

A formulation of quasi-regular non-local Dirichlet forms on F chet spaces with application to a stochastic quantization of Φ_3^4 field

Sergio ALBEVERIO^{*} and Minoru W. YOSHIDA[†]

February 20, 2019

1 Introduction

We consider a space S that is a real Banach space l^p , $1 \leq p \leq \infty$ with suitable weights. Let μ be a Borel probability measure on S . On the *real* $L^2(S; \mu)$ space, for each $0 < \alpha \leq 1$, we give an explicit formulation of α -stable type (cf., e.g., section 5 of [Fukushima,Uemura 2012] for corresponding formula on $L^2(\mathbb{R}^d)$, $d < \infty$) *non-local* strictly quasi-regular (cf. section IV-3 of [M,R 92]) Dirichlet forms $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ (with a domain $\mathcal{D}(\mathcal{E}_{(\alpha)})$), and show the existence of S -valued Hunt processes properly associated to $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$. These general theorems are applied to a stochastic quantization of (α -stable type) Euclidean Φ_3^4 field on $\mathbb{R}^3$.

The objective of the present paper is to announce the above developments that are part of *general* (e.g. for $0 < \alpha < 2$) and *detailed results* given in [A,Kagawa,Yahagi,Y 2018] (cf. also [A,Y 2018]), where the state spaces S are assumed to be either the above l^p , $1 \leq p \leq \infty$, spaces or the direct product $\mathbb{R}^{\mathbb{N}}$ (with $\mathbb{R}$ and resp. $\mathbb{N}$ the spaces of real numbers and resp. natural numbers), both understood as Fr chet spaces, and for each $0 < \alpha < 2$, an explicit formulation of α -stable type non-local quasi-regular (cf. section IV-3 of [M,R 92]) Dirichlet forms is considered.

2 Markovian symmetric forms individually adapted to each measure space

The state space S , on which we define the Markovian symmetric forms, is a weighted l^p space, denoted by $l_{(\beta_i)}^p$, such that, for some $p \in [1, \infty)$ and a weight $(\beta_i)_{i \in \mathbb{N}}$ with $\beta_i \geq 0, i \in \mathbb{N}$,

$$S = l_{(\beta_i)}^p \equiv \{\mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} : \|\mathbf{x}\|_{l_{(\beta_i)}^p} \equiv \left(\sum_{i=1}^{\infty} \beta_i |x_i|^p \right)^{\frac{1}{p}} < \infty\}. \quad (2.1)$$

^{*}Inst. Angewandte Mathematik, and HCM, Univ. Bonn, Germany, email :albeverio@iam.uni-bonn.de

[†]Dept. Information Systems Kanagawa Univ., Yokohama, Japan, email: washizuminoru@hotmail.com

We denote by $\mathcal{B}(S)$ the Borel σ -field of S . Suppose that we are given a Borel probability measure μ on $(S, \mathcal{B}(S))$. For each $i \in \mathbb{N}$, let σ_{i^c} be the sub σ -field of $\mathcal{B}(S)$ that is generated by the Borel sets

$$B = \{\mathbf{x} \in S \mid x_{j_1} \in B_1, \dots, x_{j_n} \in B_n\}, \quad j_k \neq i, \quad B_k \in \mathcal{B}^1, \quad k = 1, \dots, n, \quad n \in \mathbb{N}, \quad (2.2)$$

where $\mathcal{B}^1$ denotes the Borel σ -field of $\mathbb{R}^1$, i.e., σ_{i^c} is the smallest σ -field that includes every B given by (2.2). Namely, σ_{i^c} is the sub σ -field of $\mathcal{B}(S)$ generated by the variables $\mathbf{x} \setminus x_i$, i.e., all variables except for the i -th variable x_i . For each $i \in \mathbb{N}$, let $\mu(\cdot \mid \sigma_{i^c})$ be the conditional probability, a one-dimensional probability distribution-valued σ_{i^c} measurable function, (μ -every where defined) that is characterized by (cf. (2.4) of [A,R91])

$$\mu(\{\mathbf{x} : x_i \in A\} \cap B) = \int_B \mu(A \mid \sigma_{i^c}) \mu(d\mathbf{x}), \quad \forall A \in \mathcal{B}^1, \quad \forall B \in \sigma_{i^c}. \quad (2.3)$$

Define

$$L^2(S; \mu) \equiv \left\{ f \mid f : S \rightarrow \mathbb{R}, \text{ measurable and } \|f\|_{L^2} = \left(\int_S |f(\mathbf{x})|^2 \mu(d\mathbf{x}) \right)^{\frac{1}{2}} < \infty \right\}, \quad (2.4)$$

and

$$\mathcal{FC}_0^\infty \equiv \text{the } \mu \text{ equivalence class of } \left\{ f \mid \exists n \in \mathbb{N}, f \in C_0^\infty(\mathbb{R}^n \rightarrow \mathbb{R}) \right\} \subset L^2(S; \mu), \quad (2.5)$$

where $C_0^\infty(\mathbb{R}^n \rightarrow \mathbb{R})$ denotes the space of *real valued* infinitely differentiable functions on $\mathbb{R}^n$ with compact supports.

