

A NOTE ON MULTIVALENT FUNCTIONS

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1. INTRODUCTION

Let A_p denote the class of functions of the form

$$(1.1) \quad f(z) = \sum_{n=p}^{\infty} a_n z^n \quad (a_p = 1; p \in N = \{1, 2, 3, \dots\})$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$.

A function $f(z)$ belonging the class A_p is said to be p -valently α -convex in the unit disk U if and only if

$$(1.2) \quad \operatorname{Re}\left\{(1 - \alpha) \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} + \alpha \left(1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right)\right\} > 0$$

for some real α , and for all $z \in U$ (cf. [5]).

Denoting by $A_p(\alpha)$ the subclass of A_p consisting of functions which are p -valently α -convex in the unit disk U , we see that $A_p(\alpha)$ is the generalization class of α -convex functions studied by Miller, Mocanu and Reade [2] (or [3], [4]).

Recently, Saitoh, Nunokawa, Owa, Sekine and Fukui [6] have proved some interesting results for functions belonging to the class $A_p(\alpha)$.

2. PROPERTIES OF THE CLASS $A_p(\alpha)$

We begin with the statements of the following lemmas.

LEMMA 1 ([1]). Let $\phi(u,v)$ be a complex valued function,

$\phi: D \rightarrow \mathbb{C}$, $D \subset \mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane),

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u,v)$ satisfies the following conditions:

- (i) $\phi(u,v)$ is continuous in D ;
- (ii) $(1,0) \in D$ and $\operatorname{Re}\{\phi(1,0)\} > 0$;
- (iii) $\operatorname{Re}\{\phi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ and such that

$$v_1 \leq -(1 + u_2^2)/2.$$

Let $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ be regular in the unit disk

U such that $(p(z), zp'(z)) \in D$ for all $z \in U$.

If

$$\operatorname{Re}\{\phi(p(z), zp'(z))\} > 0 \quad (z \in U),$$

then

$$\operatorname{Re}\{p(z)\} > 0 \quad (z \in U).$$

LEMMA 2 ([6]). If $f(z) \in A_p(\alpha)$ with $\alpha \geq 1$, then

$$(2.1) \quad \operatorname{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \frac{-\alpha + \sqrt{\alpha(\alpha + 8)}}{4}$$

for $z \in U$.

PROOF. Define the function $g(z)$ by

$$(2.2) \quad \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} = \beta + (1 - \beta)g(z)$$

for $f(z) \in A_p(\alpha)$, where

$$(2.3) \quad \beta(\alpha) = \frac{-\alpha + \sqrt{\alpha(\alpha + 8)}}{4}.$$

It follows from the above that $g(z)$ is regular in the unit disk U , and that $g(z) = 1 + g_1 z + g_2 z^2 + \dots$.

Making the logarithmic differentiations of both sides in (2.2), we have

$$(2.4) \quad 1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} = \beta + (1 - \beta)g(z) + \frac{(1 - \beta)zg'(z)}{\beta + (1 - \beta)g(z)}.$$

Thus we can see that

$$(2.5) \quad \begin{aligned} & \operatorname{Re}\left\{(1 - \alpha)\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} + \alpha\left(1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right)\right\} \\ &= \operatorname{Re}\left\{\beta + (1 - \beta)g(z) + \frac{\alpha(1 - \beta)zg'(z)}{\beta + (1 - \beta)g(z)}\right\} > 0 \end{aligned}$$

for $f(z) \in A_p(\alpha)$. Letting

$$(2.6) \quad \phi(u, v) = \beta + (1 - \beta)u + \frac{\alpha(1 - \beta)v}{\beta + (1 - \beta)u},$$

(note that $u = g(z)$ and $v = zg'(z)$), we know that

- (i) $\phi(u, v)$ is continuous in $D = (C - \{\beta/(\beta - 1)\}) \times C$;
- (ii) $(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} = 1 > 0$;
- (iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -(1 + u_2^2)/2$,

$$\begin{aligned} \operatorname{Re}\{\phi(iu_2, v_1)\} &= \beta + \frac{\alpha\beta(1 - \beta)v_1}{\beta^2 + (1 - \beta)^2 u_2^2} \\ &\leq \beta - \frac{\alpha\beta(1 - \beta)(1 + u_2^2)}{2\{\beta^2 + (1 - \beta)^2 u_2^2\}} \\ &\leq 0. \end{aligned}$$

Therefore, the function $\phi(u, v)$ defined by (2.6) satisfies the conditions Lemma 1. It follows from this fact that $\operatorname{Re}\{g(z)\} > 0$, that is, that

$$(2.7) \quad \operatorname{Re}\left\{-\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \beta,$$

which completes the proof of Lemma 2.

LEMMA 3. Let k denote the real number such that $0 < k < 1$. Then we have the following inequality.

$$(2.8) \quad \cos k\theta \geq \cos^k \theta \quad (|\theta| < \pi/2).$$

PROOF. We put

$$F(\theta) = \cos k\theta - \cos^k \theta.$$

Then we have

$$F'(\theta) = k \left(\frac{\sin \theta}{\cos^{1-k} \theta} - \sin k\theta \right) \geq 0$$

and $F(0) = 0$.

