

A Radon transform in monogenic function theory .

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Abstract. In this paper we study the multiple Taylor series expansion of a monogenic function, by making use of a Radon transform in monogenic function theory.

Introduction. In (3)), Hayman proved that every harmonic function (and hence every holomorphic function) in $\{z \in \mathbb{C} \mid |z| < R\}$, $z = x + iy$, admits a multiple Taylor series expansion which converges absolutely in the domain which is given by

$$\{(x, y) \in \mathbb{R}^2 \mid |x| + |y| < R\}.$$

In the several dimensional case Siciak studied this problem in (5), by making use of the complex extension of a harmonic function in the unit ball.

In this paper we generalize the result of Hayman for holomorphic functions to the theory of monogenic functions; a theory which has been studied by Delanghe and Brackx in (2) and which is a generalization of the theory of holomorphic functions to several dimensions.

We shall prove that every analytic function, of which the multiple Taylor series converges in the interval

$$]-R_1, R_1[\times \dots \times]-R_m, R_m[, \quad R_1 > 0, \dots, R_m > 0,$$

admits a unique left monogenic extension to the domain

$$\{u \in \mathbb{R}^{m+1} \mid |u_0| + |u_1| < R_1, \dots, |u_0| + |u_m| < R_m\}.$$

Furthermore the multiple Taylor series of the extension converges absolutely in this domain, and this domain is optimal with respect to this type of convergence.

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To this end, we construct a generalization:

$$P(u, \vec{z}), (u, \vec{z}) \in \mathbb{R}^{m+1} \times \mathbb{C}^m$$

of the function

$$(1 - uz)^{-1}, (u, z) \in \mathbb{C} \times \mathbb{C},$$

which is holomorphic in $\vec{z}$ and monogenic in u , and which is used to generalize the classical Radon transform:

$$\pi : \mathcal{H}'(\bar{B}(0, 1)) \rightarrow \mathcal{H}(B(0, 1)), \text{ which is given by}$$

$$\pi(T)(u) = \langle T_z, (1 - uz)^{-1} \rangle,$$

to the theory of monogenic functions.

Preliminaries. In the sequel we always work with modules of functions with values in a complex Clifford algebra.

The complex Clifford algebra over $\mathbb{R}^m$ is defined as follows:

$$\mathcal{A} = \left\{ \sum_{A \in \{1, \dots, m\}} a_A e_A \mid a_A \in \mathbb{C} \right\}$$

where $e_A = e_{\alpha_1} \dots e_{\alpha_h}$, when $A = \{\alpha_1, \dots, \alpha_h\}$; $\alpha_1 < \dots < \alpha_h$,

and $e_\emptyset = e_0 = 1$, $e_{\{k\}} = e_k$, $k = 1, \dots, m$.

The involution in $\mathcal{A}$ is defined by $\bar{a} = \sum_{A \in \{1, \dots, m\}} \bar{a}_A \bar{e}_A$,

$e_A = e_{\alpha_1} \dots e_{\alpha_h}$, where $\bar{e}_A = \bar{e}_{\alpha_h} \dots \bar{e}_{\alpha_1}$ and $\bar{e}_{\alpha_j} = -e_{\alpha_j}$.

As $\mathcal{A}$ is isomorphic to $\mathbb{C}^{2^m}$, we may provide $\mathcal{A}$ with the $\mathbb{C}^{2^m}$ -norm.

$$\text{Hence, } |a| = \left(\sum_{A \in \{1, \dots, m\}} |a_A|^2 \right)^{1/2}.$$

Furthermore it is easy to show that for any $a, b \in \mathcal{A}$,

$$|a \cdot b| \leq 2^{m/2} |a| |b|.$$

A point $(x_0, \dots, x_m)$ of $\mathbb{R}^{m+1}$ shall be identified with the Clifford number $x_0 + \vec{x} = x_0 + \sum_{j=1}^m x_j e_j$.

In this way $\mathbb{R}^{m+1}$ is imbedded in $\mathcal{A}$.

There are several ways to define a product in a Clifford algebra. In our theory the product in $\mathcal{A}$ is defined by the relations: $e_k^2 = -1$ and $e_k e_j + e_j e_k = 0$, whenever $k \neq j$, $k, j = 1, \dots, m$.