On $L^2(S; \mu)$, for any $0 < \alpha \leq 1$ (for the case of general $0 < \alpha < 2$, cf. [A,Kagawa,Yahagi,Y 2018]), we are going to define the Markovian symmetric forms $\mathcal{E}_{(\alpha)}$ called *individually adapted Markovian symmetric forms of index α relative to the measure μ* . They have a natural analogy of the one for α -stable type (*non local* Dirichlet forms on $\mathbb{R}^d$, $d < \infty$ (cf. Remark 1 given below and (5.3), (1.4) of [Fukushima,Uemura 2012]), and can be seen as non local analogy of local classical Dirichlet forms on infinite dimensional topological vector spaces (cf. [A,R 89, 90, 91]). The latter are defined by making use of directional derivatives. The definition of our forms is as follows: Firstly, for each $0 < \alpha \leq 1$ and $i \in \mathbb{N}$, and for the variables $y_i, y'_i \in \mathbb{R}^1$, $\mathbf{x} = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) \in S$ and $\mathbf{x} \setminus x_i \equiv (x_1, \dots, x_{i-1}, x_{i+1}, \dots)$, we consider the bilinear expression

$$\begin{aligned} \Phi_\alpha(u, v; y_i, y'_i, \mathbf{x} \setminus x_i) \\ \equiv \frac{1}{|y_i - y'_i|^{\alpha+1}} \times \left\{ u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots) - u(x_1, \dots, x_{i-1}, y'_i, x_{i+1}, \dots) \right\} \\ \times \left\{ v(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots) - v(x_1, \dots, x_{i-1}, y'_i, x_{i+1}, \dots) \right\}, \end{aligned} \quad (2.6)$$

and set

$$\mathcal{E}_{(\alpha)}^{(i)}(u, v) \equiv \int_S \left\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}}(y_i) \Phi_\alpha(u, v; y_i, x_i, \mathbf{x} \setminus x_i) \mu(dy_i \mid \sigma_{i^c}) \right\} \mu(d\mathbf{x}), \quad (2.7)$$

$$\mathcal{E}_{(\alpha)}(u, v) \equiv \sum_{i \in \mathbb{N}} \mathcal{E}_{(\alpha)}^{(i)}(u, v). \quad (2.8)$$

where $I_{\{\cdot\}}$ denotes the indicator function. For $y_i \neq y'_i$, (2.6) is well defined for any real valued $\mathcal{B}(S)$ -measurable functions u and v . For the Lipschitz continuous functions $\tilde{u} \in C_0^\infty(\mathbb{R}^n \rightarrow \mathbb{R}) \subset \mathcal{FC}_0^\infty$ resp. $\tilde{v} \in C_0^\infty(\mathbb{R}^m \rightarrow \mathbb{R}) \subset \mathcal{FC}_0^\infty$, $n, m \in \mathbb{N}$ which are representations of $u \in \mathcal{FC}_0^\infty$ resp. $v \in \mathcal{FC}_0^\infty$, $n, m \in \mathbb{N}$, (2.7) and (2.8) are well defined (the right hand side of (2.8) has only a finite number of sums). In Theorem 1 given below we see that (2.7) and (2.8) are well defined for $\mathcal{FC}_0^\infty$, the space of μ -equivalent class.

Remark 1 We can also derive the following equivalent expressions for $\mathcal{E}_{(\alpha)}^i(u, v)$.

$$\begin{aligned} \mathcal{E}_{(\alpha)}^{(i)}(u, v) &= \int_S \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}} \Phi_\alpha(u, v; y_i, x_i, \mathbf{x} \setminus x_i) \mu(dy_i | \sigma_{i^c}) \right\} \mu(dx_i | \sigma_{i^c}) \mu(d\mathbf{x}) \\ &= \int_S \left\{ \int_{\mathbb{R}^2} I_{\{y_i \neq y'_i\}} \Phi_\alpha(u, v; y_i, y'_i, \mathbf{x} \setminus x_i) \mu(dy_i | \sigma_{i^c}) \mu(dy'_i | \sigma_{i^c}) \right\} \mu(d\mathbf{x}) \quad (2.9) \\ &= \int_{S \setminus x_i} \left\{ \int_{\mathbb{R}^2} I_{\{y_i \neq y'_i\}} \Phi_\alpha(u, v; y_i, y'_i, \mathbf{x} \setminus x_i) \mu(dy_i | \sigma_{i^c}) \mu(dy'_i | \sigma_{i^c}) \right\} \mu(d(\mathbf{x} \setminus x_i)), \end{aligned}$$

where $\mu(d(\mathbf{x} \setminus x_i))$ is the marginal probability distribution of the variable $\mathbf{x} \setminus x_i$, i.e., for any $A \in \sigma_{i^c}$, $\int_A \mu(d(\mathbf{x} \setminus x_i)) = \int_S I_{\mathbb{R}}(x_i) I_A(\mathbf{x} \setminus x_i) \mu(d\mathbf{x})$. The third and fourth formulas give more symmetric definitions for $\mathcal{E}_{(\alpha)}^{(i)}(u, v)$ with respect to the variables y_i and x_i (analogous to (1.2.1) of [Fukushima 80]). These will be used in section 4

The following is the main theorem on the closability part of this paper.

Theorem 1 The symmetric non-local forms $\mathcal{E}_{(\alpha)}$, $0 < \alpha \leq 1$ given by (2.8) are

- i) well-defined on $\mathcal{FC}_0^\infty$;
- ii) Markovian;
- iii) closable in $L^2(S; \mu)$.

For each $0 < \alpha \leq 1$, the closed extension of $\mathcal{E}_{(\alpha)}$ is denoted by $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ with the domain $\mathcal{D}(\mathcal{E}_{(\alpha)})$, which is a non-local Dirichlet form on $L^2(S; \mu)$.

Moreover it holds that $1 \in \mathcal{D}(\mathcal{E}_{(\alpha)})$.

