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Quantized calculus and Teichmüller space

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1. quantized calculus.

The famous duality theorem by Fefferman states that the dual space of Re$H^1(S^1)$ is $BMO(S^1)$. On the other hand, H^1 can be represented as a product of two elements in $H^2(S^1)$,

$$h \in \mathrm{Re}H^1 \longleftrightarrow h = g_1 H g_2 + (Hg_1)g_2, \quad g_j \in L^2(S^1)$$

Here, H is the Hilbert transformation. Further Fefferman showed that

$$|\int fhd\theta| \leq C\|f\|_{BMO}\|g_1\|_2\|g_2\|_2$$

Hence for every function f on S^1, set

$$[H,f](g) = H(fg) - fH(g) \qquad g \in L^2(S^1),$$

and we have

$$\int fhd\theta = \int [H,f](g_1)g_2 d\theta.$$

THEOREM ([1]). *The orerator $[H,f]$ is bounded if and only if $f \in BMO(S^1)$.*

Following A. Connes, this operator is called the quantized derivative $d^Q(f)$ of f. In fact, considering on the real line, we have

$$[H,f](g) = Const. \int_{\mathbf{R}} \frac{f(x) - f(y)}{x - y} g(y)dy$$

and hence can considered as the "polarization" of usual differentiation. Moreover we know

THEOREM ([1]). *The operator $[H,f]$ is compact if and only if $f \in VMO(S^1)$.*

Here $VMO(S^1)$ is the closure of $C(S^1)$ in $BMO(S^1)$. In particular, if $f \in C(S^1)$, then $[H,f]$ is compact. Recall that the dual space of $VMOA(S^1)$ is $H^1(S^1)$ and that an element of $VMO(S^1)$ is not necessarily continuous but only 'quasicontinuous'. More precisely,

$$L^\infty \cap VMO = QC \left(= (H^\infty + C) \cap (\overline{H^\infty} + C) = C + HC\right)$$

REMARK ([2]). *If $f \in L^\infty$ and $|f| \in C(S^1)$, then $f \in QC$.*

Now, "smoother" $f(S^1)$ in fractal sense, better f as a compact operator.

THEOREM ([3]). *The operator $[H,f]$ belongs to the Schatten class $\mathfrak{L}^p$ if and only if $f \in B_p^{1/p}(S^1)$, where $B_p^{1/p}(S^1)$ is the Besov space as below.*

Here $f \in \mathfrak{L}^p$ means that the sequence of eigenvalues of $|T| = (T^*T)^{1/2}$ belongs to ℓ^p. (In particular, $\mathfrak{L}^2$ is the Hilbert-Schmidt class.)

Next $f \in B_p^{1/p}(S^1)$ means that f satisfies the inequality

$$\iint_{S^1 \times S^1} |f(x+t) - 2f(x) + f(x-t)|^p t^{-2} dx dt < +\infty.$$

Recall that, if $p > 1$, this inequality is equivalent to

$$\iint_{S^1 \times S^1} |f(x+t) - f(x)|^p t^{-2} dx dt < +\infty.$$

On the other hand, considering the harmonic extension on D, $f \in B_p^{1/p}(S^1)$ if and only if

$$\int_D \|D^2 f\|^p (1-|z|)^{2p-2} |dz \wedge d\bar{z}| < +\infty.$$

(When $p > 1$, this is equivalent that f is p-integrable 1-form, namely

$$\int_D \|Df\|^p(1-|z|)^{p-2}|dz \wedge d\bar{z}| < +\infty.)$$

COROLLARY. *$B_2^{1/2}(S^1)$ is the Sobolev space (the harmonic Dirichlet space) $HD(D)$ $= W_1^2(D) \cap H(D)$, where D is the unit disk.*

Boundary values form $H^{1/2} = \{(a_n) \mid \sum |n||a_n|^2 < +\infty\}$ (, which S. Nag used).

2. On Hausdorff dimension of quasicircles.

A Riemann map f onto a K-quasi disk has a (1/K)-Hölder continuous boundary value. Hence, for instance (, also see Astata, to appear), we have

PROPOSITION (CF. FALCONER). *The Hausdorff dimension of a K-quasicircle is at most $2 - \frac{1}{K}$.*

On the other hand,

THEOREM (SULLIVAN). *Assume that there is a cocompact quasiFuchsian group Γ whose limit set is a quasicircle C as the limit set.Then The Hausdorff dimension of C is p if and only if a Riemman map f onto the interior of C belongs to $B_q^{1/q}(S^1)$ for every $q > p$.*

COROLLARY ([4]). *A quasicircle C as in Theorem 4 has Hausdorff dimension p if and only if*

$$p = \inf\{q \mid [H, f] \in \mathfrak{L}^q\}.$$

PROBLEM. *Characterize such quasicircles that corresponds to finitely generated Kleinian groups.*

3. Teichmüller spaces.

Here we will give new representation of the Universal Teichmüller sapce. First we recall (cf. Astala-Gehring, '86) the folllowing

THEOREM (KOEBE). *$\{\log f' \mid f$ is univalent on $D\}$ is bounded in the Bloch space*

$$\mathfrak{B} = \{f \mid \sup(1-|z|^2)|f'(z)| < +\infty\}.$$

On the other hand, the boundary value of $\log f'$, where $f \in T(1)$, does not necesarily belong to $BMO(S^1)$. (cf. Astata-Zinsmeister, '91)

Now, if f is a Riemann map onto a quasidisk, f itself has a continuous boundary value. Hence we can consider to represent Riemann maps in the above spaces.

