3 - S_3)^n = \emptyset$ holds, since f is in the case (B-6).

Table 14: One-Variable Functions f_A^1 and f_A^2 of Example 4

X or Y	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
$f_0^1(X)$	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
$f_1^1(X)$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>012</u>	<u>012</u>
$f_2^1(X)$	<u>0</u>	<u>0</u>	<u>1</u>	<u>0</u>	<u>01</u>	<u>01</u>	<u>01</u>
$f_0^2(Y)$	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
$f_1^2(Y)$	<u>0</u>	<u>1</u>	<u>0</u>	<u>01</u>	<u>0</u>	<u>01</u>	<u>01</u>
$f_2^2(Y)$	<u>0</u>	<u>2</u>	<u>1</u>	<u>02</u>	<u>012</u>	<u>12</u>	<u>012</u>

Table 15: Truth Table of f' of Example 4

$X \setminus Y$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
<u>1</u>	<u>0</u>	<u>1</u>	<u>0</u>	<u>01</u>	<u>0</u>	<u>01</u>	<u>01</u>
<u>2</u>	<u>0</u>	<u>2</u>	<u>1</u>	<u>02</u>	<u>012</u>	<u>12</u>	<u>012</u>
<u>01</u>	<u>0</u>	<u>01</u>	<u>0</u>	<u>01</u>	<u>0</u>	<u>01</u>	<u>01</u>
<u>02</u>	<u>0</u>	<u>02</u>	<u>01</u>	<u>02</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>12</u>	<u>0</u>	<u>012</u>	<u>01</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>012</u>	<u>0</u>	<u>012</u>	<u>01</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>

Table 16: Truth Table of f'' of Example 4

$X \setminus Y$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
<u>1</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
<u>2</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
<u>01</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>01</u>	<u>0</u>	<u>01</u>	<u>01</u>
<u>02</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>12</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>012</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>

$$\begin{aligned}
f_0^1(X) &= \underline{0}, & f_1^1(X) &= \underline{1}X^1 \sqcup X^2, & f_2^1(X) &= \underline{1}X^2 \\
f_0^2(Y) &= \underline{0}, & f_1^2(Y) &= \underline{1}Y^1, & f_2^2(Y) &= Y^{12}(\underline{1} \sqcup Y^{01})
\end{aligned}$$

Therefore, it follows by Eq. (19) that f' is expressed by the following formula.

$$f' = X^0 f_0^2(Y) \sqcup X^1 f_1^2(Y) \sqcup X^2 f_2^2(Y) \sqcup f_0^1(X) Y^0 \sqcup f_1^1(X) Y^1 \sqcup f_2^1(X) Y^2 \quad (30)$$

Table 15 is the truth table of f' . Next, consider f'' in Eq. (21). Since

$$\begin{aligned}
\hat{T}(\underline{01}) &= \{(\underline{01}, \underline{01}), (\underline{01}, \underline{12})\}, \\
\hat{T}(\underline{02}) &= \hat{T}(\underline{12}) = \emptyset, \text{ and} \\
\hat{T}(\underline{012}) &= \{(\underline{02}, \underline{01}), (\underline{02}, \underline{02}), (\underline{02}, \underline{12}), (\underline{12}, \underline{01}), (\underline{12}, \underline{02}), (\underline{12}, \underline{12})\},
\end{aligned}$$

it follows by Eq. (22) that we have the following formulas.

$$\begin{aligned}
f_{\hat{T}(\underline{01})}(X, Y) &= X^0 X^1 Y^0 Y^1 \sqcup X^0 X^1 Y^1 Y^2 \\
f_{\hat{T}(\underline{012})}(X, Y) &= X^0 X^2 Y^0 Y^1 \sqcup X^0 X^2 Y^0 Y^2 \sqcup X^0 X^2 Y^1 Y^2 \sqcup X^1 X^2 Y^0 Y^1 \sqcup \\
&\quad X^1 X^2 Y^0 Y^2 \sqcup X^1 X^2 Y^1 Y^2
\end{aligned}$$

We then have $f''(X, Y)$ below by Eq. (21).

$$f''(X, Y) = \underline{1}f_{\hat{T}(\underline{01})}(X, Y) \sqcup \underline{1}f_{\hat{T}(\underline{012})}(X, Y) \sqcup f_{\hat{T}(\underline{012})}(X, Y) \quad (31)$$

Table 16 is the truth table of f'' . It follows by Property 4 that $f(X, Y) = f'(X, Y) \sqcup f''(X, Y)$.
(End of Example)

Table 17: Truth Table of G of Example 5

$X \backslash Y$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>0</u>	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>012</u>	<u>012</u>
<u>1</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>
<u>2</u>	<u>0</u>	<u>2</u>	<u>1</u>	<u>02</u>	<u>012</u>	<u>12</u>	<u>012</u>
<u>01</u>	<u>02</u>	<u>12</u>	<u>2</u>	<u>012</u>	<u>02</u>	<u>012</u>	<u>012</u>
<u>02</u>	<u>0</u>	<u>12</u>	<u>12</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>12</u>	<u>02</u>	<u>2</u>	<u>12</u>	<u>02</u>	<u>012</u>	<u>12</u>	<u>012</u>
<u>012</u>	<u>02</u>	<u>12</u>	<u>12</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>