3 Proof of Theorem 1.

Suppose that $0 < \alpha \leq 1$

For the statement i), we have to show that

i-1) for any real valued $\mathcal{B}(S)$ -measurable function u on S , such that $u = 0$, $\mu - a.e.$, it holds that $\mathcal{E}_{(\alpha)}(u, u) = 0$ (cf. (3.8) given below), and

i-2) for any $u, v \in \mathcal{FC}_0^\infty$, there corresponds only one value $\mathcal{E}_{(\alpha)}(u, v) \in \mathbb{R}$,

For the statement ii), we have to show that (cf. [Fukushima 80]) for any $\epsilon > 0$ there exists a real function $\varphi_\epsilon(t)$, $-\infty < t < \infty$, such that $\varphi_\epsilon(t) = t$, $\forall t \in [0, 1]$, $-\epsilon \leq \varphi_\epsilon(t) \leq 1 + \epsilon$, $\forall t \in (-\infty, \infty)$, and $0 \leq \varphi_\epsilon(t') - \varphi_\epsilon(t) \leq t' - t$ for $t < t'$, such that for any $u \in \mathcal{FC}_0^\infty$ it holds that $\varphi_\epsilon(u) \in \mathcal{FC}_0^\infty$ and

$$\mathcal{E}_{(\alpha)}(\varphi_\epsilon(u), \varphi_\epsilon(u)) \leq \mathcal{E}_{(\alpha)}(u, u). \quad (3.1)$$

For the statement iii), we have to show the following: For a sequence $\{u_n\}_{n \in \mathbb{N}}$, $u_n \in \mathcal{FC}_0^\infty$, $n \in \mathbb{N}$, if

$$\lim_{n \rightarrow \infty} \|u_n\|_{L^2(S; \mu)} = 0, \quad (3.2)$$

and

$$\lim_{n, m \rightarrow \infty} \mathcal{E}_{(\alpha)}(u_n - u_m, u_n - u_m) = 0, \quad (3.3)$$

then

$$\lim_{n \rightarrow \infty} \mathcal{E}_{(\alpha)}(u_n, u_n) = 0. \quad (3.4)$$

i-1) can be seen as follows:

For each $i \in \mathbb{N}$ and any real valued $\mathcal{B}(S)$ -measurable function u , note that for each $\epsilon > 0$,

$$I_{\{\epsilon < |x_i - y_i|\}}(y_i) I_K(y_i) \Phi_\alpha(u, u; y_i, x_i, \mathbf{x} \setminus x_i)$$

defines a $\mathcal{B}(S \times \mathbb{R})$ -measurable function. Here we use an extension of the function $\Phi_\alpha(u, u; y_i, x_i, \mathbf{x} \setminus x_i)$, for $v = u$, $x = x_i$, defined by (2.6) to a general $\mathcal{B}(S)$ -measurable function u (instead of a function in $\mathcal{FC}_0^\infty$). $\mathcal{B}(S \times \mathbb{R})$ is the Borel σ -field of $S \times \mathbb{R}$. $\mathbf{x} = (x_i, i \in \mathbb{N}) \in S$ and $y_i \in \mathbb{R}$. Then, for any compact subset K of $\mathbb{R}$, $0 \leq I_{\{\epsilon < |x_i - y_i|\}}(y_i) I_K(y_i) \Phi_\alpha(u, u; y_i, x_i, \mathbf{x} \setminus x_i)$ converges monotonically to $I_{\{y_i \neq x_i\}}(y_i) \Phi_\alpha(u, u; y_i, x_i, \mathbf{x} \setminus x_i)$ as $K \uparrow \mathbb{R}$ and $\epsilon \downarrow 0$, for every $y_i \in \mathbb{R}$, $\mathbf{x} \in S$, and by the Fatou's Lemma, we have

$$\begin{aligned} & \int_S \left\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}}(y_i) \Phi_\alpha(u, u; y_i, x_i, \mathbf{x} \setminus x_i) \mu(dy_i \mid \sigma_{i^c}) \right\} \mu(d\mathbf{x}) \\ &= \int_S \liminf_{K \uparrow \mathbb{R}} \liminf_{\epsilon \downarrow 0} \left\{ \int_{\mathbb{R}} I_{\{\epsilon < |x_i - y_i|\}}(y_i) I_K(y_i) \Phi_\alpha(u, u; y_i, x_i, \mathbf{x} \setminus x_i) \mu(dy_i \mid \sigma_{i^c}) \right\} \mu(d\mathbf{x}) \\ &\leq \liminf_{K \uparrow \mathbb{R}} \liminf_{\epsilon \downarrow 0} \int_S \left\{ \int_{\mathbb{R}} I_{\{\epsilon < |x_i - y_i|\}}(y_i) I_K(y_i) \Phi_\alpha(u, u; y_i, x_i, \mathbf{x} \setminus x_i) \mu(dy_i \mid \sigma_{i^c}) \right\} \mu(d\mathbf{x}), \end{aligned} \quad (3.5)$$

