

Properties of the solutions of certain differential equations

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Abstract

The main object of this paper is to investigate several geometric properties of the solutions of second order ordinary differential equations.

1. Introduction

Let A denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disk

$$U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}.$$

Also, let S , S^* and $S^*(\alpha)$ denote the subclasses of A consisting of functions which are,

respectively, univalent, starlike with respect to the origin, and starlike of order α in U ($0 \leq \alpha < 1$). Thus, by definition, we have (see for detail [1, 4]),

$$S^*(\alpha) := \left\{ f : f \in A \text{ and } \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \ (z \in U; 0 \leq \alpha < 1) \right\} \quad (1.2)$$

and

$$S^* := S^*(0). \quad (1.3)$$

Furthermore, S_p denote the subclasses of A with the property

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \quad (z \in U), \quad (1.4)$$

and UCV denote the subclasses of A with the property

$$\left| \frac{zf''(z)}{f'(z)} \right| < \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \quad (z \in U). \quad (1.5)$$

Remark 1. (1) $f(z) \in UCV \Leftrightarrow zf'(z) \in S_p$.

$$(2) \quad S_p \subset S^*\left(\frac{1}{2}\right)$$

2. A class of bounded functions and earlier results

Let B_f denote the class of bounded functions

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, \quad (2.1)$$

analytic in U , for which

$$|w(z)| < J \quad (z \in U; J > 0). \quad (2.2)$$

Definition 1 Let H_J be the class of complex functions

$h(u, v)$ satisfying each of the following conditions:

- (i) $h(u, v)$ is continuous in a domain $D \subset \mathbb{C} \times \mathbb{C}$;
- (ii) $(0, 0) \in D$ and $|h(0, 0)| < J$ ($J > 0$);
- (iii) $|h(Je^{i\theta}, Ke^{i\theta})| < J$ whenever $(Je^{i\theta}, Ke^{i\theta}) \in D$
($\theta \in \mathbb{R}, K \geq J > 0$).

Example 1. It is easily seen that the functions

- (1) $h(u, v) = \gamma u + v \in H_J$, $\gamma \in \mathbb{C}$ ($\operatorname{Re} \gamma \geq 0$), $D = \mathbb{C} \times \mathbb{C}$.
- (2) $h(r, s) = r^2 + r + s \in H_J$, $D = \mathbb{C} \times \mathbb{C}$.

Definition 2 Let $h \in H_J$ with corresponding domain D .

We denote by $B_J(h)$ the class of functions $w(z)$ given by (2.1), which are analytic in U and satisfy each of following conditions:

- (i) $(w(z), zw'(z)) \in D$ ($z \in U$),
- (ii) $|h(w(z), zw'(z))| < J$ ($z \in U; J > 0$).

The function class $B_J(h)$ is not empty. Indeed, for any $h \in H_J$, we have

$$w(z) = c, z \in B_J(h), \quad (2.3)$$

for sufficiently small $|c|$ depending on h .

We need the following lemmas to prove our results.

Lemma 1 (see [6]) For any $h \in H_J$,

$$B_J(h) \subset B_J \quad (h \in H_J; 0 < J \leq 1).$$

Lemma 1 leads us immediately to the following result, which also given by [6].

Lemma 2 ([6]) Let $h \in H_J$ and let the function $b(z)$ be analytic in U with

$$|b(z)| < J \quad (z \in U; 0 < J \leq 1).$$

If the following initial-value problem:

$$h(w(z), zw'(z)) = b(z) \quad (w(0) = 0) \quad (2.4)$$

has a solution $w(z)$ analytic in U , then

$$|w(z)| < J \quad (z \in U; 0 < J \leq 1). \quad (2.5)$$

Using Lemma 2, we proved several results.

For example,

Theorem A (See [6]) Let $a(z)$ and $b(z)$ be analytic in U with

$$\left| z \left(b(z) - \frac{1}{2} a'(z) - \frac{1}{4} [a(z)]^2 \right) \right| < \frac{1}{2} \quad (2.6)$$

and

$$|a(z)| < 1. \quad (2.7)$$

Let $w(z)$ denote the solution of the initial-value problem:

$$w''(z) + a(z) w'(z) + b(z) w(z) = 0 \quad (2.8)$$

$$(z \in U, w(0) = w'(0) = 0)$$

Then $w(z)$ is starlike in U .

