

ON THE AUTOMORPHISM GROUP OF THE SUBGROUP LATTICE OF A FINITE ABELIAN p -GROUP; SOME GENERALIZATIONS

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ABSTRACT. The automorphism group $\text{Aut } \mathcal{L}(M)$ of the submodule lattice $\mathcal{L}(M)$ of a finite-length module M over complete discrete valuation ring $\mathfrak{o}$ is studied. Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be the type of M . We show that for those M with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$, $\text{Aut } \mathcal{L}(M)$ can be analyzed by computing a certain subgroup of the bijections on a quotient of the scalar ring $\mathfrak{o}$. In particular, when the residue field $k = \mathfrak{o}/\mathfrak{p}$ is a finite field $\mathbb{F}_q$, we compute the order of the group.

1. OBJECTIVE

Let $\mathfrak{o}$ be a discrete valuation ring with the maximal ideal $\mathfrak{p}$, a prime element π (i.e., $\mathfrak{o}\pi = \mathfrak{p}$) and the valuation function $v : \mathfrak{o} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$. Let $k \cong \mathfrak{o}/\mathfrak{p}$ denote the residue field. Let M be an $\mathfrak{o}$ -module of finite length. Then, since $\mathfrak{o}$ is a principal ideal domain, M can be written as a sum of cyclic $\mathfrak{o}$ -submodules:

$$M \cong \mathfrak{o}/\mathfrak{p}^{\lambda_1} \oplus \dots \oplus \mathfrak{o}/\mathfrak{p}^{\lambda_l},$$

with $\lambda = (\lambda_1, \dots, \lambda_l)$ being some partition of a non-negative integer (That is, we have $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l$). λ is called the *type* of M . Now since we have $\mathfrak{o}/\mathfrak{p}^i \cong \bar{\mathfrak{o}}/\bar{\mathfrak{p}}^i$ where $\bar{\mathfrak{o}}$ is the completion of $\mathfrak{o}$ and $\bar{\mathfrak{p}}$ its maximal ideal, without loss of generality we can assume $\mathfrak{o}$ to be complete. Let $\mathcal{L}(M)$ denote the set of $\mathfrak{o}$ -submodules of M . $\mathcal{L}(M)$ inherits a lattice structure by inclusion relation. Our main objective is to compute $\text{Aut } \mathcal{L}(M)$, the automorphism group of the lattice $\mathcal{L}(M)$, for such λ as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$.

When $\mathfrak{o} = \mathbb{Z}_p$, the ring of p -adic integers, M becomes nothing but a finite abelian p -group and $\mathcal{L}(M)$ the subgroup lattice of M . This can be generalized by considering the case $\mathfrak{o} = W[\mathbb{F}_q]$, the ring of Witt vectors over the finite field $\mathbb{F}_q$, for $W[\mathbb{F}_p] \cong \mathbb{Z}_p$. Another example of $\mathfrak{o}$ is the ring $k[[t]]$ of formal power series in one variable t .

We call $e = (e_1, \dots, e_l) \in M^l$ an *ordered basis* for M if $M = \bigoplus_{i=1}^l \mathfrak{o}e_i$ and $\mathfrak{o}e_i \cong \mathfrak{o}/\mathfrak{p}^{\lambda_i}$. Let e be fixed. We denote by $R(e)$ the set of $\varphi \in \text{Aut } \mathcal{L}(M)$ satisfying

$$\begin{cases} \varphi(\mathfrak{o}e_i) = \mathfrak{o}e_i & \forall i \in [1, l] \\ \varphi(\mathfrak{o}(e_1 + e_i)) = \mathfrak{o}(e_1 + e_i) & \forall i \in [2, l] \end{cases}$$

In most cases it boils down to computing $R(e)$ in order to analyze $\text{Aut } \mathcal{L}(M)$, in the sense we describe as follows.

Since an automorphism of $\mathfrak{o}$ -module M induces an automorphism of the lattice $\mathcal{L}(M)$, we have the natural group homomorphism

$$\Theta : \text{Aut } M \rightarrow \text{Aut } \mathcal{L}(M).$$

It can be directly checked that $\text{Ker } \Theta \cong (\mathfrak{o}/\mathfrak{p}^{\lambda_1})^\times$ and that $\text{Aut } M$ can be expressed in matrix form, as described in the sequel. Naturally $\text{Aut } \mathcal{L}(M)$ contains a subgroup isomorphic to $\text{Aut } M / \text{Ker } \Theta$, and we let $\text{PAut } M$ denote this subgroup.

It turns out that $\text{Aut } \mathcal{L}(M)$ is a product of these two subgroups $R(e)$ and $\text{PAut } M$. Namely, we have

Lemma 1.

$$\begin{aligned} R(e) \cdot \text{PAut } M &= \text{Aut } \mathcal{L}(M) \\ R(e) \cap \text{PAut } M &= 1. \end{aligned}$$

Also, we remark that if e and e' are ordered base for M , then it is easily checked that $\varphi R(e)\varphi^{-1} = R(e')$, where $\varphi \in \text{PAut } M$ is the lattice automorphism induced by the module automorphism of M defined by $e_i \mapsto e'_i$ ($1 \leq i \leq l$). Hence the isomorphism type of $R(e)$ does not depend on the choice of e . We content ourselves with computing $R(e)$ instead of computing $\text{Aut } \mathcal{L}(M)$ for our purpose.