Table 18: Truth Table of h of Example 5

$X \backslash Y$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>0</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>
<u>1</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>
<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>
<u>01</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>02</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>12</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>02</u>	<u>012</u>	<u>12</u>	<u>012</u>
<u>012</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>

Example 5 Consider the function f defined by Table 11, which is in (B-4). The formula expressing f is given by Property 5. First, consider the formula $g_A^1(X)$ and $g_A^2(Y)$ of Eq. (23). It follows by Eq. (17) and (18) that the one-variable functions $f_A^1(X)$ and $f_A^2(Y)$ are obtained below.

$$\begin{aligned} f_0^1(X) &= X^1, & f_1^1(X) &= \underline{1} \sqcup X^{12}, & f_2^1(X) &= \underline{1} \sqcup X^{01}, \\ f_0^2(Y) &= \underline{1}Y^1 \sqcup Y^2, & f_1^2(Y) &= \underline{2}, & f_2^2(Y) &= Y^{12}(\underline{1} \sqcup Y^{01}) \end{aligned}$$

Then, by Eq. (23), we have

$$\begin{aligned} g_0^1(X, Y) &= X^1 \sqcup Y^{12}, & g_1^1(X, Y) &= \underline{1} \sqcup X^{12} \sqcup Y^{02}, & g_2^1(X, Y) &= \underline{1} \sqcup X^{01} \sqcup Y^{01}, \\ g_0^2(X, Y) &= X^{12} \sqcup \underline{1}Y^1 \sqcup Y^2, & g_1^2(X, Y) &= \underline{2}, & g_2^2(X, Y) &= X^{01} \sqcup Y^{12}(\underline{1} \sqcup Y^{01}). \end{aligned}$$

By Eq. (26), we have the function $G(X, Y)$ expressed by $\wedge$ -ing of all the above $g_A^i(X, Y)$'s. Table 17 is the truth table of $G(X, Y)$. Next, consider h in Eq. (24). It follows by Eq. (17) that the functions $f^{12}(X, Y)$ and $f^{02}(X, Y)$ are expressed by the following formulas.

$$\begin{aligned} f^{12}(X, Y) &= X^1 \sqcup X^{12}Y^{12} \sqcup Y^1 \sqcup Y^2 \sqcup X^0X^1Y^0 \sqcup X^1X^2Y^0 \\ f^{02}(X, Y) &= X^0Y^{02} \sqcup X^1 \sqcup X^{01}Y^2 \sqcup X^{12}Y^{01} \sqcup Y^1 \end{aligned}$$

Thus, by Eq. (24), we have

$$h(X, Y) = \left(v(X, Y) \sqcup f^{12}(X, Y) \right) \wedge \left(v(X, Y) \sqcup f^{02}(X, Y) \sqcup \underline{1} \right),$$

where $v(X, Y) = X^0 \sqcup X^1 \sqcup X^2 \sqcup Y^0 \sqcup Y^1 \sqcup Y^2$. Table 18 is the truth table of $h(X, Y)$. Lastly, by Eq. (25), $f(X, Y)$ are expressed by the following formula.

$$\begin{aligned} f(X, Y) &= G(X, Y) \wedge h(X, Y) \\ &= g_0^1(X, Y) \wedge g_1^1(X, Y) \wedge g_2^1(X, Y) \wedge g_0^2(X, Y) \wedge g_1^2(X, Y) \wedge g_2^2(X, Y) \wedge h(X, Y) \end{aligned}$$

(End of Example)

Example 6 Consider the function f define by Table 13, which is in (B-6). The formula expressing f is given by Property 6. Since f is in the case (2) of Property 6, f is expressed by the formula given in Eq. (29).

First, consider the formula $G(X, Y)$. It follows by Eq. (17) and (18) that one-variable functions $f_A^1(X)$ and $f_A^2(Y)$ are obtained below.

Table 19: Truth Table of G of Example 6

$X \backslash Y$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>
<u>1</u>	<u>0</u>	<u>1</u>	<u>1</u>	<u>01</u>	<u>01</u>	<u>1</u>	<u>01</u>
<u>2</u>	<u>0</u>	<u>2</u>	<u>2</u>	<u>12</u>	<u>02</u>	<u>2</u>	<u>02</u>
<u>01</u>	<u>0</u>	<u>01</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>
<u>02</u>	<u>0</u>	<u>012</u>	<u>02</u>	<u>012</u>	<u>02</u>	<u>012</u>	<u>012</u>
<u>12</u>	<u>0</u>	<u>12</u>	<u>12</u>	<u>012</u>	<u>012</u>	<u>12</u>	<u>012</u>
<u>012</u>	<u>0</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>	<u>012</u>