Let $\Omega \subset \mathbb{R}^{m+1}$ be open and let $f \in C_1(\Omega, \mathcal{A})$.

Then f is left (resp. right) monogenic in Ω if

$$D f = \sum_{j=0}^m e_j \frac{\partial}{\partial x_j} f = 0 \quad (\text{resp. } f D = \sum_{j=0}^m \frac{\partial}{\partial x_j} f e_j = 0)$$

in Ω .

The right $\mathcal{A}$ -module of left monogenic functions in Ω is denoted by $M_1(\Omega, \mathcal{A})$. It is a Fréchet module for the topology of uniform convergence on the compact subsets of Ω .

For the basic elementary function theoretic theorems we refer to (2).

For any open subset Ω of $\mathbb{C}^m$, $\mathcal{H}_{(1)}(\Omega, \mathcal{A})$ (resp. $\mathcal{H}_{(r)}(\Omega, \mathcal{A})$) is the left (resp. right) module of $\mathcal{A}$ -valued holomorphic functions in Ω . Hence its dual module: $\mathcal{H}'_{(1)}(\Omega, \mathcal{A})$, consists of left linear $\mathcal{A}$ -valued analytic functionals.

ω_{m+1} is the area of the unit sphere in $\mathbb{R}^{m+1}$.

I. Special monogenic functions.

Let $u = u_0 + \vec{u}$ belong to $\mathbb{R}^{m+1}$ let $\vec{z} = \vec{x} + i\vec{y} = \sum_{j=1}^m e_j z_j$

belong to $\mathbb{C}^m$ and put $\langle \vec{u}, \vec{z} \rangle = \sum_{j=1}^m u_j z_j$.

Then one easily shows that the functions $(\langle \vec{u}, \vec{z} \rangle - u_0 \vec{z})^k$,

which are defined for $(u, \vec{z}) \in \mathbb{R}^{m+1} \times \mathbb{C}^m$, are left and right monogenic for $u \in \mathbb{R}^{m+1}$ and holomorphic for $\vec{z} \in \mathbb{C}^m$.

Furthermore, when f is a $\mathbb{C}$ -valued holomorphic function in $\{z \in \mathbb{C} \mid |z| < \rho\}$, $\rho > 0$, admitting the Taylor series expansion $f(z) = \sum_{k=0}^{\infty} c_k z^k$, then the function $f(u, z)$,

$$(u, z) \in \mathbb{C}^1, \text{ can be generalized immediately to the function}$$

$$(1) \quad F(u, \vec{z}) = \sum_{k=0}^{\infty} c_k (\langle \vec{u}, \vec{z} \rangle - u_0 \vec{z})^k, \quad (u, \vec{z}) \in \mathbb{R}^{m+1} \times \mathbb{C}^m,$$

which is left and right monogenic in u and holomorphic in $\vec{z}$.

In the following theorem we study the convergence of the

$$\text{series } \sum_{k=0}^{\infty} c_k (\langle \vec{u}, \vec{z} \rangle - u_0 \vec{z})^k.$$

Theorem I. Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be holomorphic in

$\{z \in \mathbb{C} \mid |z| < \rho\}$, $\rho > 0$. Then the series $\sum_{k=0}^{\infty} c_k (\langle \vec{u}, \vec{z} \rangle - u_0 \vec{z})^k$ converges absolutely as a multiple

Taylor series of $u_0, \dots, u_m$ in the domain

$$\left\{ u \in \mathbb{R}^{m+1} \mid |u_0| \left(\sum_{j=1}^m |z_j|^2 \right)^{1/2} + \sum_{j=1}^m |u_j| |z_j| < \rho \right\}.$$

In the case $\vec{z} = \vec{x} \in \mathbb{R}^m$, this domain is optimal.