2. HISTORICAL BACKGROUND

Let us mention the relation with earlier results. The structure of $\text{Aut } \mathcal{L}(M)$ is well-known for the case $\lambda_1 = \lambda_2 = \lambda_3$, which is essentially the result of Baer [2]. In this case, we have $\text{Aut } \mathcal{L}(M) \cong R(e) \rtimes \text{PAut } M$, and

$$R(e) \cong \text{Aut } \mathfrak{o}/\mathfrak{p}^{\lambda_3},$$

where $\text{Aut } \mathfrak{o}/\mathfrak{p}^{\lambda_3}$ is the group of automorphisms of ring $\mathfrak{o}/\mathfrak{p}^{\lambda_3}$. In particular, when $\lambda_1 = \dots = \lambda_l = 1$ ($l \geq 3$), M becomes a vector space over the residue field k of $\mathfrak{o}$, and $\text{Aut } \mathcal{L}(M)$ is isomorphic to $PGL(l, k)$, the group of projective semi-linear automorphisms. This result is a variation of so called *the Fundamental Theorem of Finite Projective Geometry*.

We next consider the case when the residue field of $\mathfrak{o}$ is the finite field $\mathbb{F}_p$. Let $M = \mathfrak{o}/\mathfrak{p} \oplus \mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_p \oplus \mathbb{F}_p$. Then $\text{Aut } \mathcal{L}(M)$ is isomorphic to the symmetric group $\mathfrak{S}_{p+1}$ and $\text{PAut } M$ isomorphic to the projective general linear group $PGL(2, p)$ (Note that $|PGL(2, p)| = (p+1)p(p-1)$). In this case, $R(e)$ is a subgroup that fixes three points and isomorphic to $\mathfrak{S}_{p-2}$. More generally, for $M = \mathbb{Z}_p/p^{\lambda_2}\mathbb{Z}_p \oplus \mathbb{Z}_p/p^{\lambda_2}\mathbb{Z}_p$ ($\mathfrak{o} = \mathbb{Z}_p$ is the ring of p -adic integers), Holmes' result [5] states that $\text{Aut } \mathcal{L}(M)$ is isomorphic to $\mathfrak{S}_p^{(\lambda_2-1)} \wr \mathfrak{S}_{p+1}$, where $\mathfrak{S}_p^{(n)}$ means $\mathfrak{S}_p \wr \dots \wr \mathfrak{S}_p$ (n times) and $\wr$ denotes the standard wreath product. In this case, $\text{PAut } M$ is nothing but $PGL_2(\mathbb{Z}_p/p^{\lambda_2}\mathbb{Z}_p)$, and we note that $|PGL_2(\mathbb{Z}_p/p^{\lambda_2}\mathbb{Z}_p)| = (p+1)p(p-1) \cdot (p^{\lambda_2-1})^3$. $R(e)$ is the subgroup that fixes three points $\mathbb{Z}_p(1, 0)$, $\mathbb{Z}_p(0, 1)$ and $\mathbb{Z}_p(1, 1)$; in fact, we have

$$R(e) \cong (\mathfrak{S}_p^{(\lambda_2-1)} \wr \mathfrak{S}_{p-2}) \times \left\{ \prod_{i=0}^{\lambda_2-2} (\mathfrak{S}_p^{(i)} \wr \mathfrak{S}_{p-1}) \right\}^3.$$

Holmes [5] also obtains a result for the case $\lambda_1 > \lambda_2 > \lambda_3 = 0$: $\text{Aut } \mathcal{L}(M) \cong G^2 \times H^{\lambda_1-\lambda_2-1}$, where $G = \mathfrak{S}_p^{\lambda_2}$ and $H = \mathfrak{S}_p^{(\lambda_2-1)} \wr \mathfrak{S}_{p-1}$.

There have been works to bridge the gap between Baer's result and Holmes'. Costantini-Holmes-Zacher[3] and Costantini-Zacher[4] treated the case of abelian p -groups in a rather general framework. Yasuda[11] studied the case of finite abelian p -groups for $\lambda_1 > \lambda_2 = \lambda_3$ with explicit computation of $R(e)$ and $\text{Aut } \mathcal{L}(M)$. In this work, we shall treat the case $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$, in the general setting of finite-length modules over (complete) discrete valuation ring.

3. NOTATIONS AND NOTIONS

Here we give some supplementary definitions and notations. Put $\mathfrak{q}_i = \mathfrak{p}^i \setminus \mathfrak{p}^{i+1}$ for $i \geq 1$; i.e., $\mathfrak{q}_i = \{a \in \mathfrak{o} \mid v(a) = i\}$. We define $\mathfrak{q}_0 = \mathfrak{o} \setminus \mathfrak{p} = \mathfrak{o}^\times$, the set of invertible elements. For $a, b \in \mathfrak{o}$ such that $v(a) \geq v(b)$ ($b \neq 0$), there exists an element $x \in \mathfrak{o}$ such that $a = xb$. As $\mathfrak{o}$ is a domain, x must be unique. We use the notation $\frac{a}{b} = x$.

Given a set X , $\text{Map}(X)$ denotes the set of maps $f : X \rightarrow X$. $\text{Sym}(X)$ denotes the set of bijections $f : X \rightarrow X$. $\text{Map}(X)$ forms a monoid with respect to function composition, whereas $\text{Sym}(X)$ forms a group. Given two sets X and Y , we define Y^X to be the set of maps $f : X \rightarrow Y$.

Let G be a group, and H a group acting on a set X . Let $f, g : X \rightarrow G$ be two maps, and define a map $f \circ g : X \rightarrow G$ by $f \circ g(x) = f(x) \cdot g(x)$ where $\cdot$ is the product in G . Then G^X becomes a group with respect to this $\circ$. Let $h \in H$ and $f \in G^X$. We define a semidirect product $G^X \rtimes H$ with respect to the group homomorphism $H \rightarrow \text{Aut } G^X$ ($h \mapsto (f \mapsto fh^{-1})$). We write $G \wr H$ to denote this semidirect product, and call it the *wreath product* of G and H .

