

DIRICHLET METHOD OF SUMMABILITY AND NONLINEAR ERGODIC THEOREMS IN HILBERT SPACE

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My purpose in this exposé is to give a brief summary of some recent results concerning the summation method of Dirichlet's type and nonlinear ergodic theorems of Dirichlet's type in Hilbert spaces. The exposé is mainly a report on the author's personal work on the subject, by a general survey. Most of the results mentioned below were discussed in [6] and [7].

Let X be a complex Banach space and let $B[X]$ be the Banach algebra of bounded linear operators from X to itself. For a given $T \in B[X]$, the resolvent set of T denoted by $\rho(T)$ is the set of $\lambda \in \mathbb{C}$ for which $(\lambda I - T)^{-1}$ exists as an operator in $B[X]$ with domain X . The spectrum of T is the complement of $\rho(T)$ and denoted by $\sigma(T)$. $\rho(T)$ is an open subset of $\mathbb{C}$ and $\sigma(T)$ is a nonempty bounded closed subset of $\mathbb{C}$. So, the spectral radius $\gamma(T)$ of T is well-defined: in fact, $\gamma(T) = \sup\{|\sigma(T)|\} = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}$. The function $R(\lambda; T)$ defined by $R(\lambda; T) = (\lambda I - T)^{-1}$ for $\lambda \in \rho(T)$ is called the resolvent of T . It is well known [3] that $R(\lambda; T)$ is analytic in $\rho(T)$ and if $T \in B[X]$ and $|\lambda| > \gamma(T)$ then $\lambda \in \rho(T)$ and

$$R(\lambda; T) = (\lambda I - T)^{-1} = \sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n,$$

the series converging in the uniform operator topology. It is also known that if $d(\lambda)$ denotes the distance from $\lambda \in \mathbb{C}$ to $\sigma(T)$, then $\|R(\lambda; T)\| \geq 1/d(\lambda)$. If we take $\lambda = e^z$, $z = s + it$ ($s, t \in \mathbb{R}$), then $|\lambda| > \gamma(T)$ implies $s > \log \gamma(T)$ whenever $\gamma(T) > 0$. This characterization is a matter of great interest in connection with the question of what is the abscissa of uniform convergence of $R(\lambda; T)$ as a series.

Given $T \in B[X]$ let $\Phi(T)$ denote the class of all functions of complex variables which are analytic in some open set containing $\sigma(T)$. The following theorem is fundamental in the theory of linear ergodic theorems.

THEOREM 1 (Dunford [2]). Let $T \in B[X]$ and let $f_n \in \Phi(T)$ satisfy $\lim_{n \rightarrow \infty} f_n(1) = 1$ and (so) $\lim_{n \rightarrow \infty} (I - T)f_n(T) = \theta$ (the null operator). Then the following statements

are equivalent.

- (1) $(\text{so}) \lim_{n \rightarrow \infty} f_n(T) = E, E^2 = E, EX = \text{Ker}(I-T).$
- (2) $\{f_n(T)x\}$ is weakly sequentially compact for each $x \in X$.
- (3) $X = \text{Ker}(I-T) \oplus \overline{(I-T)X}, \sup_n \|f_n(T)\| < \infty.$

Suppose $f_n, f \in B[X]$, where $T \in B[X]$. If $f_n(T)$ converges strongly (or uniformly) then $f(T)f_n(T)$ converges strongly (or uniformly). We are interested in the converse problem. Under what conditions on f_n, f and T does the convergence of $f(T)f_n(T)$ imply the convergence of $f_n(T)$? Theorems describing this situation are in the nature of Tauberian theorems. The condition $(\text{so}) \lim_{n \rightarrow \infty} (I-T)f_n(T) = \theta$ appearing in Theorem 1 is just the case $f(T) = I-T$. Such Tauberian conditions are indispensable in discussing ergodic theorems. It should be noted that the strong regularity of summation methods has a very close connection with Tauberian conditions.