I_K denotes the indicator function of K . Through the definition of the conditional probability distributions and conditional expectations, we see that, for any $\epsilon > 0$,

$$\begin{aligned} & \int_S \left\{ \int_{\mathbb{R}} I_{\{\epsilon < |x_i - y_i|\}}(y_i) I_K(y_i) \frac{1}{|y_i - x_i|^{\alpha+1}} (u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots))^2 \mu(dy_i \mid \sigma_{i^c}) \right\} \mu(d\mathbf{x}) \\ &\leq \frac{1}{\epsilon^{\alpha+1}} \int_S \left\{ \int_{\mathbb{R}} I_{\{\epsilon < |x_i - y_i|\}}(y_i) I_K(y_i) (u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots))^2 \mu(dy_i \mid \sigma_{i^c}) \right\} \mu(d\mathbf{x}) \\ &\leq \frac{1}{\epsilon^{\alpha+1}} \int_S \left\{ \int_{\mathbb{R}} (u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots))^2 \mu(dy_i \mid \sigma_{i^c}) \right\} \mu(d\mathbf{x}) \\ &= \frac{1}{\epsilon^{\alpha+1}} \int_S (u(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots))^2 \mu(d\mathbf{x}), \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} & \int_S (u(x_1, \dots))^2 \left\{ \int_{\mathbb{R}} I_{\{\epsilon < |x_i - y_i|\}}(y_i) I_K(y_i) \frac{1}{|y_i - x_i|^{\alpha+1}} \mu(dy_i \mid \sigma_{i^c}) \right\} \mu(d\mathbf{x}) \\ &\leq \frac{1}{\epsilon^{\alpha+1}} \int_S (u(x_1, \dots))^2 \mu(d\mathbf{x}). \end{aligned} \quad (3.7)$$

From (3.6), by making use of the Cauchy Schwarz's inequality we have

$$\begin{aligned} & \left| \int_S u(x_1, \dots, x_n) \left\{ \int_{\mathbb{R}} I_{\{\epsilon < |x_i - y_i|\}}(y_i) I_K(y_i) \frac{1}{|y_i - x_i|^{\alpha+1}} \right. \right. \\ & \quad \left. \left. \times u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots) \mu(dy_i \mid \sigma_{i^c}) \right\} \mu(d\mathbf{x}) \right| \\ & \leq \frac{1}{\epsilon^{\alpha+1}} \int_S (u(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n))^2 \mu(d\mathbf{x}). \end{aligned}$$

By this and (3.6), (3.7), from (3.5) we have proven i-1):

$$\mathcal{E}_{(\alpha)}^{(i)}(u, u) = 0, \quad \forall i \in \mathbb{N}, \quad \mathcal{E}_{(\alpha)}(u, u) = 0,$$

for any real valued $\mathcal{B}(S)$ -measurable function u such that $u = 0$, μ -a.e.. (3.8)

In order to show i-2), for $0 < \alpha \leq 1$, take any *representation* $\tilde{u} \in C_0^\infty(\mathbb{R}^n)$ of $u \in \mathcal{FC}_0^\infty$, $n \in \mathbb{N}$. Using $0 < \alpha + 1 \leq 2$, it is easy to see from the definition (2.6) that there exists an $M < \infty$ depending on $\tilde{u}$ such that

$$0 \leq \Phi_\alpha(\tilde{u}, \tilde{u}; y_i, y'_i, \mathbf{x} \setminus x_i) \leq M, \quad \forall \mathbf{x} \in S, \text{ and } \forall y_i, y'_i \in \mathbb{R}. \quad (3.9)$$

Since, $u = \tilde{u} + \bar{0}$ for some real valued $\mathcal{B}(S)$ -measurable function $\bar{0}$ such that $\bar{0} = 0$, μ -a.e., by (3.9) together with i-1) (cf. (3.8)) and the the Cauchy Schwarz's inequality, for $u \in \mathcal{FC}_0^\infty$, $\mathcal{E}_{(\alpha)}(u, u) \in \mathbb{R}$, $0 < \alpha \leq 1$, is identical with $\mathcal{E}_{(\alpha)}(\tilde{u}, \tilde{u})$ and well-defined (in fact, for only a finite number of $i \in \mathbb{N}$. we have $\mathcal{E}_{(\alpha)}^{(i)}(u, u) \neq 0$, cf. also (2.8)). Then by the Cauchy Schwarz's inequality i-2) follows.

The proof of ii) is very similar to the one given in section 1 of [Fukushima 80], and it is omitted.

iii) can be proved as follows (cf. section 1 of [Fukushima 80]): Suppose that a sequence $\{u_n\}_{n \in \mathbb{N}}$ satisfies (3.2) and (3.3). Then, by (3.2) there exists a measurable set $\mathcal{N} \in \mathcal{B}(S)$ and a sub sequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\mu(\mathcal{N}) = 0$, $\lim_{n_k \rightarrow \infty} u_{n_k}(\mathbf{x}) = 0$, $\forall \mathbf{x} \in S \setminus \mathcal{N}$. Define

$$\tilde{u}_{n_k}(\mathbf{x}) = u_{n_k}(\mathbf{x}) \quad \text{for } \mathbf{x} \in S \setminus \mathcal{N}, \quad \text{and} \quad \tilde{u}_{n_k}(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \mathcal{N}.$$

Then,

$$\tilde{u}_{n_k}(\mathbf{x}) = u_{n_k}(\mathbf{x}), \quad \mu - a.e., \quad \lim_{n_k \rightarrow \infty} \tilde{u}_{n_k}(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in S. \quad (3.10)$$