Proof. As f is holomorphic for $z \in \mathbb{C}$, $|z| < \rho$ and as

$$(\langle \vec{u}, \vec{z} \rangle - u_0 \vec{z})^k = \sum_{\sum_{j=0}^m k_j = k} \frac{k!}{k_0! \dots k_m!} (-u_0 \vec{z})^{k_0} \prod_{j=1}^m (u_j z_j)^{k_j},$$

the domain of absolute convergence of the multiple Taylor

series $\sum_{k=0}^{\infty} c_k (\langle \vec{u}, \vec{z} \rangle - u_0 \vec{z})^k$ is determined by the

condition

$$\sum_{k=0}^{\infty} |c_k| \sum_{\sum_{j=0}^m k_j = k} \frac{k!}{k_0! \dots k_m!} |u_0|^{k_0} |\vec{z}|^{k_0} \prod_{j=1}^m (|u_j| |z_j|)^{k_j} < \infty \quad (*).$$

For $k_0 = 2s$, $s \in \mathbb{N}$, $\vec{z}^{k_0} = (-1)^s \left(\sum_{j=1}^m z_j^2 \right)^s$

and for $k_0 = 2s + 1$, $s \in \mathbb{N}$, $\vec{z}^{k_0} = (-1)^s \left(\sum_{j=1}^m z_j^2 \right)^s \vec{z}$.

Hence, for any $k_0 \in \mathbb{N}$, $|\vec{z}^{k_0}| \leq \left(\sum_{j=1}^m |z_j|^2 \right)^{k_0/2}$

and when $\vec{z} = \vec{x} \in \mathbb{R}^m$, $|\vec{x}^{k_0}| = \left(\sum_{j=1}^m x_j^2 \right)^{k_0/2}$.

Hence (*) is satisfied as soon as $(u, \vec{z})$ satisfies the inequality

$$|u_0| \left(\sum_{j=1}^m |z_j|^2 \right)^{1/2} + \sum_{j=1}^m |u_j| |z_j| < \rho$$

and when $\vec{z} = \vec{x} \in \mathbb{R}^m$, (*) is equivalent with this inequality. ■

We give two important examples.

Example 1.

$$P(u, \vec{z}) = \sum_{k=0}^{\infty} (\langle \vec{u}, \vec{z} \rangle - u_0 \vec{z})^k = \frac{1 - \langle \vec{u}, \vec{z} \rangle - u_0 \vec{z}}{(1 - \langle \vec{u}, \vec{z} \rangle)^2 + u_0^2 \sum_{j=1}^m z_j^2}$$

is the generalization of $(1 - uz)^{-1}$, $(u, z) \in \mathbb{C}^2$.

Obviously $P(u, \vec{z})$ is defined in the open domain

$\mathcal{U} = (\mathbb{R}^{m+1} \times \mathbb{C}^m) \setminus S$, where S is given by the equations

$$(1 - \langle \vec{u}, \vec{x} \rangle)^2 + u_0^2 |\vec{x}|^2 = \langle \vec{u}, \vec{y} \rangle^2 + u_0^2 |\vec{y}|^2$$

$$(1 - \langle \vec{u}, \vec{x} \rangle) \langle \vec{u}, \vec{y} \rangle = u_0^2 \langle \vec{x}, \vec{y} \rangle.$$

Example 2.

$$\begin{aligned} \text{Exp}(u, \vec{z}) &= \sum_{k=0}^{\infty} \frac{1}{k!} (\langle \vec{u}, \vec{z} \rangle - u_0 \vec{z})^k \\ &= e^{\langle \vec{u}, \vec{z} \rangle} \left(\cos(u_0 \sqrt{\sum_{j=1}^m z_j^2}) - \frac{\vec{z}}{\sqrt{\sum_{j=1}^m z_j^2}} \sin(u_0 \sqrt{\sum_{j=1}^m z_j^2}) \right). \end{aligned}$$

The function $\text{Exp}(u, \sum_{j=1}^m e_j)$ has been studied explicitly

by Brackx in (1).

In (8) we introduced another way to generalize holomorphic functions. Let f be holomorphic in $\{z \in \mathbb{C} \mid |z| < \rho\}$ and admit the Taylor expansion $f(z) = \sum_{k=0}^{\infty} c_k z^k$.

Then the function $\frac{1}{z} f\left(\frac{u}{z}\right)$, $(u, z) \in \mathbb{C} \times (\mathbb{C} \setminus \{0\})$ is generalized to the function

$$(2) F(u, y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} c_k \left(\sum_{j=0}^m u_j \frac{\partial}{\partial y_j} \right)^k \frac{\bar{y}}{|y|^{m+1}},$$

which is defined for $(u, y) \in \mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \setminus \{0\})$

such that $|u| < \rho|y|$, and which is left and right monogenic in both variables u and y separately.