We now give the description of the automorphism group $\text{Aut } M$ of an $\mathfrak{o}$ -module M in matrix form, as promised. Let e be fixed. The action of $f \in \text{Aut } M$ is then determined by its action on $e = (e_1, \dots, e_l)$. Write

$$f(e_j) = \sum_{i=1}^l a_{ij} e_i$$

and express f as the matrix $(a_{ij})_{i,j=1}^l$. Rewriting $\lambda = (\lambda_1, \dots, \lambda_l) = \langle d_1^{m_1}, \dots, d_r^{m_r} \rangle$ ($d_1 > \dots > d_r$) (this means that λ contains m_r -many components equal to d_r), $\text{Aut } M$ can be expressed in matrix form as

$$\begin{pmatrix} GL_{m_1}(\mathfrak{o}/\mathfrak{p}^{d_1}) & \dots & \text{Hom}((\mathfrak{o}/\mathfrak{p}^{d_r})^{\oplus m_r}, (\mathfrak{o}/\mathfrak{p}^{d_1})^{\oplus m_1}) \\ \vdots & \ddots & \vdots \\ \text{Hom}((\mathfrak{o}/\mathfrak{p}^{d_1})^{\oplus m_1}, (\mathfrak{o}/\mathfrak{p}^{d_r})^{\oplus m_r}) & \dots & GL_{m_r}(\mathfrak{o}/\mathfrak{p}^{d_r}) \end{pmatrix},$$

with respect to the ordered basis e . Here, the block matrix in the diagonal

$$A \in GL_{m_i}(\mathfrak{o}/\mathfrak{p}^{d_i})$$

is of size $m_i \times m_i$ and has elements of $\mathfrak{o}/\mathfrak{p}^{d_i}$ in its components, satisfying $\pi \nmid \det A$. Also, the block matrix at (i, j) -position ($i \neq j$)

$$A \in \text{Hom}((\mathfrak{o}/\mathfrak{p}^{d_j})^{\oplus m_j}, (\mathfrak{o}/\mathfrak{p}^{d_i})^{\oplus m_i})$$

is of size $m_i \times m_j$ and in its components has elements of $\mathfrak{p}^{d_i - \min(d_j, d_i)}(\mathfrak{o}/\mathfrak{p}^{d_i})$, that is, for $i < j$ ($\implies d_i > d_j$) elements of $\mathfrak{p}^{d_i - d_j}(\mathfrak{o}/\mathfrak{p}^{d_i})$, and for $i > j$ ($\implies d_i < d_j$) elements of $\mathfrak{o}/\mathfrak{p}^{d_i}$.

4. MAIN RESULTS

For the case $\lambda_1 > \lambda_2 = \lambda_3$, we can state our main result as follows:

Theorem 2. Assume $\lambda_2 = \lambda_3$. Then $R(e)$ contains a normal subgroup N such that

$$R(e)/N \cong \text{Aut } \mathfrak{o}/\mathfrak{p}^{\lambda_3},$$

$$N \cong \begin{cases} k^{\lambda_1 - \lambda_2} & \lambda_2 = \lambda_3 > 2, \\ (k^\times)^{\lambda_1 - \lambda_2} & \lambda_2 = \lambda_3 = 1. \end{cases}$$

The case $\lambda_1 \geq \lambda_2 > \lambda_3$ turns out to be rather complicated. The rest of this section is dedicated to explain our main result for this case.

Let $i \geq 1$. For $a, b \in \mathfrak{o}$, we write

$$a \equiv b \pmod{\mathfrak{p}^i}$$

to mean $a - b \in \mathfrak{p}^i$. With abuse of notation, we write $\mathfrak{p}^i$ also to denote this equivalence relation. Then obviously we have $\mathfrak{p} \supset \mathfrak{p}^2 \supset \mathfrak{p}^3 \supset \dots$. On the other hand, put $u_i = 1 + \mathfrak{p}^i \subset \mathfrak{o}$ ($i \geq 1$). For $a, b \in \mathfrak{o}$, write

$$a \sim b \pmod{u_i}$$

if $a \in u_i b$. Clearly this defines an equivalence relation on $\mathfrak{o}$. Again with abuse of notation, we just write u_i to denote this relation. Then note that we have $u_1 \supset u_2 \supset u_3 \supset \dots$. Also note that $\mathfrak{p}^i \supset u_i$ holds for all $i \geq 1$.

Lemma 3. The union of relations $\mathfrak{p}^i \cup u_j$ is an equivalence relation for all $i, j \geq 1$.

Hence we have $\mathfrak{p}^i \vee u_j = \mathfrak{p}^i \cup u_j$, and it makes sense to denote the quotient set by $\mathfrak{o}/\mathfrak{p}^i/u_j = \mathfrak{o}/u_j/\mathfrak{p}^i = \mathfrak{o}/\mathfrak{p}^i \vee u_j$ for all $i, j \geq 1$.

Now we proceed to the following lemma:

Lemma 4. Let $\varphi \in R(e)$ be given. There exist bijective maps $\tau : \mathfrak{o} \rightarrow \mathfrak{o}$ and $\sigma : \mathfrak{o} \rightarrow \mathfrak{o}$ such that $\varphi(ae_1 + e_2) = \mathfrak{o}(\tau(a)e_1 + e_2)$ and $\varphi\mathfrak{o}(e_1 + ae_2) = \mathfrak{o}(e_1 + \sigma(a)e_2)$ for all $a \in \mathfrak{o}$. τ and σ induce bijections $\tau : \mathfrak{o}/\mathfrak{p}^{\lambda_1}/u_{\lambda_2} \rightarrow \mathfrak{o}/\mathfrak{p}^{\lambda_1}/u_{\lambda_2}$ and $\sigma : \mathfrak{o}/\mathfrak{p}^{\lambda_2} \rightarrow \mathfrak{o}/\mathfrak{p}^{\lambda_2}$, respectively, which are uniquely determined by φ .

Let $\varphi \in R(e)$ be given and τ, σ as in the preceding lemma. We list in the following lemma some of the properties satisfied by τ and σ .