Now we consider the Dirichlet series of the following type

$$D[f, \mu; z](T) = \sum_{n=0}^{\infty} e^{-\mu_n z} f_n(T),$$

where $z \in \mathbb{C}$, $f = \{f_n\} \subset \Phi(T)$ and $\mu = \{\mu_n\}$, $0 \leq \mu_0 < \mu_1 < \dots < \mu_n \rightarrow \infty$ as $n \rightarrow \infty$.

THEOREM 2 ([6]). Let $T \in B[X]$ and $f_n \in \Phi(T)$ be such that $\sup_n \|\sum_{k=0}^n f_k(T)\| > 0$ and define

$$a_{\mu}(f; T) = \limsup_{n \rightarrow \infty} \frac{\log \|\sum_{k=0}^n f_k(T)\|}{\mu_n}$$

with $f = \{f_n\}$ and $\mu = \{\mu_n\}$. Then the following statements hold.

- (1) If $s > 0$ and $D[f, \mu; z](T)$ converges in the uniform operator topology for any $z \in \mathbb{C}$ with $z = s+it$, $t \in \mathbb{R}$, then $s \geq a_{\mu}(f; T)$.
- (2) The Dirichlet series $D[f, \mu; z](T)$ converges in the uniform operator topology for any $z \in \mathbb{C}$ with $\text{Re}(z) > \max(0, a_{\mu}(f; T))$ when $a_{\mu}(f; T) < \infty$.

If $0 \leq a_{\mu}(f; T) < \infty$ in Theorem 2, we say that the number $a_{\mu}(f; T)$ is the abscissa of uniform convergence of the Dirichlet series $D[f, \mu; z](T)$. This theorem plays a fundamental role in investigating ergodic theorems of Dirichlet's type.

COROLLARY 3. Let $T \in B[X]$ satisfy the conditions $\sup_n \|\sum_{k=0}^n T^k\| > 0$ and $(\text{uo}) \lim_{n \rightarrow \infty} T^n/n^{\omega} = \theta$ for some $0 < \omega \leq 1$. Let $f = \{f_n\}$, $f_n(T) = T^n$ ($n \geq 0$) and $\mu = \{\mu_n\}$, $\mu_n = n+1$ ($n \geq 0$). Then $a_{\mu}(f; T) \leq 0$ and $R(\lambda; T) = \sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n$ converges in the

uniform operator topology for $\lambda \in \mathbb{C}$ with $\log |\lambda| > 0$.

Proof. In view of Theorem 2, all that is required is to show that $a_\mu(f; T) \leq 0$. For a sufficiently small $\varepsilon > 0$, there exists by assumption an integer $N \geq 1$ such that

$$\|T^n\| < \varepsilon n^\omega \quad \text{for all } n > N.$$

Then we have

$$\begin{aligned} a_\mu(f; T) &= \limsup_{n \rightarrow \infty} \frac{\log \left\| \sum_{k=0}^n T^k \right\|}{n+1} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\log(n+1) + \log \left\{ \max_{0 \leq k \leq N} \|T^k\| + \varepsilon n^\omega \right\}}{n+1} = 0, \end{aligned}$$

as desired.

Let $\mu = \{\mu_n\}$ ($n \geq 0$) be a sequence of real numbers satisfying the following conditions:

- (i) $0 \leq \mu_0 < \mu_1 < \dots < \mu_n \rightarrow \infty$ as $n \rightarrow \infty$;
- (ii) $\inf_{n \geq 0} \{\mu_{n+1} - \mu_n\} = \delta$ for some $\delta > 0$;
- (iii) $\lim_{n \rightarrow \infty} \{\mu_{n+1} / \mu_n\} = 1$;
- (iv) $\lim_{s \rightarrow 0+} g(s) = +\infty$;
- (v) $\sup_{s > 0} \frac{1}{g(s)} \sum_{n=0}^{\infty} n \{e^{-\mu_n s} - e^{-\mu_{n+1} s}\} < \infty$,

where $g(s) = \sum_{n=0}^{\infty} e^{-\mu_n s}$ which converges for $s > 0$. Such a sequence $\mu = \{\mu_n\}$ determines a strongly regular method of summability. This new summation method will be called the (D, μ) -method (Dirichlet method of summability). Let H be a real Hilbert space and let C be a nonempty bounded closed convex subset of H . A mapping $T: C \rightarrow C$ is called asymptotically nonexpansive with Lipschitz constants $\{\alpha_n\}$ if