By the fact i-1), precisely by (3.8), shown above and (3.10), for each i , we see that

$$\begin{aligned} & \int_S \left\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}}(y_i) \Phi_\alpha(u_n, u_n; y_i, x_i, \mathbf{x} \setminus x_i) \mu(dy_i \mid \sigma_{i^c}) \right\} \mu(d\mathbf{x}) \\ & = \int_S \left\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}}(y_i) \lim_{n_k \rightarrow \infty} \Phi_\alpha(u_n - \tilde{u}_{n_k}, u_n - \tilde{u}_{n_k}; y_i, x_i, \mathbf{x} \setminus x_i) \mu(dy_i \mid \sigma_{i^c}) \right\} \mu(d\mathbf{x}) \\ & \leq \liminf_{n_k \rightarrow \infty} \int_S \left\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}} \Phi_\alpha(u_n - \tilde{u}_{n_k}, u_n - \tilde{u}_{n_k}; y_i, x_i, \mathbf{x} \setminus x_i) \mu(dy_i \mid \sigma_{i^c}) \right\} \mu(d\mathbf{x}) \\ & = \liminf_{n_k \rightarrow \infty} \int_S \left\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}} \Phi_\alpha(u_n - u_{n_k}, u_n - u_{n_k}; y_i, x_i, \mathbf{x} \setminus x_i) \mu(dy_i \mid \sigma_{i^c}) \right\} \mu(d\mathbf{x}) \\ & \equiv \liminf_{n_k \rightarrow \infty} \mathcal{E}_{(\alpha)}^{(i)}(u_n - u_{n_k}, u_n - u_{n_k}). \end{aligned} \quad (3.11)$$

Now, by using the assumption (3.3) on the right hand side of (3.11), we get

$$\lim_{n \rightarrow \infty} \mathcal{E}_{(\alpha)}^{(i)}(u_n, u_n) = 0, \quad \forall i \in \mathbb{N}. \quad (3.12)$$

(3.12) together with i) show that for each $i \in \mathbb{N}$, $\mathcal{E}_{(\alpha)}^{(i)}$ with the domain $\mathcal{F}C_0^\infty$ is closable in $L^2(S; \mu)$. Since, $\mathcal{E}_{(\alpha)} \equiv \sum_{i \in \mathbb{N}} \mathcal{E}_{(\alpha)}^{(i)}$, by using Fatou's Lemma, from (3.12) and the assumption (3.3) we see that

$$\mathcal{E}_{(\alpha)}(u_n, u_n) = \sum_{i \in \mathbb{N}} \lim_{m \rightarrow \infty} \mathcal{E}_{(\alpha)}^{(i)}(u_n - u_m, u_n - u_m) \leq \liminf_{m \rightarrow \infty} \mathcal{E}_{(\alpha)}(u_n - u_m, u_n - u_m) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This proves (3.4) (cf. Proposition I-3.7 of [M,R 92] for a general argument of this type). This completes the proof of iii). Thus, by the closed extension the non-local Dirichlet form $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ is defined.

In order to see that $1 \in \mathcal{D}(\mathcal{E}_{(\alpha)})$, we take $\eta \in C_0^\infty(\mathbb{R} \rightarrow \mathbb{R})$ such that $\eta(x) \geq 0$, $|\frac{d}{dx}\eta(x)| \leq 1$ for $x \in \mathbb{R}$, and $\eta(x) = 1$ for $|x| < 1$; $\eta(x) = 0$ for $|x| > 3$, and define $u_M(x_1, x_2, \dots) \equiv \eta(x_1 \cdot M^{-1}) \prod_{i \geq 2} I_{\mathbb{R}}(x_i) \in \mathcal{F}C_0^\infty \subset \mathcal{D}(\mathcal{E}_{(\alpha)})$ for each $M \in \mathbb{N}$. Then it is possible to show that (cf. (2.6) and (2.7)) $\sup_{M \in \mathbb{N}} \mathcal{E}_{(\alpha)}(u_M, u_M) < \infty$. Since, $\lim_{M \rightarrow \infty} u_M(\mathbf{x}) = 1 = \prod_{i \geq 1} I_{\mathbb{R}}(x_i)$ point wise, and hence $\mu - a.e.$, from Lemma I-2.12 of [M,R 92] we have $1 \in \mathcal{D}(\mathcal{E}_{(\alpha)})$. This complete the proof of Theorem 1. ■

4 Quasi-regularity

For each $i \in \mathbb{N}$, we denote by X_i the random variable (i.e., measurable function) on $(S, \mathcal{B}(S), \mu)$, that represents the coordinate x_i of $\mathbf{x} = (x_1, x_2, \dots)$, precisely,

$$X_i : S \ni \mathbf{x} \longmapsto x_i \in \mathbb{R}. \quad (4.1)$$

By making use of the random variable X_i , we have the following probabilistic expression:

$$\int_S 1_B(x_i) \mu(d\mathbf{x}) = \mu(X_i \in B), \quad \text{for } B \in \mathcal{B}(S). \quad (4.2)$$

Theorem 2 *Let $0 < \alpha \leq 1$, and let $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ be the closed Markovian symmetric form defined through Theorem 1 on the state space S . For $S = l_{(\beta_i)}^p$, $1 \leq p < \infty$, if there exists a positive l^p sequence $\{\gamma_i^{-\frac{1}{p}}\}_{i \in \mathbb{N}}$, and an $0 < M < \infty$ such that*

$$\sum_{i=1}^{\infty} \beta_i^{\frac{2}{p}} \gamma_i^{\frac{2}{p}} \cdot \mu\left(\beta_i^{\frac{1}{p}} |X_i| > M \cdot \gamma_i^{-\frac{1}{p}}\right) < \infty, \quad (4.3)$$

holds, then $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ is a (strictly) quasi-regular Dirichlet form.