Lemma 5. We have

- (1): $\tau(1) \sim 1 \pmod{\mathfrak{p}^{\lambda_1} \vee u_{\lambda_2}}$, $\sigma(1) \equiv 1 \pmod{\mathfrak{p}^{\lambda_2}}$,
- (2): $\tau(\mathfrak{p}) \subset \mathfrak{p}$, $\sigma(\mathfrak{p}) \subset \mathfrak{p}$,
- (3): $\tau(ab) \sim \tau(a)\tau(b) \pmod{\mathfrak{p}^{\lambda_1} \vee u_{\lambda_3}}$ for all $a, b \in \mathfrak{o}$,
- (4): $\sigma(ab) \sim \sigma(a)\sigma(b) \pmod{\mathfrak{p}^{\lambda_2} \vee u_{\lambda_3}}$ for all $a, b \in \mathfrak{o}$,
- (5): $\tau(a - b) \sim \tau(a) - \tau(b) \pmod{\mathfrak{p}^{\lambda_1} \vee \mathfrak{p}^{\lambda_2 + v(b)} \vee u_{\lambda_3}}$ for all $a, b \in \mathfrak{o}$,

- (6): $\sigma(a - b) \sim \sigma(a) - \sigma(b) \bmod \mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3}$ for all $a, b \in \mathfrak{o}$,
 (7): $\tau(a) \equiv \sigma(a^{-1})^{-1} \bmod \mathfrak{p}^{\lambda_2}$ for all $a \in \mathfrak{o}^\times$,
 (8): $\tau(a) \sim \sigma(a) \bmod \mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3}$ for all $a \in \mathfrak{o}$.

Given three positive integers $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$, let

$$\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})$$

denote the set of bijections $\tau : \mathfrak{o} \rightarrow \mathfrak{o}$ that satisfy the following three conditions:

Valuation law: $\tau(\mathfrak{p}) \subset \mathfrak{p}$,

Strict product law: $\tau(ab) \sim \tau(a)\tau(b) \bmod \mathfrak{p}^{\lambda_1} \vee \mathfrak{u}_{\lambda_3}$ for all $a, b \in \mathfrak{o}$,

Difference law: $\tau(a - b) \sim \tau(a) - \tau(b) \bmod \mathfrak{p}^{\lambda_1} \vee \mathfrak{p}^{\lambda_2 + v(b)} \vee \mathfrak{u}_{\lambda_3}$ for all $a, b \in \mathfrak{o}$.

In this section we shall prove that this set forms a group and a $\tau \in \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})$ induces a bijection $\tau : \mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2} \rightarrow \mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}$. It turns out that $R(e)$ can be described by using this group.

Lemma 6. *Let $\tau : \mathfrak{o} \rightarrow \mathfrak{o}$ be a bijective map that satisfies the valuation law and the strict product law. Then we have*

$$\tau(\mathfrak{p}^i) = \mathfrak{p}^i$$

for all $i \in [0, \lambda_1]$; that is, we have

$$v(\tau(a)) = v(a)$$

for all $a \in \mathfrak{o} \setminus \mathfrak{p}^{\lambda_1}$.

Lemma 7. *Let $i \leq \lambda_1$ and $j \leq \lambda_2$. Let $\tau : \mathfrak{o} \rightarrow \mathfrak{o}$ be a bijective map that satisfies the difference law and the condition $v(\tau(a)) = v(a)$ for all $a \in \mathfrak{o} \setminus \mathfrak{p}^{\lambda_1}$. For $a, b \in \mathfrak{o}$, we have $a \sim b \bmod \mathfrak{p}^i \vee \mathfrak{u}_j$ if and only if $\tau(a) \sim \tau(b) \bmod \mathfrak{p}^i \vee \mathfrak{u}_j$. That is, there exists a unique bijective map $\bar{\tau}$ that makes the diagram below commutative:*

$$\begin{array}{ccc} \mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2} & \xrightarrow{\tau} & \mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2} \\ \downarrow & & \downarrow \\ \mathfrak{o}/\mathfrak{p}^i/\mathfrak{u}_j & \xrightarrow{\bar{\tau}} & \mathfrak{o}/\mathfrak{p}^i/\mathfrak{u}_j \end{array}$$

Proposition 8. $\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})$ forms a subgroup of $\text{Sym}(\mathfrak{o})$.

Now denote by $\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}}$ the stabilizer of $\mathfrak{u}_{\lambda_2}$ in $\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})$. That is,

$$\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}} = \{\tau \in \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o}) \mid \tau(\mathfrak{u}_{\lambda_2}) = \mathfrak{u}_{\lambda_2}\}.$$

Then define

$$\Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}} \times \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}})$$

to be the set of $(\tau, \sigma) \in \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}} \times \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}}$ satisfying the conditions

$$\begin{cases} \tau(a)^{-1} \equiv \sigma(a^{-1}) \bmod \mathfrak{p}^{\lambda_2} & \forall a \in \mathfrak{o}^\times, \\ \tau(a) \sim \sigma(a) \bmod \mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3} & \forall a \in \mathfrak{o}. \end{cases}$$

Note that since $\tau(a)\tau(a^{-1}) \sim 1 \bmod \mathfrak{p}^{\lambda_1} \vee \mathfrak{u}_{\lambda_3}$ whence $\tau(a)^{-1} \sim \tau(a^{-1}) \bmod \mathfrak{p}^{\lambda_1} \vee \mathfrak{u}_{\lambda_3}$ for all $a \in \mathfrak{o}^\times$, the first condition $\tau(a)^{-1} \equiv \sigma(a^{-1}) \bmod \mathfrak{p}^{\lambda_2}$ implies the second condition $\tau(a) \sim \sigma(a) \bmod \mathfrak{p}^{\lambda_1} \vee \mathfrak{u}_{\lambda_3}$.

Lemma 9. *The set*

$$\Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}} \times \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}})$$

forms a subgroup of $\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}} \times \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}}$.

