

On the asymptotic expansion of the Kashaev invariant and the twisted Reidemeister torsion of two-bridge knots

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1 Introduction

This note is a survey of the joint work [8] with Tomotada Ohtsuki.

In [2, 3], Kashaev defined the Kashaev invariant $\langle L \rangle_N \in \mathbb{C}$ of a link L for $N = 2, 3, \dots$ by using the quantum dilogarithm at $q = e^{2\pi\sqrt{-1}/N}$. In [4], he conjectured that, for any hyperbolic link L , $\frac{2\pi}{N} \log |\langle L \rangle_N|$ goes to the hyperbolic volume of $S^3 - L$ as $N \rightarrow \infty$. In [6], Ohtsuki proposed the following refined conjecture:

Conjecture 1 ([6]). *For any hyperbolic knot K , the asymptotic expansions of the Kashaev invariant of K is presented by the following form,*

$$\langle K \rangle_N = e^{N\varsigma(K)} N^{3/2} \omega(K) \cdot \left(1 + \sum_{i=1}^d \kappa_i(K) \cdot \left(\frac{2\pi\sqrt{-1}}{N} \right)^i + O\left(\frac{1}{N^{d+1}}\right) \right), \quad (1)$$

for any d , where $\omega(K)$ and $\kappa_i(K)$'s are some scalars only depending on K . Here $\varsigma(K) = \frac{1}{2\pi\sqrt{-1}} (\text{cs}(S^3 - K) + \sqrt{-1} \text{vol}(S^3 - K))$, where “cs” and “vol” denote the Chern-Simons invariant and the hyperbolic volume.

It is shown in [6, 9, 7] that, for any hyperbolic knot K with up to 7 crossings, Conjecture 1 holds. Moreover, the following is conjectured for $\omega(K)$ of (1):

Conjecture 2. *For any hyperbolic knot K ,*

$$2\pi\sqrt{-1} \omega(K)^2 = \pm\tau(K),$$

where $\tau(K)$ is the twisted Reidemeister torsion associated with the holonomy representation of the hyperbolic structure of the complement of K .

For the figure-eight knot, this conjecture was shown by Andersen and Hansen [1] and H. Murakami [5]. We show

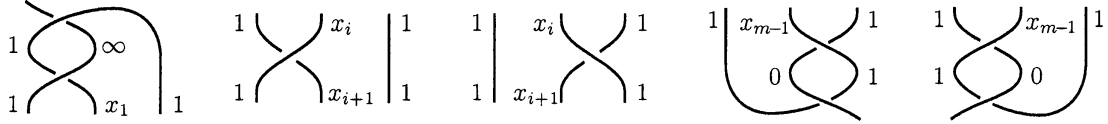
Theorem 1 ([8]). *For any hyperbolic knot K with up to 7 crossings, Conjecture 2 holds.*

2 Results

Let us review a parameterized knot diagram of an open knot, where an open knot is a 1-tangle whose closure is a knot. We parameterize edges of an open knot diagram by parameters in $\mathbb{C} \cup \{\infty\}$. We parameterize edges adjacent to unbounded regions by 1. We parameterize edges next to the terminal edges by 0 or ∞ ; we parameterize such an edge by ∞ (resp. 0) if it is connected to the terminal edge by an under-path (resp. an over-path). We parameterize the other edges in such a way that the parameters belong to $\mathbb{C} - \{0\}$, and satisfy the *hyperbolicity equations*

$$\frac{u'}{u} \Big| \frac{x}{v} \Big| \frac{v'}{v} \quad \left(1 - \frac{x}{u}\right)\left(1 - \frac{v'}{x}\right) = \left(1 - \frac{x}{u'}\right)\left(1 - \frac{v}{x}\right). \quad (2)$$

We consider a hyperbolic two-bridge knot K . Any open two-bridge knot can be presented by a plat closure of a 3-braid of a product of copies of σ_1 and σ_2^{-1} , *i.e.*, any open two-bridge knot diagram D (or its mirror image) can be obtained by gluing copies of the following tangle diagrams, which we call *elementary diagrams*.