In the following theorem we give a relation between the above introduced generalizations of a holomorphic function.

Theorem 2. Let $f(z) = \sum_{k=0}^{\infty} c_k u^k$ be holomorphic for $|z| < \rho$ and let $F(u, \vec{z})$ and $F(u, y)$ be the generalizations of $f(u.z)$ and $\frac{1}{z} f\left(\frac{u}{z}\right)$, introduced in (1) and (2) respectively. Then for sufficiently small $r > 0$ and $|u| < \rho r$,

$$F(u, \vec{z}) = \frac{1}{\omega_{m+1}} \int_{\partial B(0, r)} F(u, y) d\sigma_y P(y, \vec{z}).$$

Proof. Let $|z_j| < R$; $j=1, \dots, m$.

Then $P(u, \vec{z})$ is left monogenic in

$$A = \left\{ u \in \mathbb{R}^{m+1} \mid |u_0| \sqrt{m} + \sum_{j=1}^m |u_j| < R^{-1} \right\}.$$

Choose $r > 0$ such that $\bar{B}(0, r) = \{u \in \mathbb{R}^{m+1} \mid |u| < r\} \subset A$.

Then $\frac{1}{\omega_{m+1}} \int_{\partial B(0, r)} F(u, y) d\sigma_y P(y, \vec{z})$

is defined for $|u| < \rho r$ and in view of (7) it admits a Taylor series expansion which is exactly equal to

$$\sum_{k=0}^{\infty} c_k (\langle \vec{u}, \vec{z} \rangle - u_0 \vec{z})^k. \blacksquare$$

2. The Radon transform

In this section we study the monogenic version of the Radon transform.

Definition 1. Let $\Omega \subset \mathbb{C}^m$ be a domain of holomorphy and let $T \in \mathcal{H}'_{(1)}(\Omega, \mathcal{A})$.

Then we define the Radon transform by

$$\Pi(T)(u) = \langle T_{\vec{z}}, P(u, \vec{z}) \rangle.$$

Definition 2. Let $\Omega \subset \mathbb{R}^{m+1}$ be open and let $T \in \mathcal{M}'_A(\Omega, \mathcal{A})$.

Then we define the converse Radon transform by

$$\mu(T)(\vec{z}) = \langle T_u, P(u, \vec{z}) \rangle.$$

Observe that both Π and μ are generalizations of the classical Radon transform

$\Pi(T)(z) = \langle T_u, (1 - u \cdot z)^{-1} \rangle$, $T \in \mathcal{H}'(\Omega)$,
and that $\Pi(T \cdot a) = \Pi(T) \cdot a$ and $\mu(a \cdot T) = a \cdot \mu(T)$
for all $a \in \mathcal{A}$.

Π maps complex analytic functionals into left monogenic functions and μ maps analytic functionals in the monogenic sense into holomorphic functions.

In this paper we study the image of the transform Π in some special interesting cases.

Let $R_1 > 0, \dots, R_m > 0$. Then we put:

- a. $B(R_1, \dots, R_m) = \{ \vec{z} \in \mathbb{C}^m \mid |z_j| < R_j \}$
 - b. $P(R_1, \dots, R_m) = \{ u \in \mathbb{R}^{m+1} \mid \sum_{j=1}^m R_j |u_j| + \sqrt{\sum_{j=1}^m R_j^2} |u_0| < 1 \}$
 - c. $b(R_1, \dots, R_m) = \{ \vec{z} \in \mathbb{C}^m \mid \sum_{j=1}^m R_j |z_j| < 1 \}$
 - d. $p(R_1, \dots, R_m) = p_1(R_1) \cap \dots \cap p_m(R_m)$
- where $p_j(R_j) = \{ u \in \mathbb{R}^{m+1} \mid |u_j| + |u_0| < R_j \}$.

Furthermore we put

$$\mathcal{H}_{(1)}(\bar{B}(R_1, \dots, R_m)) = \lim_{\varepsilon > 0} \inf \mathcal{H}_{(1)}(B(R_1 + \varepsilon, \dots, R_m + \varepsilon), d)$$

and

$$\mathcal{H}_{(1)}(\bar{B}(R_1, \dots, R_m)) = \lim_{\varepsilon > 0} \inf \mathcal{H}_{(1)}(b(R_1 + \varepsilon, \dots, R_m + \varepsilon), d)$$

In Theorem 3 and Theorem 4 we give a characterization of $\pi(\mathcal{H}'_{(1)}(\bar{B}(R_1, \dots, R_m)))$.