We have observed that $\tau \in \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})$ induces a bijective map $\tau : \mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2} \rightarrow \mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}$. That is to say, there exists a natural group homomorphism $\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o}) \rightarrow \text{Sym}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})$. Let us define

$$\begin{aligned} &\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}) \\ &\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2}) \end{aligned}$$

to be the images of $\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o}) \rightarrow \text{Sym}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})$ and $\text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(\mathfrak{o}) \rightarrow \text{Sym}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})$, respectively. Furthermore, let

$$\begin{aligned} & \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \\ & \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1 \end{aligned}$$

be the subgroups of the above two, corresponding to $\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}}$ and $\text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}}$, respectively. Lastly, we denote by

$$\Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1)$$

the subgroup of $\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1$ that corresponds to $\Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}} \times \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}})$. Now we can state our:

Theorem 10 (Main Isomorphism Theorem). *We have*

$$R(e) \cong \Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1)$$

if $\lambda_2 > \lambda_3$.

Note that one way of isomorphism

$$\Phi : R(e) \rightarrow \Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1)$$

is already given, by sending $\Phi : \varphi \mapsto (\tau, \sigma)$. In order to compute $R(e)$, this theorem allows us to compute $\Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1)$ instead. Let

$$\Lambda : \Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1) \rightarrow \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1$$

be the "projection" map to the first component; i.e., $\Lambda : (\tau, \sigma) \mapsto \tau$. Then $\text{Ker } \Lambda$ is the set of $(1, \sigma)$ satisfying

$$\begin{cases} \sigma(a) \equiv a \pmod{\mathfrak{p}^{\lambda_2}} & a \in \mathfrak{o}^\times/\mathfrak{p}^{\lambda_2}, \\ \sigma(a) \sim a \pmod{\mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3}} & a \in \mathfrak{p}/\mathfrak{p}^{\lambda_2}. \end{cases}$$

Let K be the kernel of the natural map $\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1 \rightarrow \mathfrak{o}^\times/\mathfrak{p}^{\lambda_2}$, that is,

$$K = \{\sigma \in \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1 \mid \sigma(a) = a \text{ for all } a \in \mathfrak{o}^\times/\mathfrak{p}^{\lambda_2}\}.$$

Lemma 11. *We have*

$$\text{Ker } \Lambda \cong K.$$

We shall show that the group in question, $\Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1)$, is isomorphic to a semidirect product of K and the first component $\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1$:

Proposition 12. *The sequence*

$$1 \rightarrow K \rightarrow \Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1) \rightarrow \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \rightarrow 1$$

is exact and splitting. In other words, we have

$$\Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1) \cong \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \ltimes K.$$

This result divides our investigation into two parts: the analysis of the structure of K and that of $\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1$. We begin with the former.

Recall that

$$\begin{aligned} K &= \{\sigma \in \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1 \mid \sigma(a) = a \ \forall a \in (\mathfrak{o}^\times/\mathfrak{p}^{\lambda_2})\} \\ &= \{\sigma \in \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1 \mid \sigma(a) = a \ \forall a \in (\mathfrak{o}^\times/\mathfrak{p}^{\lambda_2}) \text{ and } \sigma(a) \sim a \pmod{\mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3}} \ \forall a \in \mathfrak{p}/\mathfrak{p}^{\lambda_2}\}. \end{aligned}$$

For the sake of convenience, we shall analyze groups slightly larger than K ; namely,

$$\tilde{K} = \{\sigma \in \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1 \mid \sigma(a) \sim a \pmod{\mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3}} \ \forall a \in \mathfrak{o}/\mathfrak{p}^{\lambda_2}\},$$

and

$$\tilde{K}_1 = \{\sigma \in \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_2})_1 \mid \sigma(a) \sim a \pmod{\mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3}} \ \forall a \in \mathfrak{o}/\mathfrak{p}^{\lambda_2}\}.$$

Of course we have $\tilde{K}_1 = \{\sigma \in \tilde{K} \mid \sigma(1) = 1\}$.

Proposition 13. $\tilde{K}$ decomposes into a direct product as

$$\tilde{K} \cong Q_0 \times Q_1 \times \cdots \times Q_{\lambda_2 - \lambda_3 - 1}$$

where each factor Q_i (defined for $0 \leq i \leq \lambda_2 - 1$) is given by

$$Q_i = \left\{ \sigma \in \tilde{K} \mid \sigma(a) = a \text{ for all } a \in (\mathfrak{o} \setminus \mathfrak{q}_i) / \mathfrak{p}^{\lambda_2} \right\}.$$

Corollary 14. $\tilde{K}_1$ and K decompose into direct products as

$$\begin{aligned} \tilde{K}_1 &\cong \bar{Q}_0 \times Q_1 \times \cdots \times Q_{\lambda_2 - \lambda_3 - 1} \text{ and} \\ K &\cong Q_1 \times Q_2 \times \cdots \times Q_{\lambda_2 - \lambda_3 - 1}, \end{aligned}$$

respectively, where

$$\bar{Q}_0 = \left\{ \sigma \in \tilde{K}_1 \mid \sigma(a) = a \ \forall a \in \mathfrak{p} / \mathfrak{p}^{\lambda_2} \right\} = \left\{ \sigma \in Q_0 \mid \sigma(1) = 1 \right\}.$$

We now focus on the calculation of each Q_i . First, we give a description of generators of Q_i . We begin with the following lemma.

Lemma 15. Let $\sigma \in \tilde{K}$ and $0 \leq j \leq \lambda_2 - \lambda_3$. If $a \equiv b \pmod{\mathfrak{p}^j}$, then $\sigma(a) - a \equiv \sigma(b) - b \pmod{\mathfrak{p}^{j+\lambda_3}}$.