From the hyperbolicity equations among parameters of the resulting tangle diagram, the values of x_i are recursively determined by

$$x_{i+1} = \begin{cases} x_i + 1 - \frac{x_i}{x_{i-1}} & \text{if the strand of } x_i \text{ is between } \sigma_1 \text{ and } \sigma_1 \\ & \text{or between } \sigma_2^{-1} \text{ and } \sigma_2^{-1}, \\ x_i + \frac{(x_i - 1)^2}{1 - \frac{x_i}{x_{i-1}}} & \text{otherwise.} \end{cases}$$

It is known that a hyperbolic structure of the complement of K is obtained from a parametrized diagram ([11], [13]). Calculating the monodromy representation, from the definition of $\tau(K)$, we can obtain a reformulation of $\tau(K)$. Explicitly, we define $\Phi(D)$ to be the composition of Φ of elementary diagrams whose values are given as follows,

$$\begin{aligned} \Phi \left(\begin{array}{c} 1 \\ 1 \end{array} \Big| \begin{array}{c} \infty \\ x_1 \end{array} \Big| 1 \right) &= x_1(x_1 - 1) \begin{pmatrix} 1 & 2x_1 & 0 \\ 0 & -x_1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \\ \Phi \left(\begin{array}{c} 1 \\ 1 \end{array} \Big| \begin{array}{c} x_i \\ x_{i+1} \end{array} \Big| 1 \right) &= x_{i+1} \begin{pmatrix} 1 & 2x_{i+1} & 1 \\ 0 & -x_{i+1} & -1 \\ 0 & 0 & 1 \end{pmatrix}, \\ \Phi \left(\begin{array}{c} 1 \\ 1 \end{array} \Big| \begin{array}{c} x_i \\ x_{i+1} \end{array} \Big| 1 \right) &= x_{i+1} \begin{pmatrix} 1 & 0 & 0 \\ -1 & -x_{i+1} & 0 \\ 1 & 2x_{i+1} & 1 \end{pmatrix}, \end{aligned}$$

$$\Phi \left(\begin{array}{c} 1 \\ x_{m-1} \\ 0 \\ 1 \end{array} \right) = \frac{x_{m-1}^3}{(x_{m-1} - 1)^3} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix},$$

$$\Phi \left(\begin{array}{c} 1 \\ x_{m-1} \\ 1 \\ 0 \end{array} \right) = \frac{x_{m-1}^3}{(x_{m-1} - 1)^3} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Then, we have that $\frac{2}{\tau(K)} = \Phi(D)$.

Let us review the definition of the Kashaev invariant. Let K be an oriented knot and $N \geq 2$. We put $q = \exp(2\pi\sqrt{-1}/N)$, $(x)_n = (1-x)(1-x^2)\cdots(1-x^n)$ and $\mathcal{N} = \{0, 1, \dots, N-1\}$.

For $i, j, k, l \in \mathcal{N}$, we put

$$R_{kl}^{ij} = \frac{N q^{-\frac{1}{2}+i-k} \theta_{kl}^{ij}}{(q)_{[i-j]}(\bar{q})_{[j-l]}(q)_{[l-k-1]}(\bar{q})_{[k-i]}}, \quad \bar{R}_{kl}^{ij} = \frac{N q^{\frac{1}{2}+j-l} \theta_{kl}^{ij}}{(\bar{q})_{[i-j]}(q)_{[j-l]}(\bar{q})_{[l-k-1]}(q)_{[k-i]}},$$

where $[m] \in \mathcal{N}$ is the residue of m modulo N , and we put

$$\theta_{kl}^{ij} = \begin{cases} 1 & \text{if } [i-j] + [j-l] + [l-k-1] + [k-i] = N-1, \\ 0 & \text{otherwise.} \end{cases}$$