Theorem 3. Let $T \in \mathcal{H}'_{(1)}(\bar{B}(R_1, \dots, R_m))$.

Then $\Pi(T)(u)$ is left monogenic in $P(R_1, \dots, R_m)$ and its multiple Taylor series converges absolutely in $P(R_1, \dots, R_m)$. Furthermore $P(R_1, \dots, R_m)$ is optimal for the absolute convergence of multiple Taylor series of monogenic functions.

Proof. Let $T \in \mathcal{H}'_{(1)}(\bar{B}(R_1, \dots, R_m))$. Then for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|\langle T, \psi \rangle| \leq C_\varepsilon \sup_{\vec{z} \in B(R_1, \dots, R_m)} |\psi(\vec{z})|$$

for every $\psi \in \mathcal{H}_{(1)}(\bar{B}(R_1 + 2\varepsilon, \dots, R_m + 2\varepsilon))$.

Let $u \in P(R_1, \dots, R_m)$ be fixed and choose $\varepsilon > 0$ such that $u \in P(R_1 + 2\varepsilon, \dots, R_m + 2\varepsilon)$. Then for fixed u ,

$P(u, \vec{z}) \in \mathcal{H}_{(1)}(\bar{B}(R_1 + 2\varepsilon, \dots, R_m + 2\varepsilon))$ and hence,

$\Pi(T)(u)$ is defined in $P(R_1, \dots, R_m)$.

Furthermore,

$$\begin{aligned} & \langle T_{\vec{z}}, (\langle \vec{u}, \vec{z} \rangle - u_0 \vec{z})^k \rangle \\ &= \sum_{\substack{\vec{k} \\ j=0}}^m \frac{k!}{k_0! \dots k_m!} u_0^{k_0} \dots u_m^{k_m} \langle T_{\vec{z}}, (-\vec{z})^{k_0} \prod_{j=1}^m z_j^{k_j} \rangle \end{aligned}$$

and

$$\begin{aligned}
 & | \langle T_{\vec{z}}, (-\vec{z})^{k_0} \prod_{j=1}^m z_j^{k_j} \rangle | \\
 & \leq C_{\varepsilon} \sup_{\bar{B}(R_1+\varepsilon, \dots, R_m+\varepsilon)} |\vec{z}|^{k_0} \prod_{j=1}^m |z_j|^{k_j} \\
 & \leq C_{\varepsilon} \left(\sum_{j=1}^m (R_j + \varepsilon)^2 \right)^{k_0/2} \prod_{j=1}^m (R_j + \varepsilon)^{k_j}
 \end{aligned}$$

Hence,

$$\begin{aligned}
 & \sum_{\substack{j=1 \\ k_j=k}}^m \frac{k!}{k_0! \dots k_m!} |u_0|^{k_0} \dots |u_m|^{k_m} | \langle T_{\vec{z}}, (-\vec{z})^{k_0} \prod_{j=1}^m z_j^{k_j} \rangle | \\
 & \leq C_{\varepsilon} \left(\sum_{j=1}^m |u_j| (R_j + \varepsilon) + |u_0| \left(\sum_{j=1}^m (R_j + \varepsilon)^2 \right)^{1/2} \right)^k \\
 & \leq C_{\varepsilon} (1 - \delta_{\varepsilon})^k, \text{ for some } \delta_{\varepsilon} > 0; \text{ which implies that}
 \end{aligned}$$

the multiple Taylor series of $\Pi(T)(u)$ converges

absolutely in $P(R_1, \dots, R_m)$.

Furthermore, $\delta_{(R_1, \dots, R_m)} \in \mathcal{H}'_{(1)}(\bar{B}(R_1, \dots, R_m))$

and

$$\Pi(\delta_{(R_1, \dots, R_m)})(u) = \frac{1 - \sum_{j=1}^m R_j u_j - u_0 \sum_{j=1}^m R_j e_j}{(1 - \sum_{j=1}^m R_j u_j)^2 + u_0^2 \sum_{j=1}^m R_j^2}$$

admits a multiple Taylor series, which, in view of

Theorem I, converges absolutely in $P(R_1, \dots, R_m)$,

but not in any point outside of $P(R_1, \dots, R_m)$.