We apply this lemma particularly to Q_i ($0 \leq i \leq \lambda_2 - \lambda_3 - 1$). For $j \in [i+1, \lambda_2 - \lambda_3]$, let $(\mathfrak{p}^{j+\lambda_3-1} / \mathfrak{p}^{\lambda_2})^{\mathfrak{q}_i / \mathfrak{p}^j}$ denote the set of maps $z : \mathfrak{q}_i / \mathfrak{p}^j \rightarrow \mathfrak{p}^{j+\lambda_3-1} / \mathfrak{p}^{\lambda_2}$. Note that since $\mathfrak{p}^{j+\lambda_3-1} / \mathfrak{p}^{\lambda_2}$ is an abelian group, so becomes $(\mathfrak{p}^{j+\lambda_3-1} / \mathfrak{p}^{\lambda_2})^{\mathfrak{q}_i / \mathfrak{p}^j}$ naturally. Given $j \in [i+1, \lambda_2 - \lambda_3]$ and $z : \mathfrak{q}_i / \mathfrak{p}^j \rightarrow \mathfrak{p}^{j+\lambda_3-1} / \mathfrak{p}^{\lambda_2}$, define a map $g_{j,z} : \mathfrak{o} / \mathfrak{p}^{\lambda_2} \rightarrow \mathfrak{o} / \mathfrak{p}^{\lambda_2}$ by

$$g_{j,z}(a) = \begin{cases} a + z(a \bmod \mathfrak{p}^j) & a \in \mathfrak{q}_i, \\ a & \text{otherwise.} \end{cases}$$

We shall show that $g_{j,z}$ is in Q_i ; more precisely,

Proposition 16. We have

$$Q_i = \langle g_{j,z} \rangle_{\substack{j \in [i+1, \lambda_2 - \lambda_3] \\ z \in (\mathfrak{p}^{j+\lambda_3-1} / \mathfrak{p}^{\lambda_2})^{\mathfrak{q}_i / \mathfrak{p}^j}}} = \langle g_{j,z} \rangle_{\substack{j \in [i+1, \lambda_2 - \lambda_3] \\ z \in (S\pi^{j+\lambda_3-1})^{\mathfrak{q}_i / \mathfrak{p}^j}}},$$

where $(S\pi^{j+\lambda_3-1})^{\mathfrak{q}_i / \mathfrak{p}^j}$ denotes the set of maps $z : \mathfrak{q}_i / \mathfrak{p}^j \rightarrow S\pi^{j+\lambda_3-1} \subset \mathfrak{p}^{j+\lambda_3-1} / \mathfrak{p}^{\lambda_2}$.

Using these generators, we give two sorts of descriptions of Q_i . The former turns out to be useful particularly for the case $\lambda_3 = 1$, whereas the latter being useful for the case $\lambda_3 \geq \frac{1}{2}\lambda_2$.

For $j \in [i+1, \lambda_2 - \lambda_3 + 1]$, define

$$L_i^{(j)} = \left\{ \sigma \in Q_i \mid \sigma(a) \equiv a \pmod{\mathfrak{p}^{j+\lambda_3-1}} \ \forall a \in \mathfrak{o} / \mathfrak{p}^{\lambda_2} \right\}.$$

Then clearly we have $Q_i \supset L_i^{(j)}$ for each j whence obtain a chain of normal subgroups $L_i^{(j)}$.

Proposition 17. We have a normal series of Q_i :

$$Q_i = L_i^{(i+1)} \supset L_i^{(i+2)} \supset \cdots \supset L_i^{(\lambda_2 - \lambda_3 + 1)} = 1,$$

where the factors of the series are given by

$$L_i^{(j)} / L_i^{(j+1)} \cong (\mathfrak{o} / \mathfrak{p})^{\mathfrak{q}_i / \mathfrak{p}^j},$$

for all $j \in [i+1, \lambda_2 - \lambda_3]$.

Proposition 18. We have

$$L_i^{(j)} = \langle g_{n,z} \rangle_{\substack{n \in [j, \lambda_2 - \lambda_3] \\ z \in (\mathfrak{p}^{n+\lambda_3-1} / \mathfrak{p}^{\lambda_2})^{\mathfrak{q}_i / \mathfrak{p}^n}}} = \langle g_{n,z} \rangle_{\substack{n \in [j, \lambda_2 - \lambda_3] \\ z \in (S\pi^{n+\lambda_3-1})^{\mathfrak{q}_i / \mathfrak{p}^n}}}.$$

Lemma 19. We have the exact sequence of groups

$$1 \rightarrow L_i^{(\lambda_2 - \lambda_3)} \rightarrow Q_i \rightarrow Q_{i+1} \rightarrow 1.$$

This sequence splits if $\lambda_3 = 1$.

Proposition 20. If $\lambda_3 = 1$, then Q_i decomposes into a wreath product as

$$Q_i \simeq \underbrace{k \wr k \wr \cdots \wr k}_{\lambda_2 - i - 1} \wr 1$$

where 1 is to act on $k^\times$ trivially and k on k by addition. To put it more concisely,

$$Q_i \simeq (k^{l(\lambda_2 - i - 1)})^{k^\times}.$$

We present another way of describing the structure of Q_i . In order to do this, let us define

$$U_i^{(j)} = \{ \sigma \in Q_i \mid \sigma(a) - a \equiv \sigma(b) - b \pmod{\mathfrak{p}^{\lambda_2}} \text{ if } a \equiv b \pmod{\mathfrak{p}^j} \},$$

where $i \leq j \leq \lambda_2 - \lambda_3$. This gives us a filtration of Q_i :

$$Q_i = U_i^{(\lambda_2 - \lambda_3)} \supset U_i^{(\lambda_2 - \lambda_3 - 1)} \supset \cdots \supset U_i^{(i+1)} \supset U_i^{(i)} = T_i.$$

Here, T_i is the group of translations, i.e.,

$$T_i = \{ \sigma \in Q_i \mid \sigma(a) - a = \sigma(b) - b \text{ for all } a, b \in \mathfrak{o}/\mathfrak{p}^{\lambda_2} \}.$$