Example 2 Let $a(z) = -z$, $b(z) = \frac{z^2}{4}$ in Theorem A, then a solution of

$$w''(z) - z w'(z) + \frac{z^2}{4} w(z) = 0 \quad (2.9)$$

is $w(z) = \sqrt{2} e^{\frac{z^2}{4}} \sin \frac{z}{\sqrt{2}}$. This function $w(z)$ is starlike function.

3 Main results and their consequences

Theorem 1 Let $a(z)$ and $b(z)$ be analytic in U with

$$|z(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2)| < J \quad (z \in U; 0 < J < 1) \quad (3.1)$$

and

$$|a(z)| \leq K \quad (0 < K \leq 2 - 2J). \quad (3.2)$$

Let $w(z)$ ($z \in U$) be the solution of the initial-value problem (2.8). Then, $w(z)$ is starlike of order $1 - J - \frac{K}{2}$.

Proof. The transformation

$$w(z) = \exp\left(-\frac{1}{2} \int_0^z a(\xi) d\xi\right) v(z) \quad (3.3)$$

leads to the normal form

$$v''(z) + \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2\right) v(z) = 0 \quad (3.4)$$

and $v(0) = v'(0) - 1 = 0$. If we put

$$u(z) = \frac{zv'(z)}{v(z)} - 1 \quad (z \in U), \quad (3.5)$$

then $u(z)$ is analytic in U , $u(0) = 0$ and (3.4)

becomes

$$[u(z)]^2 + u(z) + zu'(z) = -z^2\left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2\right), \quad (3.6)$$

or equivalently

$$h(u(z), zu'(z)) = -z^2 \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2 \right), \quad (3.7)$$

where $h(r, s) = r^2 + r + s$. It is easy to check

$h(r, s) \in H_J$, i.e.,

(i) $h(r, s)$ is continuous in $\mathbb{C} \times \mathbb{C}$;

(ii) $(0, 0) \in \mathbb{C} \times \mathbb{C}$, $|h(0, 0)| = 0 < J$;

(iii) $|h(Je^{i\theta}, Ke^{i\theta})| \geq J$ ($K \geq J$).

From (3.1), we have

$$|-z^2 \left(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2 \right)| < J \quad (z \in U).$$

By using Lemma 2, we obtain $|u(z)| < J$ ($z \in U$).

Therefore, we have

$$\left| \frac{zv'(z)}{v(z)} - 1 \right| < J \quad (z \in U).$$

This implies

$$1 - J < \operatorname{Re} \left\{ \frac{zv'(z)}{v(z)} \right\} < 1 + J \quad (z \in U). \quad (3.8)$$

From (3.3), we have

$$\exp\left(\frac{1}{2}\int_0^z a(\xi)d\xi\right) \cdot w(z) = v(z). \quad (3.9)$$

Logarithmically differentiating of (3.9) leads to

$$\frac{zw'(z)}{w(z)} = \frac{zv'(z)}{v(z)} - \frac{z}{2}a(z). \quad (3.10)$$

Combining (3.8), (3.10) and (3.2), we have

$$\begin{aligned} \operatorname{Re} \left\{ \frac{zw'(z)}{w(z)} \right\} &\geq \operatorname{Re} \left\{ \frac{zv'(z)}{v(z)} \right\} - \left| \frac{z}{2} a(z) \right| \\ &> 1 - J - \frac{K}{2} \quad (z \in U), \end{aligned} \quad (3.11)$$

where $0 < 2J + K \leq 2$, and thus $w(z)$ is starlike of order $1 - J - \frac{K}{2}$. Q.E.D.

Example 3. Let $a(z) = -\frac{2}{3}z$, $b(z) = \frac{z^2}{9}$ in Theorem 1, then the solution of

$$w''(z) - \frac{2}{3}z w'(z) + \frac{z^2}{9} w(z) = 0 \quad (3.12)$$

$$\text{is } w(z) = \sqrt{3} e^{\frac{z^2}{6}} \cdot \sin \frac{z}{\sqrt{3}} \in S^* \left(\frac{1}{3} \right).$$

Next, we prove

Theorem 2 Let $a(z)$ and $b(z)$ be analytic in U with

$$|z(b(z) - \frac{1}{2}a'(z) - \frac{1}{4}[a(z)]^2)| < J \quad (z \in U, 0 < J < \frac{1}{2}) \quad (3.13)$$

and

$$|a(z)| \leq K \quad (0 < K \leq 1 - 2J). \quad (3.14)$$

Let $w(z)$ ($z \in U$) be the solution of the initial-value problem (2.8). Then, $w(z) \in S_p$.