Let D be an 1-tangle diagram of an open knot whose closure is the knot K . A *labeling* is an assignment of an element of $\mathcal{N}$ to each edge of D . We define the *weights* of labeled elementary tangle diagrams by

$$\begin{aligned} W \left(\begin{array}{c} i \quad j \\ \diagdown \quad \diagup \\ k \quad l \end{array} \right) &= R_{kl}^{ij}, & W \left(\begin{array}{c} \curvearrowright \\ k \quad l \end{array} \right) &= q^{-1/2} \delta_{k,l-1}, & W \left(\begin{array}{c} \curvearrowleft \\ k \quad l \end{array} \right) &= \delta_{k,l}, \\ W \left(\begin{array}{c} i \quad j \\ \diagup \quad \diagdown \\ k \quad l \end{array} \right) &= \bar{R}_{kl}^{ij}, & W \left(\begin{array}{c} \curvearrowright \\ i \quad j \end{array} \right) &= q^{1/2} \delta_{i,j+1}, & W \left(\begin{array}{c} \curvearrowleft \\ i \quad j \end{array} \right) &= \delta_{i,j}. \end{aligned} \quad (3)$$

Then, the Kashaev invariant $\langle K \rangle_N$ of K is defined by

$$\langle K \rangle_N = \sum_{\text{labelings}} \prod_{\substack{\text{crossings} \\ \text{of } D}} W(\text{crossings}) \prod_{\substack{\text{critical} \\ \text{points of } D}} W(\text{critical points}) \in \mathbb{C}.$$

We define the potential function for an open knot diagram parametrized by hyperbolicity parameters. We consider an angle consisting of two adjacent edges at a crossing, and associate such an angle with the value

$$\begin{array}{c} x \quad y \\ \diagdown \quad \diagup \end{array} \rightsquigarrow \text{Li}_2\left(\frac{x}{y}\right) - \text{Li}_2(1) \quad \begin{array}{c} x \quad y \\ \diagup \quad \diagdown \end{array} \rightsquigarrow \text{Li}_2(1) - \text{Li}_2\left(\frac{y}{x}\right)$$

where $\text{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt$. We define the potential function V to be the sum of such values for all angles except for the constant terms. We remark that the equations

$$\frac{\partial}{\partial x_i} V = 0 \quad \text{for all } i$$

give the hyperbolicity equations and so, a solution of the hyperbolicity equations gives a critical point of V . Furthermore, it is known that

$$\log(q)_n \sim -\frac{N}{2\pi\sqrt{-1}} \text{Li}_2(e^{2\pi\sqrt{-1}\frac{n}{N}}),$$

So, from the definition of the potential function, formally, we obtain the following approximation:

$$\langle K \rangle_N \sim \sum_{i_1, \dots, i_m} \exp \left(\frac{N}{2\pi\sqrt{-1}} V(e^{2\pi\sqrt{-1}\frac{i_1}{N}}, \dots, e^{2\pi\sqrt{-1}\frac{i_m}{N}}) \right).$$

Putting $\frac{i_1}{N} = t_1, \dots, \frac{i_m}{N} = t_m$ and using the Poisson summation formula formally,

$$\langle K \rangle_N \sim N^m \int \exp \left(\frac{N}{2\pi\sqrt{-1}} V(e^{2\pi\sqrt{-1}t_1}, \dots, e^{2\pi\sqrt{-1}t_m}) \right) dt_1 \cdots dt_m.$$

Moreover, putting $x_i = e^{2\pi\sqrt{-1}t_i}$, we obtain

$$\langle K \rangle_N \sim N^m \int \exp \left(\frac{N}{2\pi\sqrt{-1}} V(x_1, \dots, x_m) \right) dx_1 \cdots dx_m.$$

By using the saddle point method formally and more calculations of the expansions, we obtain

$$\langle K \rangle_N \sim e^{N\varsigma(K)} \cdot N^{3/2} \cdot \omega(K),$$

where $\varsigma(K) = \frac{1}{2\pi\sqrt{-1}} V(x_{1;c}, \dots, x_{m;c})$ for a critical point $(x_{1;c}, \dots, x_{m;c})$ of V and $\omega(K)$ can be written in terms of the Hessian of V at the critical point $(x_{1;c}, \dots, x_{m;c})$.