This means that $P(R_1, \dots, R_m)$ is optimal for the absolute convergence of multiple Taylor series of monogenic functions. ■

Theorem 4. Let f be left monogenic in $P(R_1, \dots, R_m)$

such that the multiple Taylor series of f converges

absolutely in $P(R_1, \dots, R_m)$. Then $f = \Pi(T)$ for

some $T \in \mathcal{H}'_{(1)}(\bar{B}(R_1, \dots, R_m))$.

Proof. In view of (4), one easily shows that the mapping

$$\tilde{\pi} : T_{\vec{z}} \longrightarrow \langle T_{\vec{z}}, (1 - \sum_{j=1}^m \{ \}_j z_j)^{-1} \rangle = \tilde{\pi}(T)(\vec{z})$$

is a topological isomorphism between $\mathcal{H}'_{(1)}(\bar{B}(R_1, \dots, R_m))_b$ and $\mathcal{H}_{(r)}(b(R_1, \dots, R_m))$. (and also between

$$\mathcal{H}'_{(1)}(\bar{B}(R_1, \dots, R_m))_b \text{ and } \mathcal{H}_{(r)}(B(R_1, \dots, R_m)) .)$$

On the other hand, $f|_{u_0=0}$ admits a multiple Taylor series expansion:

$$f|_{u_0=0}(u_1, \dots, u_m) = \sum_{k=0}^{\infty} \sum_{\substack{j=1 \\ \sum k_j=k}}^m u_1^{k_1} \dots u_m^{k_m} a_{k_1, \dots, k_m}$$

which converges absolutely in $\{ \vec{u} \in \mathbb{R}^m \mid \sum_{j=1}^m |u_j| R_j < 1 \}$.

$$\text{Hence, } f(\{ \}_1, \dots, \{ \}_m) = \sum_{k=0}^{\infty} \sum_{\substack{j=1 \\ \sum k_j=k}}^m \{ \}_1^{k_1} \dots \{ \}_m^{k_m} a_{k_1, \dots, k_m} ,$$

which is the holomorphic extension of $f|_{u_0=0}(u_1, \dots, u_m)$, belongs to $\mathcal{H}_{(r)}(b(R_1, \dots, R_m))$.

Hence $f|_{u_0=0}(u_1, \dots, u_m) = \langle T_{\vec{z}}, P(u, \vec{z}) \rangle|_{u_0=0}$, for some $T \in \mathcal{H}'_{(1)}(\bar{B}(R_1, \dots, R_m))$.

As analytic functions in open subsets of $\mathbb{R}^m$ admit unique left monogenic extensions to open subsets of $\mathbb{R}^{m+1}$ (Theorem 6.), we obtain that $f(u) = \langle T_{\vec{z}}, P(u, \vec{z}) \rangle$. ■

In view of Theorem 3. and Theorem 4 , $\pi(\mathcal{H}'_{(1)}(\bar{B}(R_1, \dots, R_m)))$ coincides with the right module of left monogenic functions in $P(R_1, \dots, R_m)$ of which the multiple Taylor series converges absolutely in $P(R_1, \dots, R_m)$.

In an analogous way one shows:

Theorem 5. $\Pi(\mathcal{H}'_{(1)}(\mathcal{B}(R_1, \dots, R_m)))$ coincides with the right module of left monogenic functions in $p(R_1, \dots, R_m)$ of which the multiple Taylor series converges absolutely in $p(R_1, \dots, R_m)$.

3. The Cauchy-Kowalewski extension theorem .

Let $\Omega \subset \mathbb{R}^m$ be open. Then an open subset $\tilde{\Omega}$ of $\mathbb{R}^{m+1}$ is called a normal open neighbourhood of Ω in $\mathbb{R}^{m+1}$ when for each point $u \in \tilde{\Omega}$, $u = u_0 + \vec{u}$, the set $\{x \in \mathbb{R}^{m+1} \mid x = x_0 + \vec{u} \text{ and } x \in \tilde{\Omega}\}$ is connected and contains one point of Ω .