Proof. From (3.10) in the proof of Theorem 1,

$$\frac{zw'(z)}{w(z)} - 1 = \frac{zv'(z)}{v(z)} - 1 - \frac{z}{2}a(z).$$

Then we have

$$\begin{aligned} \left| \frac{zw'(z)}{w(z)} - 1 \right| &\leq \left| \frac{zv'(z)}{v(z)} - 1 \right| + \left| \frac{z}{2}a(z) \right| \\ &< J + \frac{K}{2} \quad (z \in U). \end{aligned} \quad (3.15)$$

From (3.11) and (3.15), we obtain

$$\operatorname{Re} \left\{ \frac{zw'(z)}{w(z)} \right\} > \left| \frac{zw'(z)}{w(z)} - 1 \right| \quad (0 < 2J + K \leq 1, \quad z \in U), \quad (3.16)$$

that is, $w(z) \in S_p$. Q.E.D.

Example 4 Let $a(z) = -\frac{z}{2}$ and $b(z) = \frac{z^2}{16}$ in

Theorem 2, the solution of

$$w''(z) - \frac{z}{2}w'(z) + \frac{z^2}{16}w(z) = 0 \quad (3.17)$$

is $w(z) = 2e^{\frac{z^2}{8}} \sin \frac{z}{2} \in S_p$.

Also, $2e^{\frac{z^2}{8}} \sin \frac{z}{2} \in S^*(\frac{1}{2})$.

Furthermore, we prove the following theorems.

Theorem 3 Let $z p(z)$ be analytic in U with

$|z p(z)| < J$ ($z \in U$; $0 < J \leq 1$). Let $w(z)$, $z \in U$, be the

solution of the following differential equation

$$w''(z) + p(z)w(z) = 0 \quad (3.18)$$

with $w(0) = 0$ and $w'(z) \neq 0$. Then the solution $w(z)$ is starlike of order $1-J$, that is,

$$\operatorname{Re} \left\{ \frac{zw'(z)}{w(z)} \right\} > 1-J \quad (z \in U; 0 < J \leq 1). \quad (3.19)$$

Proof. We put

$$u(z) = \frac{zw'(z)}{w(z)} - 1 \quad (z \in U). \quad (3.20)$$

Then $u(z)$ is analytic in U , $u(0) = 0$ and (3.18) becomes

$$[u(z)]^2 + u(z) + zu'(z) = -z^2p(z), \quad (3.21)$$

or equivalently

$$h(u(z), zu'(z)) = -z^2p(z), \quad (3.22)$$

where $h(r, s) = r^2 + r + s$. It is easy to check

$h(r, s) \in H_J$. From assumption, we have

$$|z^2p(z)| < J \quad (z \in U; 0 < J \leq 1).$$

By using Lemma 2, we obtain

$$|u(z)| < J \quad (z \in U; 0 < J \leq 1),$$

which, in view of the relationship (3.20),

yields

$$\left| \frac{zw'(z)}{w(z)} - 1 \right| < J \quad (z \in U; 0 < J \leq 1), \quad (3.23)$$

that is,

$$\operatorname{Re} \left\{ \frac{zw'(z)}{w(z)} \right\} > 1 - J.$$

This means $w(z) \in S^*(1-J)$. Q.E.D.

Remark 2 In [7], Shams, Kulkarni and Jahangiri introduced the following class $SD(\alpha, \beta)$.

Let $SD(\alpha, \beta)$ be the family of functions $f(z) \in A$ satisfying the inequality

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \quad (3.24)$$

$$(z \in U, \alpha \geq 0, 0 \leq \beta < 1).$$

We can see $SD(1, 0) = S_p$.

Next, we prove the following theorem.