Moreover, we define $\Psi(D)$ to be the composition of Ψ of elementary diagrams whose values are given as follows,

$$\begin{aligned} \Psi \left(\begin{array}{c|c} 1 & \infty \\ \hline 1 & x_1 \\ \hline & 1 \end{array} \right) &= \left(1 \quad \frac{x_1}{1-x_1} \right), \\ \Psi \left(\begin{array}{c|c} 1 & x_i \\ \hline 1 & x_{i+1} \\ \hline & 1 \end{array} \right) &= \frac{x_{i+1}}{x_i} \begin{pmatrix} -\frac{x_i(x_{i+1}-1)}{(x_i-1)x_{i+1}} & 1 \\ \frac{x_i-x_{i+1}}{x_{i+1}} & -\frac{x_i-1}{x_{i+1}-1} \end{pmatrix}, \\ \Psi \left(\begin{array}{c|c} 1 & x_i \\ \hline 1 & x_{i+1} \\ \hline & 1 \end{array} \right) &= \frac{x_{i+1}}{x_i} \begin{pmatrix} \frac{x_i(x_{i+1}-1)}{(x_i-1)x_{i+1}} & 1 \\ \frac{x_i-x_{i+1}}{x_{i+1}} & \frac{x_i-1}{x_{i+1}-1} \end{pmatrix}, \end{aligned}$$

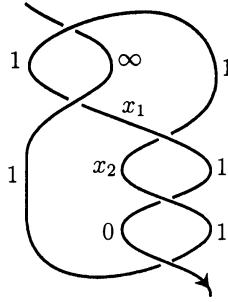
$$\Psi \left(\begin{array}{c} 1 \\ x_{m-1} \\ 0 \end{array} \middle| \begin{array}{c} 1 \\ 1 \end{array} \right) = \left(\begin{array}{c} 1 \\ 1-x_{m-1} \\ 1 \end{array} \right), \Psi \left(\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \middle| \begin{array}{c} x_{m-1} \\ 0 \end{array} \right) = \left(\begin{array}{c} 1 \\ x_{m-1}-1 \\ 1 \end{array} \right).$$

Noting that $\omega(K)^2$ can be presented in terms of the Hessian of the potential function defined from a parametrized open diagram, it follows that $\frac{1}{\sqrt{-1}\omega(K)^2} = \Psi(D)$.

Showing that the values of $\Phi(D)$ and $\Psi(D)$ satisfy the same recursion formula, we prove Theorem 1.

3 Example

In this section, we explain our results for the $\overline{5}_2$ knot K , which is presented by the following diagram D :



From (2), the hyperbolicity equations are presented by

$$(1 - x_1)(1 - \frac{1}{x_1}) = 1 - \frac{x_2}{x_1}, \quad (1 - \frac{x_2}{x_1})(1 - \frac{1}{x_2}) = 1 - x_2.$$

Hence,

$$x_1^3 - 2x_1^2 + 3x_1 - 1 = 0.$$

Corresponding to the holonomy representation of the hyperbolic structure of the knot complement, we choose a solution

$$x_1 = 0.784920145... + \sqrt{-1} \cdot 1.307141278... ,$$

which gives the complex hyperbolic volume by

$$\varsigma(K) = \frac{1}{2\pi\sqrt{-1}} V(x_1, x_2) = 0.450109610... - \sqrt{-1} \cdot 0.4813049796... .$$

Then, from the definition of $\Phi(D)$, we obtain

$$\frac{2}{\tau(K)} = x_1(x_1 - 1) \begin{pmatrix} 1 & 2x_1 & 0 \end{pmatrix} \cdot x_2 \begin{pmatrix} 1 & 0 & 0 \\ -1 & -x_2 & 0 \\ 1 & 2x_2 & 1 \end{pmatrix} \cdot \frac{x_2^3}{(x_2 - 1)^3} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad (4)$$

$$= -0.6323164993... + \sqrt{-1} \cdot 2.2345852998... , \quad (5)$$

and, hence, the value of the twisted Reidemeister torsion of K is give by

$$\tau(K) = -0.2344867659... - \sqrt{-1} \cdot 0.8286683659... \quad (6)$$

Let us confirm that the above value is also obtained from [12], by transforming the Reidemeister torsion associated with the longitude (of [12]) to the Reidemeister torsion associated with the meridian (the above value) as mentioned in [5].