$$\|T^n x - T^n y\| \leq (1 + \alpha_n) \|x - y\| \quad \text{for all } n \geq 0 \text{ and all } x, y \in C,$$

where $\alpha_n \geq 0$ for all $n \geq 0$ and $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$ (see Goebel and Kirk [4]). In particular, if $\alpha_n = 0$ for all $n \geq 0$ then T is said to be nonexpansive. If T is an asymptotically nonexpansive mapping on C , then for any $x \in C$

$$\left\| \frac{1}{g(s)} \sum_{n=p}^{p+q} e^{-\mu_n s} T^n x \right\| \leq M_C \left(\frac{1}{g(s)} \sum_{n=p}^{p+q} e^{-\mu_n s} \right) \rightarrow 0$$

as $p, q \rightarrow \infty$, where $M_C = \sup(\|x\|: x \in C)$. This means that for any $x \in C$,

$$D_s^{(\mu)}[T]x = \frac{1}{g(s)} \sum_{n=0}^{\infty} e^{-\mu_n s} T^n x$$

is well defined for $s > 0$. Furthermore, for each $x \in C$ there exists a unique point $x_0 \in C$ such that

$$\limsup_{n \rightarrow \infty} \|T^n x - x_0\| = \inf_{y \in C} [\limsup_{n \rightarrow \infty} \|T^n x - y\|].$$

Such a point x_0 is called the asymptotic center of the sequence $\{T^n x\}$ (see Lim [5] and Brézis and Browder [1]). We are particularly interested in the weak and strong convergence of $D_s^{(\mu)}[T]x$ when $s \rightarrow 0+$.

THEOREM 4 ([7]). Let C be a nonempty bounded closed convex subset of H and let T be an asymptotically nonexpansive mapping of C into itself. Let $\mu = \{\mu_n\}$ be the (D, μ) -method. Then for any $x \in C$, $D_s^{(\mu)}[T]x$ converges weakly to the asymptotic center of $\{T^n x\}$ as $s \rightarrow 0+$.

Following the idea of Brézis and Browder [1], we say that the (D, μ) -method is proper if for each $\{\beta(n)\} \in \ell^\infty$ with $\beta(\cdot) \geq 0$, $[g(s)]^{-1} \sum_{n=0}^{\infty} e^{-\mu_n s} \beta(n)$ converges to some δ as $s \rightarrow 0+$, then

$$\lim_{s \rightarrow 0+} \left(\frac{1}{g(s)} \right)^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} e^{-(\mu_n + \mu_k)s} \beta(|n-k|) = \delta.$$

For example, the (D, μ) -method $\mu = \{\mu_n\}$ given by $\mu_n = an + b$, where $a > 0$ and $b > 0$, satisfies the properness condition just mentioned.

THEOREM 5 ([7]). Let C be a nonempty bounded closed convex subset of H and let T be a nonexpansive nonlinear mapping of C into itself. Let $\mu = \{\mu_n\}$ be the proper (D, μ) -method. Suppose that

- (i) $0 \in C$ and $T(0) = 0$;
- (ii) For some $c > 0$, T satisfies for all $u, v \in C$ the inequality

$$|\langle Tu, Tv \rangle - \langle u, v \rangle| \leq c \{ \|u\|^2 - \|Tu\|^2 + \|v\|^2 - \|Tv\|^2 \};$$

(iii) there is an element $\{\beta(n)\} \in \ell^\infty$ with $\beta(\cdot) \geq 0$ such that for any $x \in C$

$$|\langle T^p x, T^q x \rangle - \beta(|p-q|)| \leq \gamma_{\min(p,q)},$$

where $\gamma_{\min(p,q)} \rightarrow \infty$ as $\min(p,q) \rightarrow \infty$.

Then for each $x \in C$, $D_s^{(\mu)}[T]x$ converges strongly as $s \rightarrow 0+$ to the asymptotic center of $\{T^n x\}$.