Proof of Theorem 2. It is possible to verify that the Dirichlet forms $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ satisfy the definition of the quasi-regularity given by Definition 3.1 in section IV-3 of [M,R 92]. Namely, by using the same notions adopted in [M,R 92], we have to certify that the following i), ii) and iii) are satisfied by $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$:

- i) There exists an $\mathcal{E}_{(\alpha)}$ -nest $(D_M)_{M \in \mathbb{N}}$ consisting of compact sets.
- ii) There exists a subset of $\mathcal{D}(\mathcal{E}_{(\alpha)})$, that is dense with respect to the norm $\|\cdot\|_{L^2(S;\mu)} + \sqrt{\mathcal{E}_{(\alpha)}}$. And the elements of this subset have $\mathcal{E}_{(\alpha)}$ -quasi continuous versions.
- iii) There exists $u_n \in \mathcal{D}(\mathcal{E}_{(\alpha)})$, $n \in \mathbb{N}$, having $\mathcal{E}_{(\alpha)}$ -quasi continuous μ -versions $\tilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}_{(\alpha)}$ -exceptional set $\mathcal{N} \subset S$ such that $\{\tilde{u}_n : n \in \mathbb{N}\}$ separates the points of $S \setminus \mathcal{N}$. The fact that the quasi-regular Dirichlet form $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ is looked upon a *strictly* quasi-regular Dirichlet form can be guaranteed by showing (cf. Proposition V-2.15 of [M,R 92])
- iv) $1 \in \mathcal{D}(\mathcal{E}_{(\alpha)})$

In fact, by Theorem 1 in section 2, the above ii) and iii) hold for $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$: since $\mathcal{FC}_0^\infty \subset C(S \rightarrow \mathbb{R})$, and $\mathcal{D}(\mathcal{E}_{(\alpha)})$ is the closure of $\mathcal{FC}_0^\infty$ by Theorem 1, we can take $\mathcal{FC}_0^\infty$ as the subset of $\mathcal{D}(\mathcal{E}_{(\alpha)})$ mentioned in the above ii). Moreover, since $\mathcal{FC}_0^\infty$ separates the points S , we see that the above iii) holds. Also, iv) is the last statement of Theorem 1.

Hence, we have only to show that the above i) holds for $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$. Equivalently (cf. Definition 2.1. in section III-2 of [M,R 92]), we have to show that there exists an increasing sequence $(D_M)_{M \in \mathbb{N}}$ of compact subsets of S such that $\cup_{m \geq 1} \mathcal{D}(\mathcal{E}_{(\alpha)})_{D_M}$ is dense in $\mathcal{D}(\mathcal{E}_{(\alpha)})$ (with respect to the norm $\|\cdot\|_{L^2(S;\mu)} + \sqrt{\mathcal{E}_{(\alpha)}}$), where $\mathcal{D}(\mathcal{E}_{(\alpha)})_{D_M}$ is the subspace of $\mathcal{D}(\mathcal{E}_{(\alpha)})$ the elements of which are functions with supports belonging to D_M . For this, by Theorem 1, since $\mathcal{D}(\mathcal{E}_{(\alpha)})$ is the closure of $\mathcal{FC}_0^\infty$, it suffices to show the following: there exists a sequence of compact sets

$$D_M \subset S, \quad M \in \mathbb{N} \quad (4.4)$$

and a subset $\tilde{\mathcal{D}}(\mathcal{E}_{(\alpha)}) \subset L^2(S;\mu)$ that satisfies

$$\tilde{\mathcal{D}}(\mathcal{E}_{(\alpha)}) \subset \bigcup_{M \geq 1} \mathcal{D}(\mathcal{E}_{(\alpha)})_{D_M}; \quad (4.5)$$

for any $u \in \mathcal{FC}_0^\infty$ there exists a sequence $\{u_n\}_{n \in \mathbb{N}}$, $u_n \in \tilde{\mathcal{D}}(\mathcal{E}_{(\alpha)})$, $n \in \mathbb{N}$, such that

$$\lim_{n \rightarrow \infty} u_n = u, \quad \text{in } \mathcal{D}(\mathcal{E}_{(\alpha)}) \quad \text{with respect to the norm } \|\cdot\|_{L^2(S;\mu)} + \sqrt{\mathcal{E}_{(\alpha)}}. \quad (4.6)$$

■

5 Associated Markov processes and a standard procedure of application of stochastic quantizations on S'

Let $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, $0 < \alpha \leq 1$, be the family of strictly quasi-regular Dirichlet forms on $L^2(S;\mu)$ with a state space S defined by Theorems 2. By Theorem IV-3.5 and Proposition V-2.15 of [M,R 92] we conclude that to $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, there exists a properly associated S -valued Hunt process

$$\mathbb{M} \equiv \left(\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_{\mathbf{x}})_{\mathbf{x} \in S_\Delta} \right). \quad (5.1)$$

Δ is a point adjoined to S as an isolated point of $S_\Delta \equiv S \cup \{\Delta\}$. Let $(T_t)_{t \geq 0}$ be the strongly continuous contraction semigroup associated with $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, and $(p_t)_{t \geq 0}$ be the

corresponding transition semigroup of kernels of the Hunt process $(X_t)_{t \geq 0}$. Then for any $u \in \mathcal{FC}_0^\infty \subset \mathcal{D}(\mathcal{E}_{(\alpha)})$ the following holds:

$$\frac{d}{dt} \int_S (p_t u)(\mathbf{x}) \mu(d\mathbf{x}) = \frac{d}{dt} (T_t u, 1)_{L^2(S; \mu)} = \mathcal{E}_{(\alpha)}(T_t u, 1) = 0. \quad (5.2)$$

By this, we see that

$$\int_S (p_t u)(\mathbf{x}) \mu(d\mathbf{x}) = \int_S u(\mathbf{x}) \mu(d\mathbf{x}), \quad \forall t \geq 0, \quad \forall u \in \mathcal{FC}_0^\infty, \quad (5.3)$$

and hence,

$$\int_S P_{\mathbf{x}}(X_t \in B) \mu(d\mathbf{x}) = \mu(B), \quad \forall B \in \mathcal{B}(S). \quad (5.4)$$

Thus, we have proven the following Theorem 3.