Table 20: Truth Table of w of Example 6

$X \backslash Y$	<u>0</u>	<u>1</u>	<u>2</u>	<u>01</u>	<u>02</u>	<u>12</u>	<u>012</u>
<u>0</u>	<u>1</u>	<u>1</u>	<u>2</u>	<u>1</u>	<u>12</u>	<u>12</u>	<u>12</u>
<u>1</u>	<u>1</u>	<u>1</u>	<u>2</u>	<u>1</u>	<u>12</u>	<u>12</u>	<u>12</u>
<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>	<u>2</u>
<u>01</u>	<u>1</u>	<u>1</u>	<u>2</u>	<u>1</u>	<u>12</u>	<u>12</u>	<u>12</u>
<u>02</u>	<u>12</u>	<u>12</u>	<u>2</u>	<u>12</u>	<u>12</u>	<u>12</u>	<u>12</u>
<u>12</u>	<u>12</u>	<u>12</u>	<u>2</u>	<u>12</u>	<u>12</u>	<u>12</u>	<u>12</u>
<u>012</u>	<u>12</u>	<u>12</u>	<u>2</u>	<u>12</u>	<u>12</u>	<u>12</u>	<u>12</u>

$$\begin{aligned}
f_0^1(X) &= \underline{0} & f_1^1(X) &= (\underline{1} \wedge X^{12}) \sqcup X^2 & f_2^1(X) &= X^{12} \wedge (\underline{1} \sqcup X^{02}) \\
f_0^2(Y) &= \underline{0}, & f_1^2(Y) &= \underline{1} \wedge Y^{12} & f_2^2(Y) &= Y^{12}
\end{aligned}$$

Thus, we obtain $g_A^1(X, Y)$ and $g_A^2(X, Y)$ of Eq. (23) below.

$$\begin{aligned}
g_0^1(X, Y) &= Y^{12} & g_1^1(X, Y) &= (\underline{1} \wedge X^{12}) \sqcup X^2 \sqcup Y^{02} & g_2^1(X, Y) &= X^{12} \wedge (\underline{1} \sqcup X^{02}) \sqcup Y^{01} \\
g_0^2(X, Y) &= X^{12} & g_1^2(X, Y) &= X^{02} \sqcup (\underline{1} \wedge Y^{12}) & g_2^2(X, Y) &= X^{01} \sqcup Y^{12}
\end{aligned}$$

By Eq. (26), we have the function $G(X, Y)$ expressed by $\wedge$ -ing of all the above $g_A^i(X, Y)$'s. Table 19 is the truth table of G .

Next, consider $w(X, Y)$ of Eq. (28). Because

$$Q_{02} = \{(2, \underline{01}), (2, \underline{02}), (2, \underline{012}), (02, \underline{02})\},$$

it follow by Eq. (28) that the formula w is obtained below.

$$w(X, Y) = \underline{1} \sqcup X^2 \sqcup Y^2$$

Table 20 is the truth table of w . Lastly, it follows by Eq. (29) that the following formula expresses the function f .

$$\begin{aligned}
f(X, Y) &= G(X, Y) \wedge w(X, Y) \\
&= g_0^1(X, Y) \wedge g_1^1(X, Y) \wedge g_2^1(X, Y) \wedge g_0^2(X, Y) \wedge g_1^2(X, Y) \wedge g_2^2(X, Y) \wedge w(X, Y).
\end{aligned}$$

(End of Example)

7 Conclusions

This paper discussed functions over P_r that preserves the set inclusion relation $\subseteq$. We referred the three kinds of operations Min, Max, and Literals over E_r , because they are functionally complete on the r -valued set E_r . This paper then proved some of the mathematical properties of functions over P_r that can be expressed by formulas. It is one of the open problems that which set of operations $\hat{o}_1, \hat{o}_2, \dots, \hat{o}_m$ over P_r can realize any function over P_r preserving the set inclusion relation $\subseteq$.

References

- [1] S. C. Kleene, *Introduction to metamathematics*, North-Holland Pub., pp. 332-340, 1952.
- [2] M. Goto, "Application of three-valued logic to construct the theory of relay networks," (in Japanese), *Proc. Joint Meeting IEE, IECE, and I. of Illum. E. of Japan*, 1948.
- [3] M. Mukaidono, "Regular ternary logic functions – Ternary logic functions suitable for treating ambiguity," *IEEE Trans. on Computers*, vol. c-35, no. 2, pp. 179-183, 1986.
- [4] M. Mukaidono, "On the B-ternary logical function," *IECE Trans.*, vol. 55-D, no. 6, pp. 355-362, 1972 (in Japanese)
- [5] J. C. Muzio and T. C. Wesselkamper, *Multiple-Valued Switching Theory*.
- [6] M. Mukaidono: "The B-ternary logic and its applications to the detection of hazards in combinational switching circuits," *Proc. of the 8th ISMVL*, pp. 269-275, 1978.
- [7] N. Takagi, Y. Nakamura and K. Nakashima: "Set-valued functions and regularity," *Proc. of the 27th ISMVL*, pp. 89-94, 1997.
- [8] I. G. Rosenberg, "Multiple-valued hyperstructures," *Proc. of the 28th ISMVL*, pp. 326-333, 1998.