Next we consider the convergence of the sequences $\{x_n\} \subset C$ generated by the iteration procedures (called Mann's type and Halpern-Wittmann's type) by the Dirichlet method.

THEOREM 6 ([7]). Let C be a nonempty bounded closed convex subset of H and let T be a nonexpansive nonlinear mapping of C into itself. Let $\mu = \{\mu_n\}$ be the (D, μ) -method. Define (the Mann's type sequence)

$$\begin{aligned} x_1 &= x \in C \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) D_{s_n}^{(\mu)}[T]x_n \quad \text{for } n \geq 1, \end{aligned}$$

where $\{\alpha_n\}$ is a sequence in $[0, a]$ for some $0 < a < 1$ and $s_n \rightarrow 0+$ as $n \rightarrow \infty$. Then the sequence $\{x_n\}$ so defined converges weakly to the asymptotic center of $\{T^n x\}$.

THEOREM 7. Let C be a nonempty bounded closed convex subset of H and let T be a nonexpansive nonlinear mapping of C into itself. Let $\mu = \{\mu_n\}$ be the (D, μ) -method. Define (the Halpern-Wittmann's type sequence)

$$\begin{aligned} x_0 &= x \in C \\ x_{n+1} &= \beta_n x + (1 - \beta_n) D_{s_n}^{(\mu)}[T]x_n \quad \text{for } n \geq 0, \end{aligned}$$

where $s_n \rightarrow 0+$ as $n \rightarrow \infty$ and $\{\beta_n\}$ is a sequence in $[0, 1]$ satisfying the conditions

$$\lim_{n \rightarrow \infty} \beta_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \beta_n = \infty.$$

Then the sequence $\{x_n\}$ so defined converges strongly to Px , where P is the metric projection of H onto $\text{Fix}(T)$. Moreover, Px coincides with the asymptotic center of $\{T^n x\}$.

Proof. Note that $\text{Fix}(T) \neq \emptyset$ by Theorem 4 and let $z \in \text{Fix}(T)$. Then

$$\begin{aligned} \|x_1 - z\| &= \|\beta_0 x + (1 - \beta_0) D_{s_0}^{(\mu)}[T]x - z\| \\ &\leq \beta_0 \|x - z\| + (1 - \beta_0) \|D_{s_0}^{(\mu)}[T]x - z\| \\ &\leq \beta_0 \|x - z\| + (1 - \beta_0) \|x - z\| \\ &= \|x - z\|, \end{aligned}$$

and so, by the induction argument, $\|x_n - z\| \leq \|x - z\|$ for all $n \geq 0$. This implies that $\{x_n\}$ and $\{D_{s_n}^{(\mu)}[T]x_n\}$ are both bounded. Next we claim that

$$\limsup_{n \rightarrow \infty} \langle x - Px, D_{s_n}^{(\mu)}[Tx_n] - Px \rangle \leq 0.$$

Since $\{D_{s_n}^{(\mu)}[Tx_n]\}$ is bounded, there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle x - Px, D_{s_n}^{(\mu)}[Tx_n] - Px \rangle = \lim_{i \rightarrow \infty} \langle x - Px, D_{s_{n_i}}^{(\mu)}[Tx_{n_i}] - Px \rangle.$$

We may assume that $(w)\lim_{i \rightarrow \infty} D_{s_{n_i}}^{(\mu)}[Tx_{n_i}] = z_0$ for some $z_0 \in C$ (through a subsequence of $\{n_i\}$, if necessary). Using Lemma 2 in [7] we have

$$\lim_{i \rightarrow \infty} \|D_{s_{n_i}}^{(\mu)}[Tx_{n_i}] - TD_{s_{n_i}}^{(\mu)}[Tx_{n_i}]\| = 0,$$

and thus, the demiclosedness of $I-T$ at 0 yields $z_0 \in \text{Fix}(T)$. Hence

$$\lim_{i \rightarrow \infty} \langle x - Px, D_{s_{n_i}}^{(\mu)}[Tx_{n_i}] - Px \rangle = \langle x - Px, z_0 - Px \rangle \leq 0.$$