Theorem 3 *Let $0 < \alpha \leq 1$, and let $\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)})$ be a strictly quasi-regular Dirichlet form on $L^2(S; \mu)$ that is defined through Theorem 2. Then for $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, there exists a properly associated S -valued Hunt process (cf. Definitions IV-1.5, 1.8 and 1.13 of [M,R 92] for its precise definition) $\mathbb{M}$ defined by (5.1), the invariant measure of which is μ (cf. (5.4)).* ■

We shall now present some examples.

Consider

$$H^{-1} \equiv (|x|^2 + 1)^{-\frac{d+1}{2}} (-\Delta + 1)^{-\frac{d+1}{2}} (|x|^2 + 1)^{-\frac{d+1}{2}}, \quad (5.5)$$

as a pseudo differential operators on $\mathcal{S}'(\mathbb{R}^d \rightarrow \mathbb{R}) \equiv \mathcal{S}'(\mathbb{R}^d)$, where Δ is the d -dimensional Laplace operator Δ . Let

$$\mathcal{H}_{-n} \text{ be the completion of } \mathcal{S}'(\mathbb{R}^d) \text{ with respect to the norm } \|f\|_{-n}, \quad f \in \mathcal{S}'(\mathbb{R}^d), \quad (5.6)$$

where $\|f\|_{-n}^2 = (f, f)_{-n}$ with

$$(f, g)_{-n} = ((H^{-1})^n f, (H^{-1})^n g)_{\mathcal{H}_0}, \quad f, g \in \mathcal{S}(\mathbb{R}^d). \quad (5.7)$$

Now, the restriction of H^{-1} to Borel functions in $\mathcal{H}_0 = L^2(\mathbb{R}^d \rightarrow \mathbb{R})$ is a strictly positive self-adjoint operator in $L^2(\mathbb{R}^d \rightarrow \mathbb{R})$, which is a Hilbert-Schmidt operator and thus a compact operator. By *Hilbert-Schmidt theorem* (cf., e.g., Theorem VI 16, Theorem VI 22 of [Reed, Simon 80]) we have an orthonormal base (O.N.B.) of $\mathcal{H}_0$. The spectrum of H^{-1} consists of eigenvalues $1 \geq \lambda_1 \geq \lambda_2 \geq \dots > 0$, and we have

$$\sum_{i \in \mathbb{N}} (\lambda_i)^2 < \infty, \quad \text{i.e.,} \quad \{\lambda_i\}_{i \in \mathbb{N}} \in l^2. \quad (5.8)$$

Let $\{\varphi_i\}_{i \in \mathbb{N}}$ be the system of normalized eigen functions corresponding to the eigenvalues λ_i , $i \in \mathbb{N}$ (adequately indexed corresponding to the finite multiplicity of each λ_i), which forms an O.N.B. of $\mathcal{H}_0$.

By the definition (5.6) and (5.7), for each $n \in \mathbb{N} \cup \{0\}$, we have that

$$\{(\lambda_i)^{-n} \varphi_i\}_{i \in \mathbb{N}} \text{ is an O.N.B. of } \mathcal{H}_{-n} \quad (5.9)$$

Thus, by denoting $\mathbb{Z}$ the set of integers, by the Fourier series expansion of functions in $\mathcal{H}_m$, $m \in \mathbb{Z}$ (cf. (5.6), (5.7)), such that for $f \in \mathcal{H}_m$,

$$f = \sum_{i \in \mathbb{N}} a_i(\lambda_i^m \varphi_i), \quad \text{with} \quad a_i \equiv (f, (\lambda_i^m \varphi_i))_m = \lambda_i^{-m} (f, \varphi_i)_{L^2}, \quad i \in \mathbb{N}, \quad (5.10)$$

we have an *isometric isomorphism* τ_m from $\mathcal{H}_m$ to $l^2_{(\lambda_i^{-2m})}$ defined by, for each $m \in \mathbb{Z}$

$$\tau_m : \mathcal{H}_m \ni f \longmapsto (\lambda_1^m a_1, \lambda_2^m a_2, \dots) \in l^2_{(\lambda_i^{-2m})}, \quad (5.11)$$

where $l^2_{(\lambda_i^{-2m})}$ is the weighted l^2 space defined by (2.1) with $p = 2$, and $\beta_i = \lambda_i^{-2m}$.

By making use of the results given by [Brydges,Föhlich,Sokal 83] and applying the Bochner-Minlos's Theorem the Φ_3^4 Euclidean field measure can be realized as a Borel probability measure discussed in [Brydges,Föhlich,Sokal 83] ν on $\mathcal{H}_{-3}$. We can then define a probability measure μ on $l^2_{(\lambda_i^6)}$ such that

$$\mu(B) \equiv \nu \circ \tau_{-3}^{-1}(B) \quad \text{for} \quad B \in \mathcal{B}(l^2_{(\lambda_i^6)}). \quad (5.12)$$

We set $S = l^2_{(\lambda_i^6)}$ in Theorems 1, 2 and 3, with the weight $\beta_i = \lambda_i^6$. We can take $\gamma_i^{-\frac{1}{2}} = \lambda_i^2$ in Theorem 2 with $p = 2$, then, from (5.9) we have

$$\sum_{i=1}^{\infty} \beta_i \gamma_i \cdot \mu\left(\beta_i^{\frac{1}{2}} |X_i| > M \cdot \gamma_i^{-\frac{1}{2}}\right) \leq \sum_{i=1}^{\infty} \beta_i \gamma_i = \sum_{i=1}^{\infty} (\lambda_i)^2 < \infty \quad (5.13)$$

(5.15) shows that the condition (4.3) holds.