In (6) we showed the following Cauchy-Kowalewski type extension theorem.

Theorem 6. Let $\Omega \subset \mathbb{R}^m$ be open and let f be an $\mathcal{A}$ -valued analytic function in Ω . Then there exists a normal open neighbourhood $\tilde{\Omega}$ of Ω in $\mathbb{R}^{m+1}$ and a left monogenic function f' in $\tilde{\Omega}$ such that $f'(\vec{x} + x_0)|_{x_0=0} = f(\vec{x})$.

Furthermore, if f'_1 and f'_2 are left monogenic extensions of f in open normal neighbourhoods $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ of Ω , then $\tilde{\Omega}_1 \cap \tilde{\Omega}_2$ is a normal open neighbourhood of Ω and

$f'_1|_{\tilde{\Omega}_1 \cap \tilde{\Omega}_2} = f'_2|_{\tilde{\Omega}_1 \cap \tilde{\Omega}_2}$. Hence there exists a unique left monogenic extension of f which is defined in a maximal open and normal neighbourhood of Ω .

In the following theorem we give a characterization of the multiple Taylor series convergence of the Cauchy-Kowalewski extensions of a special class of analytic functions.

Theorem 7. Let $f(z_1, \dots, z_m)$ be an $\mathcal{A}$ -valued holomorphic function in $B(R_1, \dots, R_m)$ and let $f(x_1, \dots, x_m)$ be its restriction to $\mathbb{R}^m$. Then $f(x_1, \dots, x_m)$ admits a unique left monogenic extension $f(x_0 + \vec{x})$ in $p(R_1, \dots, R_m)$, which admits an absolutely converging multiple Taylor series expansion in $p(R_1, \dots, R_m)$ and which is given by

$$f(x_0 + \vec{x}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x_0^{2k} \Delta_m^k f(x_1, \dots, x_m) - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x_0^{2k+1} \left(\sum_{j=1}^m e_j \frac{\partial}{\partial x_j} \right) \Delta_m^k f(x_1, \dots, x_m) ..$$

Furthermore $p(R_1, \dots, R_m)$ is optimal.

Proof. As $f(z_1, \dots, z_m) \in \mathcal{H}_{(r)}(B(R_1, \dots, R_m))$, $f(z_1, \dots, z_m) = \langle T_{\vec{z}}, (1 - \sum_{j=1}^m \{z_j\} z_j)^{-1} \rangle$, for some $T \in \mathcal{H}'_{(1)}(B(R_1, \dots, R_m))$.

Hence by Theorem 5, $f(x_1, \dots, x_m) = \langle T_{\vec{x}}, P(x, \vec{x}) \rangle \big|_{x_0=0}$

admits the left monogenic extension:

$f(x_0 + \vec{x}) = \langle T_{\vec{x}}, P(x_0 + \vec{x}, \vec{x}) \rangle$ in $p(R_1, \dots, R_m)$, which admits an absolutely converging multiple Taylor series expansion in $p(R_1, \dots, R_m)$.

On the other hand, one can easily show that

$$f'(x_0 + \vec{x}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x_0^{2k} \Delta_m^k f(x_1, \dots, x_m) - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x_0^{2k+1} \left(\sum_{j=1}^m e_j \frac{\partial}{\partial x_j} \right) \Delta_m^k f(x_1, \dots, x_m)$$

is left monogenic in $p(R_1, \dots, R_m)$ and that

$$f(x_1, \dots, x_m) = f'(x_0 + \vec{x}) \big|_{x_0=0}.$$

Hence by Theorem 6, $f = f'$.

We now show that $p(R_1, \dots, R_m)$ is optimal.

Let $u \notin p(R_1, \dots, R_m)$. Then for some $j = 1, \dots, m$;

$u \notin p_j(R_j)$. One can easily show that the function

$f(x) = (R_j - (x_j - x_0 e_j))^{-1}$ is left monogenic in

$p(R_1, \dots, R_m)$ and that its multiple Taylor series,

which is given by $R_j^{-1} \sum_{k=0}^{\infty} (R_j^{-1}(x_j - x_0 e_j))^k$,

converges absolutely for any $x \in p(R_1, \dots, R_m)$,

but not for $x = u$. ■

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