The knot group $\pi_1(K)$ of K is presented by $\pi_1(K) = \langle a, b \mid aw^2 = w^2b \rangle$, where $w = ab^{-1}a^{-1}b$. The meridian longitude system (μ, λ) is presented in $\pi_1(K)$ by

$$\mu = a, \quad \lambda = (ab^{-1}a^{-1}b)^2(ba^{-1}b^{-1}a)^2.$$

A non-abelian representation $\rho : \pi_1(K) \rightarrow \mathrm{SL}_2\mathbb{C}$ is parametrized by two parameters u and s as follows:

$$\rho(a) = \begin{pmatrix} \sqrt{s} & \frac{1}{\sqrt{s}} \\ 0 & \frac{1}{\sqrt{s}} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} \sqrt{s} & 1 \\ -\sqrt{s}u & \frac{1}{\sqrt{s}} \end{pmatrix},$$

where s and u satisfies the Riley's equation $\phi_K(s, u) = 0$. The Riley's polynomial $\phi_K(s, u)$ [10] is given by

$$\phi_K(s, u) = -\frac{1}{s^2}(-2s + 3s^2 - 2s^3 + u - 3su + 6s^2u - 3s^3u + s^4u - 2su^2 + 3s^2u^2 - 2s^3u^2 + s^2u^3).$$

The holonomy representation ρ_0 corresponds to the case $s = 1$ and $\phi_K(1, u) = 1 - 2u + u^2 - u^3$. By [12], the Reidemeister torsion $T_\lambda^{\rho_0}(K)$ associated with the longitude is given by

$$T_\lambda^{\rho_0}(K) = -\frac{(2+u)(2+7u)}{u^3(4+u^2)}.$$

Let $l_{1,1}(s, u)$ be the $(1, 1)$ -entry of $\rho(\lambda)$. As mentioned in [5], we can transform $T_\lambda^{\rho_0}(K)$ to the Reidemeister torsion $\tau(K)$ associated with the meridian by the formula

$$\pm\tau(K) = 2 \left(\frac{\partial l_{1,1}}{\partial s} + \frac{\partial l_{1,1}}{\partial u} \frac{du}{ds} \right) \Big|_{s=1} \frac{1}{T_\lambda^{\rho_0}(K)}.$$

Then, choosing the solution $u = 1 - x_1 = 0.21508 - \sqrt{-1} \cdot 1.30714$ of $\phi_K(1, u) = 0$ (see [8, Appendix D]), we obtain

$$\begin{aligned} \pm\tau(K) &= \frac{2u^4(2+u^2)(4+u^2)(2+4u^2+u^4)}{(2+u)(2+7u)} \\ &= -0.234487 - \sqrt{-1} \cdot 0.828668, \end{aligned}$$

which coincides with (6). Moreover, from the definition of $\Psi(D)$,

$$\begin{aligned} \frac{1}{\sqrt{-1} \omega(K)^2} &= \begin{pmatrix} 1 & \frac{x_1}{1-x_1} \end{pmatrix} \cdot \frac{x_2}{x_1} \cdot \begin{pmatrix} \frac{x_1(x_2-1)}{(x_1-1)x_2} & 1 \\ \frac{x_1-x_2}{x_2} & \frac{x_1-1}{x_2-1} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{1-x_2} & 1 \end{pmatrix} \\ &= -0.632316... + \sqrt{-1} \cdot 2.23459..., \end{aligned}$$

which agrees with (5). On the other hand, in [6], it is rigorously shown that

$$\langle K \rangle_N \sim e^{N\varsigma(K)} \cdot N^{3/2} \cdot \omega(K). \quad (7)$$

Hence, we confirm Conjecture 2 for K .

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