Thus, by Theorem 2 and Theorem 4, for each $0 < \alpha \leq 1$, there exists an $l^2_{(\lambda_i^6)}$ -valued Hunt process

$$\mathbb{M} \equiv (\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_{\mathbf{x}})_{\mathbf{x} \in S_{\Delta}}), \quad (5.14)$$

associated to the non-local Dirichlet form $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$. We can then define an $\mathcal{H}_{-3}$ -valued process $(Y_t)_{t \geq 0}$ such that $(Y_t)_{t \geq 0} \equiv (\tau_{-2}^{-1}(X_t))_{t \geq 0}$.

Equivalently, by (5.13) for $X_t = (X_1(t), X_2(t), \dots) \in l^2_{(\lambda_i^6)}$, $P_{\mathbf{x}} - a.e.$, by setting $A_i(t)$ such that $A_i(t) = \lambda_i^3 X_i(t)$ (cf. (5.11) and (5.12)), then Y_t is given by

$$Y_t = \sum_{i \in \mathbb{N}} A_i(t) (\lambda_i^{-3} \varphi_i) = \sum_{i \in \mathbb{N}} X_i(t) \varphi_i \in \mathcal{H}_{-3}, \quad \forall t \geq 0, \quad P_{\mathbf{x}} - a.e.. \quad (5.15)$$

By (5.4) and (5.13), it is an $\mathcal{H}_{-3}$ -valued Hunt process that can be looked upon a *stochastic quantization* with respect to the non-local Dirichlet form $(\tilde{\mathcal{E}}_{(\alpha)}, \mathcal{D}(\tilde{\mathcal{E}}_{(\alpha)}))$ on $L^2(\mathcal{H}_{-3}, \nu)$, that is defined through $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, by making use of τ_{-3} . See [A,Kagawa,Yahagi,Y 2018] for more details.

References

- [A,H-K 76] Albeverio, S., Høegh-Krohn, R., Quasi invariant measures, symmetric diffusion processes and quantum fields. Les méthodes mathématiques de la théorie quantique des champs. (*Colloq. Internat. CNRS, No. 248, Marseille, 1975*) Éditions Centre Nat. Recherche Sci., Paris (1976), 11-59.
- [A,H-K 77] Albeverio, S., Høegh-Krohn, R., Dirichlet forms and diffusion processes on rigged Hilbert spaces. *Z. Wahrscheinlichkeitstheor. Verv. Geb.* **40** (1977), 1-57.
- [A,Kagawa,Yahagi,Y 2018] Albeverio, S., Kagawa, T., Yahagi, Y., Yoshida, M.W., Non-local Markovian symmetric forms on infinite dimensional spaces, part 1, The closability and quasi-regularity. (2018) Pre-print.
- [A,Ma,R 2015] Albeverio, S., Ma, Z. M., Röckner, M., Quasi regular Dirichlet forms and the stochastic quantization problem. Festschrift Masatoshi Fukushima, *Interdiscip. Math. Sci.*, **17** (2015), 27-58, World Sci. Publ., Hackensack, NJ.
- [A,R 89] Albeverio, S., Röckner, M., Classical Dirichlet forms on topological vector spaces- the construction of the associated diffusion processes, *Probab. Theory Related Fields* **83** (1989), 405-434.
- [A,R 90] Albeverio, S., Röckner, M., Classical Dirichlet forms on topological vector spaces- closability and a Cameron-Martin formula, *J. Functional Analysis* **88** (1990), 395-43.
- [A,R 91] Albeverio, S., Röckner, M., Stochastic differential equations in infinite dimensions: solution via Dirichlet forms, *Probab. Theory Related Fields* **89** (1991), 347-386.
- [A,Y 2018] Albeverio, S., Yoshida, M.W., Non-local Dirichlet forms on infinite dimensional topological vector spaces. (2018) Pre-print.
- [Brydges,Fröhlich,Sokal 83] Brydges, D., Fröhlich, J., Sokal, A., A New proof of the existence and non triviality of the continuum φ_2^4 and φ_3^4 quantum field theories, *Comm. Math. Phys.* **91** (1983), 141-186.
- [Fukushima 80] Fukushima, M., *Dirichlet forms and Markov processes*, North-Holland Mathematical Library, **23**, North-Holland Publishing Co., Amsterdam-New York, 1980.
- [F,Oshima,Takeda 2011] Fukushima, M., Oshima, Y., Takeda, M., *Dirichlet Forms and Symmetric Markov Processes, second revised and extended edition*, de Gruyter, Berlin, 2011.
- [F,Uemura 2012] Fukushima, M., Uemura, T., Jump-type Hunt processes generated by lower bounded semi- Dirichlet forms, *Ann. Probab.* **40** (2012), 858-889
- [Reed,Simon 80] Reed, M., Simon, B., *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, 1978.
- [Hida 80] Hida, T., *Brownian motion*, Springer-Verlag, New York Heidelberg Berlin 1980.
- [M,R 92] Ma, Z. M., Röckner, M., *Introduction to the theory of (Non-Symmetric) Dirichlet Forms*, Springer-Verlag, Berlin, 1992.