Theorem 4 Let $zp(z)$ be analytic in U with $|zp(z)| < J$ ($z \in U; 0 < J \leq 1$). Let $w(z), z \in U$,

be the solution of differential equation (3.18).

Then the solution $w(z)$ is in $SD(\alpha, \beta)$, i.e.,

$$\operatorname{Re}\left\{\frac{zw'(z)}{w(z)}\right\} > \alpha \left| \frac{zw'(z)}{w(z)} - 1 \right| + \beta \quad (3.25)$$

$$(z \in U; 0 < J \leq \frac{1-\beta}{1+\alpha}),$$

where $\alpha \geq 0$ and $0 \leq \beta < 1$.

Proof. According to the proof of Theorem 3, we have (3.23). Therefore,

$$\operatorname{Re}\left\{\frac{zw'(z)}{w(z)}\right\} > 1 - J \geq \alpha J + \beta \geq \alpha \left| \frac{zw'(z)}{w(z)} - 1 \right| + \beta$$

$$(0 < J \leq \frac{1-\beta}{1+\alpha}, \alpha \geq 0, 0 \leq \beta < 1).$$

That is $w(z) \in SD(\alpha, \beta)$. Q.E.D.

Example 5 (1) Let $p(z) = \frac{6}{7} - \frac{4}{49}z^2$ in Theorem 3

and Theorem 4. Then

$$|z p(z)| < \frac{46}{49},$$

therefore,

$$\operatorname{Re}\left\{\frac{zw'(z)}{w(z)}\right\} > \frac{3}{49}, \text{ that is, } w(z) \in S^*\left(\frac{3}{49}\right).$$

And the solution $w(z)$ of

$$w''(z) + \left(\frac{6}{7} - \frac{4}{49} z^2 \right) w(z) = 0$$

is $w(z) = z e^{-\frac{z^2}{7}}$. Also,

$$\operatorname{Re} \left\{ \frac{zw'(z)}{w(z)} \right\} > \frac{3}{14} \left| \frac{zw'(z)}{w(z)} - 1 \right|, \text{ i.e., } w(z) \in SD\left(\frac{3}{14}, 0\right).$$

Furthermore, $w(z) \in SD\left(\frac{1}{7}, \frac{1}{49}\right)$.

(2) Let $p(z) = \frac{6}{13} - \frac{4}{169} z^2$ in Theorem 3, Theorem 4.

Then $|zp(z)| < \frac{82}{169}$, therefore

$$\operatorname{Re} \left\{ \frac{zw'(z)}{w(z)} \right\} > \frac{87}{169}, \text{ that is, } w(z) \in S^*\left(\frac{87}{169}\right).$$

And the solution $w(z)$ of

$$w''(z) + \left(\frac{6}{13} - \frac{4}{169} z^2 \right) w(z) = 0$$

is $w(z) = z e^{-\frac{z^2}{13}}$. Furthermore,

$$\operatorname{Re} \left\{ \frac{zw'(z)}{w(z)} \right\} > 3 \left| \frac{zw'(z)}{w(z)} - 1 \right| + \frac{9}{169}, \text{ i.e.,}$$

$$w(z) \in SD\left(3, \frac{9}{169}\right). \text{ Also, } w(z) \in SD\left(2, \frac{35}{169}\right)$$

$$\text{and } w(z) \in SD\left(1, \frac{61}{169}\right).$$

Putting $p(z) = \lambda + \frac{1}{2} - \frac{z^2}{4}$ in Theorem 3, Theorem 4,

we have

Corollary 1 We consider the Weber's differential equation

$$v''(z) + \left(\lambda + \frac{1}{2} - \frac{z^2}{4}\right) v(z) = 0 \quad (3.26)$$

If $\left|\lambda + \frac{1}{2} - \frac{z^2}{4}\right| < J$ ($z \in U$; $0 < J \leq 1$),

then the solution $v(z)$ is starlike of order $1-J$, that is, $v(z) \in S^*(1-J)$.

Corollary 2 We consider the Weber's differential equation (3.23).

If $\left|\lambda + \frac{1}{2} - \frac{z^2}{4}\right| < J$ ($z \in U$; $0 < J \leq \frac{1-\beta}{1+\alpha}$),

where $\alpha \geq 0$ and $0 \leq \beta < 1$, then $w(z) \in SD(\alpha, \beta)$.

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