Now, given $\varepsilon > 0$ sufficiently small, we can choose an integer $n_0 \geq 1$, no matter how large, such that for all $n \geq n_0$

$$\beta_n \|x - Px\|^2 \leq \frac{\varepsilon}{2} \quad \text{and} \quad \langle x - Px, D_{s_n}^{(\mu)}[Tx_n] - Px \rangle \leq \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned} \|x_{n+1} - Px\|^2 &= \|\beta_n x + (1-\beta_n) D_{s_n}^{(\mu)}[Tx_n] - Px\|^2 \\ &= \beta_n^2 \|x - Px\|^2 + (1-\beta_n)^2 \|D_{s_n}^{(\mu)}[Tx_n] - Px\|^2 \\ &\quad + 2\beta_n(1-\beta_n) \langle x - Px, D_{s_n}^{(\mu)}[Tx_n] - Px \rangle \\ &\leq \beta_n \varepsilon + (1-\beta_n) \|x_n - Px\|^2, \end{aligned}$$

and inductively

$$\begin{aligned} \|x_{n+1} - Px\|^2 &\leq \left\{1 - \prod_{i=n_0}^n (1-\beta_i)\right\} \varepsilon + \prod_{i=n_0}^n (1-\beta_i) \|x_{n_0} - Px\|^2 \\ &\leq \varepsilon + \exp\left\{-\sum_{i=n_0}^n \beta_i\right\} \|x_{n_0} - Px\|^2. \end{aligned}$$

Hence since $\sum_{n=0}^{\infty} \beta_n = \infty$ we have

$$\limsup_{n \rightarrow \infty} \|x_n - Px\|^2 \leq \varepsilon.$$

The final stage of the proof is to show that Px coincides with the asymptotic center of the sequence $\{T^n x\}$. From the definition of the sequence $\{x_n\}$ it follows that

$$Px \in \text{Fix}(T) \cap \left(\bigcap_{n \geq 0} \overline{\text{co}} \{T^k x : k \geq n\} \right).$$

Let u be the asymptotic center of $\{T^n x\}$. Then $u \in \text{Fix}(T)$ (cf. Brézis and Browder [1]). We claim that $Px = u$. Suppose, for a contradiction, that $Px \neq u$. We define

$$\rho(x : z) = \liminf_{n \rightarrow \infty} \|T^n x - z\|$$

for $z \in C$. Then there exists a subsequence $\{n_i\}$ of $\{n\}$ for which $\lim_{i \rightarrow \infty} \|T^{n_i} x - Px\| = \rho(x : Px)$. So, for any $\varepsilon > 0$ we can find an integer $i_0 = i_0(x, Px, \varepsilon)$, no matter how large, such that $\|T^{n_{i_0}} x - Px\| < \rho(x : Px) + \varepsilon$. Therefore, since T is nonexpansive, we have

$$\|T^{n+n_{i_0}} x - Px\| \leq \|T^{n_{i_0}} x - Px\| < \rho(x : Px) + \varepsilon$$

for all $n \geq 0$. This gives $\lim_{n \rightarrow \infty} \|T^n x - Px\| = \rho(x : Px)$. Similarly we get $\lim_{n \rightarrow \infty} \|T^n x - u\| = \rho(x : u)$. Hence

$$\begin{aligned} \rho(x : u) &= \lim_{n \rightarrow \infty} \|T^n x - u\| \\ &= \inf \left[\limsup_{n \rightarrow \infty} \|T^n x - y\| : y \in C \right] \\ &< \limsup_{n \rightarrow \infty} \|T^n x - Px\| \\ &= \lim_{n \rightarrow \infty} \|T^n x - Px\| = \rho(x : Px). \end{aligned}$$

Taking into account that H is a Hilbert space, let K be the closed convex set of all $z \in H$ such that $\|z - u\| \leq \|z - Px\|$. Then one can find an integer m_0 for which $\{T^n x : n \geq m_0\} \subset K$, and hence

$$\overline{\text{co}} \{T^n x : n \geq m_0\} \subset K.$$

Whereas K does not contain Px , in reality Px belongs to $\overline{\text{co}} \{T^n x : n \geq m_0\}$. This is a contradiction and $Px = u$. This completes the proof of the theorem.

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