First we set

$$\Sigma = \{f \mid f \text{ is univalent on } D \text{ and has a form } = \frac{1}{z} + \sum_{n=1}^{\infty} c_n z^n \quad \text{near} \quad z = \infty, \}$$

and equip Σ with the Bers topology. Then Σ has the subset Σ_1 which we can identify with the universal Teichmüller space $T(1)$.

THEOREM. *Σ can be mapped injectively in $VMO(S^1)$.*

This injection is continuous at least on Σ_1.

In general, $BMO(S^1) \subset \mathfrak{B}$ and hence $VMO(S^1) \subset \mathfrak{B}_0$, and it is known that, for $g \in \mathfrak{B}_0$, g has a finite angular limit on a set of Hausdorff dimension 1 (Makarov '89). Also recall that $AD(D) \subset VMOA(D)$ (S.Yamashita '82. Further, see Aulaskari '88), and that $f \in \Sigma$ has a finite angular limit almost everywhere as is seen by classical Plessner's theorem.

On the other hand, Pommerenke ([6]) showed that, under the locally uniformly boundedness assumption of average multiplicity,

$$f \in BMOA(S^1) \text{ if and only if } f \in \mathfrak{B}, \quad f \in VMOA(S^1) \text{ if and only if } f \in \mathfrak{B}_0.$$

On the other hand, since multiplication by z is an invertible VMO-multiplier, we can identify Σ with $z\Sigma \subset VMOA(S^1)$. In particular, we have the following

COROLLARY. *Σ can be mapped injectively in $\mathfrak{B}_0$.*

This injection is continuous at least on Σ_1.

REMARK. *Recall that a Riemann map has a continuous boundary value if and only if the complement is locally connected. Hence the locally connectedness conjecture of the limit set (cf. Abikoff([7])) can be restated as follows;*

The image of $\Sigma(G)$ is contained in $C(S^1)$ for a finitely generated Kleinian gorup G, where $\Sigma(G)$ corresponds to $T(G)$?

It seems interesting to characterize Riemann maps, or elements of Σ, belonging to $VMO(S^1) - C(S^1)$ geometrically.

Now to prove Theorem, we note the following fact, which follows at once from the equivalence of $VMO(S^1)$ and $\mathfrak{B}_0$, and from the geometrical characterization of Bloch functions by Pommerenke ([5]).

PROPOSITION. *Let f be a holomorphic injection of D. If $f(D)$ is bounded, then the boundary value f belongs to $VMO(S^1)$.*

But this fact has an interesting

COROLLARY. *Let G be any Kleinian group which has ∞ as an ordinary point, and f be a Riemann map onto a simply connected component of G. Then $f \in VMO(S^1)$.*

Here we note that VMO-ness is a local property.

LEMMA (GOTOH). *Let f be meromorphic on D and has no poles near ∂D. If, for every $\zeta \in \partial D$, there is a neighborhood U of ζ such that $f \circ \phi_\zeta \in VMOA(S^1)$, where ϕ_ζ is a Riemann map onto $U \cap D$, then $f \in VMOA(S^1)$.*

COROLLARY. *Let f be a meromorphic injection of D. If $\infty \in f(D)$, then the boundary value f belongs to $VMO(S^1)$.*

PROOF OF THEOREM: Since the injectivity is clear, the first assertion follows from the above Corollary.

Next suppose that f_n converges to f in Σ_1. Then by uniform convergence proverty of normalized quasiconformal maps, we can see that f_n converges to f uniformly on $\overline{D}$. In particuler, f_n converges to f in $L^\infty(S^1)$ and hence in $BMO(S^1)$, which shows the second assertion, continuity of injection on Σ_1.

REMARK. *Local character of functions in $VMO(S^1)$ can be restated as Axler-Shapiro's theorem ([8]). On the other hand, when C is the limit set of a b-group, every prime end of the invariant component is area 0 by Ahlfors-Thurston's $0-1$ theorem. Hence these facts give another proof of the above Theorem for this case.*

PROBLEM. *Is the above injection, say E, continuous on the whole Σ ? If not, determine the corona, i.e. the set $\overline{E(\Sigma_1)} - E(\overline{\Sigma_1})$.*

Some further discussion on this problem will appear elsewhere.

Next, another representation can be obtained by considering the set

$$\tilde{S} = \{f \mid f \text{ is univalent and holomorphic on } D\}$$

Again we write as $\tilde{S}_1$ the set corresponding to $T(1)$, namely, the set of Riemann maps which admits a quasiconformal extension. Then the 'VMO-ness at a point' can measure the local complexity at the point metrically. For instance, we have

PROPOSITION. *Suppose that f is a Riemann map onto a component B of a Kleinian group G. If ∞ belongs to the boundary of B and is fixed by an element of G with infinite order, then f does not belongs to $BMO(S^1)$.*

PROOF: If ∞ is a parabolic fixed point, then the existence of a cusp neighborhood implies that $f \notin BMO(S^1)$ by Pommerenke's characterization of Bloch functions

([5]).

If ∞ is a loxodromic fixed point, then from self-similarity (invarience) of the limit set, we can conclude the assertion again by Pommerenke's characterization.

Outside of the fixed points set, the limit set of G may have high complexity, at least, in the finitely generated case. Hence the Riemann map f may also behave very wildly. So we may put the following

PROBLEM. *If G is a finitely generated Kleinian group with a component $f(D)$ and ∞ is not fixed by any non-trivial element of G, does f belong to $VMO(S^1)$?*

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