Lemma 21. We have

$$U_i^{(j)} = \langle g_{n,z} \rangle_{\substack{n \in [i+1, j] \\ z \in (\mathfrak{p}^{n+\lambda_3-1}/\mathfrak{p}^{\lambda_2})^{q_i/\mathfrak{p}^n}}}.$$

Now for each $j \in [i+1, \lambda_2 - \lambda_3]$ define

$$H_i^{(j)} = \{ \sigma \in U_i^{(j)} \mid \sigma(a) = a \text{ if } a_{j-1} = 0 \in S \},$$

with $a \in \mathfrak{o}/\mathfrak{p}^{\lambda_2}$ being written as $a = \sum_{n=1}^{\lambda_2-1} a_n \pi^n$ with $a_n \in S$. Obviously the definition of $H_i^{(j)}$ depends on the choice of S .

Proposition 22. The subgroups $H_i^{(j)}$ are abelian; more precisely, we have

$$\begin{aligned} H_i^{(j)} &\cong (\mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2})^{(q_i/\mathfrak{p}^{j-1})^\times \times k^\times} \\ &\cong (\mathfrak{o}/\mathfrak{p}^{\lambda_2-\lambda_3-j+1})^{(\mathfrak{o}/\mathfrak{p}^{j-i-1})^\times \times k^\times}. \end{aligned}$$

Proposition 23. Q_i decomposes into a product of abelian subgroups $H_i^{(j)} \subset Q_i$ ($i+1 \leq j \leq \lambda_2 - \lambda_3$) as

$$Q_i = H_i^{(i+1)} H_i^{(i+2)} \cdots H_i^{(\lambda_2 - \lambda_3)},$$

with the properties

$$\begin{cases} (H_i^{(i+1)} H_i^{(i+2)} \cdots H_i^{(j)}) \cap H_i^{(j+1)} = 1, \\ (H_i^{(i+1)} H_i^{(i+2)} \cdots H_i^{(j)}) H_i^{(j+1)} = H_i^{(j+1)} (H_i^{(i+1)} H_i^{(i+2)} \cdots H_i^{(j)}). \end{cases}$$

Lemma 24. If $i + \lambda_3 \geq \lambda_2 - \lambda_3$, then we have

$$Q_i \cong \bigoplus_{j=i}^{\lambda_2 - \lambda_3 - 1} (\mathfrak{o}/\mathfrak{p}^{\lambda_2 - \lambda_3 - j})^{(\mathfrak{o}/\mathfrak{p}^{j-i})^\times \times k^\times}.$$

Proposition 25. If $\lambda_3 \geq \frac{1}{2}(\lambda_2 - 1)$, then K is abelian and

$$K \cong \bigoplus_{i=1}^{\lambda_2 - \lambda_3 - 1} \bigoplus_{j=i}^{\lambda_2 - \lambda_3 - 1} (\mathfrak{o}/\mathfrak{p}^{\lambda_2 - \lambda_3 - j})^{(\mathfrak{o}/\mathfrak{p}^{j-i})^\times \times k^\times}.$$

Also, if $\lambda_3 \geq \frac{1}{2}\lambda_2$, then Q_0 is abelian and

$$Q_0 \cong \bigoplus_{j=0}^{\lambda_2 - \lambda_3 - 1} (\mathfrak{o}/\mathfrak{p}^{\lambda_2 - \lambda_3 - j})^{(\mathfrak{o}/\mathfrak{p}^j)^\times \times k^\times}.$$

Now we describe the structure of $\overline{Q}_0$, which we shall need later in computing $\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1$. So for each $j \in [1, \lambda_2 - \lambda_3]$, put

$$\overline{L}_0^{(j)} = \{\sigma \in \overline{Q}_0 \mid \sigma(a) \equiv a \pmod{\mathfrak{p}^{j+\lambda_3-1}} \forall a \in \mathfrak{o}/\mathfrak{p}^{\lambda_2}\}.$$

Evidently we have $\overline{Q}_0 \supset \overline{L}_0^{(j)}$ for each j .

Proposition 26. *We have a normal series of $\overline{Q}_0$:*

$$\overline{Q}_0 = \overline{L}_0^{(1)} \supset \overline{L}_0^{(2)} \supset \dots \supset \overline{L}_0^{(\lambda_2 - \lambda_3 + 1)} = 1,$$

where the factors of the series are given by

$$\overline{L}_0^{(j)} / \overline{L}_0^{(j+1)} \cong (\mathfrak{o}/\mathfrak{p})^{(\mathfrak{o}^\times \setminus \{1\})/\mathfrak{p}^j}$$

for each $j \in [1, \lambda_2 - \lambda_3]$.

Proposition 27. *Assume $\lambda_3 = 1$. Then $\overline{Q}_0$ decomposes into a semidirect product as*

$$\overline{Q}_0 \simeq k^{(\mathfrak{o}^\times \setminus \{1\})/\mathfrak{p}^{\lambda_2-1}} \rtimes k^{(\mathfrak{o}^\times \setminus \{1\})/\mathfrak{p}^{\lambda_2-2}} \rtimes \dots \rtimes k^{(\mathfrak{o}^\times \setminus \{1\})/\mathfrak{p}}.$$

Proposition 28. *$\overline{Q}_0$ decomposes into a product of abelian subgroups as*

$$\overline{Q}_0 \cong \overline{H}_0^{(1)} H_0^{(2)} \dots H_0^{(\lambda_2 - \lambda_3)},$$

where

$$\begin{aligned} \overline{H}_0^{(1)} &= \{\sigma \in H_0^{(1)} \mid \sigma(1) = 1\} \\ &\cong (\mathfrak{o}/\mathfrak{p}^{\lambda_2 - \lambda_3})^{k^\times \setminus \{1\}}. \end{aligned}$$