It follows from the above that

$$F(\theta) \geq 0.$$

Therefore, we have

$$\cos k\theta \geq \cos^k \theta.$$

Consequently, we complete the proof of Lemma 3.

Applying the above lemmas, we prove

THEOREM. If $f(z) \in A_p(\alpha)$ with $\alpha \geq 1$, then

$$(2.9) \quad \operatorname{Re} \left\{ \frac{f^{(p-1)}(z)}{z} \right\}^k > \left\{ \frac{1}{3 - 2\beta(\alpha)} \right\}^k \quad (z \in U),$$

where $\beta(\alpha)$ is given by (2.3) and $0 < k \leq 1$ (k ; real number).

PROOF. Step 1. First, we prove for $k = 1$.

Define the function $g(z)$ by

$$(2.10) \quad \frac{f^{(p-1)}(z)}{z} = \gamma + (1 - \gamma)g(z)$$

with

$$(2.11) \quad \gamma = \frac{1}{3 - 2\beta(\alpha)}.$$

Then $g(z) = 1 + g_1 z + g_2 z^2 + \dots$ is regular in U . Making use of

the logarithmic differentiations of both sides in (2.10), we have

$$(2.12) \quad \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} = 1 + \frac{(1-\gamma)zg'(z)}{\gamma + (1-\gamma)g(z)}.$$

Using Lemma 2, (2.12) leads to

$$(2.13) \quad \begin{aligned} \operatorname{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - \beta(\alpha)\right\} \\ = \operatorname{Re}\left\{1 - \beta(\alpha) + \frac{(1-\gamma)zg'(z)}{\gamma + (1-\gamma)g(z)}\right\} \\ > 0. \end{aligned}$$

Let $u = u_1 + iu_2$, $v = v_1 + iv_2$, and

$$(2.14) \quad \phi(u, v) = 1 - \beta(\alpha) + \frac{(1-\gamma)v}{\gamma + (1-\gamma)u}$$

(note that $u = g(z)$ and $v = zg'(z)$). Then, it follows from (2.14) that

- (i) $\phi(u, v)$ is continuous in $D = (C - \{\frac{\gamma}{\gamma-1}\}) \times C$;
- (ii) $(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} = 1 - \beta(\alpha) > 0$;
- (iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -(1 + u_2^2)/2$,

$$\begin{aligned} \operatorname{Re}\{\phi(iu_2, v_1)\} &= 1 - \beta(\alpha) + \frac{\gamma(1-\gamma)v_1}{\gamma^2 + (1-\gamma)^2u_2^2} \\ &\leq 1 - \beta(\alpha) - \frac{\gamma(1-\gamma)(1+u_2^2)}{2\{\gamma^2 + (1-\gamma)^2u_2^2\}} \\ &= \frac{(1-\gamma)(1-2\beta(\alpha))u_2^2}{2\{\gamma^2 + (1-\gamma)^2u_2^2\}} \\ &\leq 0, \end{aligned}$$

because $1 - 2\beta(\alpha) \leq 0$ for $\alpha \geq 1$. Thus the function $\phi(u, v)$ defined by (2.14) satisfies the conditions in Lemma 1. This implies that $\operatorname{Re}\{g(z)\} > 0$ ($z \in U$), that is, that

$$(2.15) \quad \operatorname{Re}\left\{\frac{f^{(p-1)}(z)}{z}\right\} > \gamma.$$

Therefore, we complete the assertion for $k = 1$.

Step 2. In the next place, we prove for $0 < k < 1$.

Letting

$$(2.16) \quad \frac{f^{(p-1)}(z)}{z} = h(z)$$

and

$$(2.17) \quad \gamma = \frac{1}{3 - 2\beta(\alpha)} > 0.$$

In Step 1, we have

$$(2.18) \quad \operatorname{Re}\{h(z)\} > \gamma > 0.$$

Now, we put

$$h(z) = \rho(\cos\theta + i\sin\theta).$$

From (2.18), we can see that

$$(2.19) \quad \rho\cos\theta > \gamma > 0 \quad (|\theta| < \pi/2).$$

Therefore,

$$\begin{aligned} \operatorname{Re}\{h(z)\}^k &= \operatorname{Re}\{\rho(\cos\theta + i\sin\theta)\}^k \\ &= \operatorname{Re} \rho^k(\cos k\theta + i\sin k\theta) \\ &= \rho^k \cos k\theta \\ &\geq \rho^k \cos^k \theta \quad (\text{by Lemma 3}) \\ &= (\rho \cdot \cos\theta)^k \\ &> \gamma^k. \end{aligned}$$

Hence, we complete the assertion for $0 < k < 1$.

Accordingly, we complete the proof of Theorem.

Making $\alpha = 1$ and $p = 1$ in Theorem, we have

COROLLARY. If the function $f(z) = z + a_2 z^2 + \dots$ is convex in U , then

$$(2.20) \quad \operatorname{Re} \left\{ \frac{f(z)}{z} \right\}^k > \left(\frac{1}{2} \right)^k \quad (z \in U)$$

for all real number k ($0 < k \leq 1$).

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