Proposition 29. *If $\lambda_3 \geq \frac{1}{2}\lambda_2$, then we have*

$$\overline{Q}_0 \cong (\mathfrak{o}/\mathfrak{p}^{\lambda_2 - \lambda_3})^{k^\times \setminus \{1\}} \oplus \bigoplus_{j=1}^{\lambda_2 - \lambda_3 - 1} (\mathfrak{o}/\mathfrak{p}^{\lambda_2 - \lambda_3 - j})^{(\mathfrak{o}/\mathfrak{p}^j)^\times \times k^\times}.$$

Now we shift our attention to calculating the structure of $\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1$. Let N and $\overline{N}$ be the kernels of the natural homomorphisms

$$\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \rightarrow \text{Aut } \mathfrak{o}/\mathfrak{p}^{\lambda_3}$$

and

$$\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \rightarrow \text{Aut}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3}),$$

respectively. That is,

$$N = \{\tau \in \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \mid \tau(a) \equiv a \pmod{\mathfrak{p}^{\lambda_3}} \forall a \in \mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\}, \text{ and}$$

$$\overline{N} = \{\tau \in \text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \mid \tau(a) \sim a \pmod{\mathfrak{p}^{\lambda_1}} \vee \mathfrak{u}_{\lambda_3} \forall a \in \mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\}.$$

Thus we have a normal series

$$\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \supset N \supset \overline{N} \supset 1.$$

Theorem 30. *The following holds.*

(1): *We have*

$$\text{Aut}_{\lambda_3}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 / N \cong \text{Aut } \mathfrak{o}/\mathfrak{p}^{\lambda_3}.$$

(2): *We have*

$$N/\overline{N} \cong \begin{cases} k & \lambda_3 \geq 2, \\ k^\times & \lambda_3 = 1. \end{cases}$$

(3): *We have*

$$\overline{N} \cong \overline{Q}_0 \times Q_0^{\lambda_1 - \lambda_2} \times K.$$

Hence in particular $\overline{N}$ is abelian if $\lambda_3 \geq \frac{1}{2}\lambda_2$.

We can show that N is abelian for certain types of λ :

Proposition 31. *If $\lambda_3 > \frac{1}{2}\lambda_1$, then N is abelian.*

The structure of $\overline{N}$ resembles that of K ; We obtain a decomposition of $\overline{N}$ similar to that of K :

Proposition 32. $\overline{N}$ decomposes into a direct product as

$$\overline{N} \cong V_0 \times V_1 \times \cdots \times V_{\lambda_1 - \lambda_3 - 1},$$

where each factor V_i (defined for $0 \leq i \leq \lambda_1 - 1$) is given by

$$V_i = \{\tau \in N \mid \tau(a) = a^{\forall a \in (\mathfrak{o} \setminus \mathfrak{q}_i)/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}}\}.$$

Lemma 33. We have the following.

- (1): $V_i \cong Q_{i-\lambda_1+\lambda_2}$ for $i \in [\lambda_1 - \lambda_2 + 1, \lambda_1 - \lambda_3 - 1]$,
- (2): $V_1 \cong V_2 \cong \cdots \cong V_{\lambda_1 - \lambda_2} \cong Q_0$ where $\lambda_1 > \lambda_2$,
- (3): $V_0 \cong \overline{Q}_0$.

Lastly, we consider the situation in which the residue field k is the finite field $\mathbb{F}_q$. Then $\text{Aut } \mathcal{L}(M)$ is evidently finite, and by the structural theorem we can compute the order of the group. There is not much to do for case $\lambda_2 = \lambda_3$, so assume $\lambda_2 > \lambda_3$. We start with computing the order $|Q_i|$. We can use either the L -sequence of Q_i or H -decomposition. Let us choose the former this time:

$$\begin{aligned} |Q_i| &= q^{(q-1)+(q-1)q+(q-1)q^2+\cdots+(q-1)q^{\lambda_2-\lambda_3-i-1}} \\ &= q^{-1+q^{\lambda_2-\lambda_3-i}}. \end{aligned}$$

In particular, we get $|Q_0| = q^{-1+q^{\lambda_2-\lambda_3}}$. Since $K = \prod_{i=1}^{\lambda_2-\lambda_3-1} Q_i$, we see that

$$|K| = \prod_{i=1}^{\lambda_2-\lambda_3-1} |Q_i| = q^{\sum_{i=1}^{\lambda_2-\lambda_3-1} (-1+q^i)} = q^{-\lambda_2+\lambda_3+1+\sum_{i=1}^{\lambda_2-\lambda_3-1} q^i}.$$

Also, by L -sequence or H -decomposition of $\overline{Q}_0$, we see that $|\overline{Q}_0|q^{\lambda_2-\lambda_3} = |Q_0|$. So we compute:

$$\begin{aligned} |\overline{N}| &= |\overline{Q}_0| \cdot |Q_0|^{\lambda_1-\lambda_2} \cdot |K| \\ &= q^{-\lambda_2+\lambda_3} q^{-1+q^{\lambda_2-\lambda_3}} \left(q^{-1+q^{\lambda_2-\lambda_3}} \right)^{\lambda_1-\lambda_2} \cdot q^{\sum_{i=1}^{\lambda_2-\lambda_3-1} (-1+q^i)} \\ &= q^{-\lambda_2+\lambda_3} q^{-1+q^{\lambda_2-\lambda_3}} q^{-\lambda_1+\lambda_2+(\lambda_1-\lambda_2)q^{\lambda_2-\lambda_3}} q^{-\lambda_2+\lambda_3+1+\sum_{i=1}^{\lambda_2-\lambda_3-1} q^i} \\ &= q^{-\lambda_2-\lambda_1+2\lambda_3+(\lambda_1-\lambda_2)q^{\lambda_1-\lambda_2}+\sum_{i=1}^{\lambda_2-\lambda_3} q^i}. \end{aligned}$$

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