

Filtered de Rham-Witt complexes and wildly ramified higher class field theory over finite fields

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(report on joint work with Shuji Saito and Yigeng Zhao)

Let k be a finite field of characteristic $p > 0$, and let X be a smooth projective variety of dimension d over k . Let $D \subset X$ be a divisor with simple normal crossings, and let $U = X - D$ be the open complement. Then for any prime $\ell \neq p$, each natural number m , and each integer j we have a perfect pairing of finite groups for the étale cohomology groups

$$H^i(U, \mathbb{Z}/\ell^m(j)) \times H_c^{2d+1-i}(U, \mathbb{Z}/\ell^m(d-j)) \longrightarrow H_c^{2d+1}(U, \mathbb{Z}/\ell^m(d)) \cong \mathbb{Z}/\ell^m,$$

where $\mathbb{Z}/\ell^m(j)$ denotes the j -th Tate twist of the constant étale sheaf $\mathbb{Z}/\ell^m$, and $H_c^m(U, -)$ is the cohomology with compact support. This can be used to describe the quotient $\pi_1^{ab}(U)/\ell^n$ of the abelianized fundamental group $\pi_1^{ab}(U)$, and, by passing to the inverse limit, the maximal abelian ℓ -adic quotient of $\pi_1^{ab}(U)$. In fact, for $j = 0$ and $i = 1$ we get isomorphisms for all m

$$H_c^{2d}(U, \mathbb{Z}/\ell^m(d)) \cong H^1(U, \mathbb{Z}/\ell^m)^\vee \cong \pi_1^{ab}(U)/\ell^m,$$

and an exact sequence

$$H^{2d-1}(D, \mathbb{Z}/\ell^m(d)) \rightarrow H_c^{2d}(U, \mathbb{Z}/\ell^m(d)) \rightarrow H^{2d}(X, \mathbb{Z}/\ell^m(d))$$

which provides a certain description of the middle group.

Now we consider p -coefficients. If D is empty, Milne obtained a perfect duality of finite groups

$$H^i(X, \nu_m^r) \times H^{d+1-i}(X, \nu_m^{d-r}) \rightarrow H^{d+1}(X, \nu_m^d) \cong \mathbb{Z}/p^r$$

where $\nu_m^r = \nu_{m,X}^r = W_m \Omega_{X,\log}^r \subset W_m \Omega_X^r$ are Illusie's logarithmic de Rham-Witt sheaves inside the components of the de Rham-Witt sheaves, which can be defined as the isomorphic image of the $d \log$ map

$$(1) \quad d \log : \mathcal{K}_{r,X}^M / p^m \xrightarrow{\cong} \nu_m^r \subset W_m \Omega_X^r,$$

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on the Milnor K -sheaf of X sending $\{a_1, \dots, a_r\}$ to $d \log[a_1]_m \wedge \dots \wedge d \log[a_r]_m$, where $[a]_m = (a, 0, \dots, 0) \in W_m(\mathcal{O}_X)$ is the Teichmüller representative. For $r = 0$ one has $\nu_m^0 = \mathbb{Z}/p^m$, and Milne's duality induces isomorphisms

$$H^d(X, \nu_m^d) \cong H^1(X, \mathbb{Z}/p^m)^\vee \cong \pi_1^{ab}(X)/p^m$$

If one tries to extend the above duality to the case where D is non-empty, one encounters the problem that there is no obvious analog of cohomology with compact support for de Rham-Witt sheaves or logarithmic de Rham-Witt sheaves. We propose the following approach. Let $\{D_\lambda\}_{\lambda \in \Lambda}$ be the (smooth) irreducible components of D . For $r \geq 1$ and $D = \sum_\lambda n_\lambda D_\lambda$ with $n_\lambda \in \mathbb{N}_0$, not all zero, let

$$\nu_{m,X|D}^r = W_m \Omega_{X|D, \log}^r \subset j_* W_m \Omega_{U, \log}^r$$

be the étale subsheaf generated étale locally by $d \log[x_1]_m \wedge \dots \wedge d \log[x_r]_m$ with $x_\nu \in \mathcal{O}_U^\times$ for all ν and $x_1 \in 1 + \mathcal{O}_X(-D)$, and let $\nu_{m,X|D}^0 = 0$.

As in the classical situation we have the following theorem:

Theorem 1. *The map $d \log$ induces an isomorphism*

$$d \log[-] : \mathcal{K}_{r,X|D}^M/p^m \xrightarrow{\cong} W_m \Omega_{X|D, \log}^r ; \{x_1, \dots, x_r\} \mapsto d \log[x_1]_m \wedge \dots \wedge d \log[x_r]_m.$$

Here $\mathcal{K}_{r,X|D}^M$ is the sheaf of relative (to D) Milnor K -groups which has been studied by one of the authors (S. Saito) with K. Rülling in [1].

The following property is immediate.

Lemma 1. *For $D_1 \leq D_2$ we have $\nu_{m,X|D_2}^r \subseteq \nu_{m,X|D_1}^r$.*

Moreover we show

Theorem 2. *There is an exact sequence*

$$0 \rightarrow \nu_{m-1,X|[D/p]}^r \rightarrow \nu_{m,X|D}^r \rightarrow \nu_{1,X|D}^r \rightarrow 0 ,$$

where $[D/p] = \sum_{\lambda \in \Lambda} [n_\lambda/p] D_\lambda$, with $[n_\lambda/p] = \min\{n' \in \mathbb{Z} \mid n' \geq n/p\}$.

By the isomorphism (1) above this is reduced to (difficult) calculations in Milnor K -theory of local rings (see [1]). Moreover, in analogy to Illusie's exact sequence

$$(2) \quad 0 \rightarrow \nu_{1,X}^r \rightarrow \Omega_X^r \xrightarrow{1-C^{-1}} \Omega_X^r / d\Omega_X^{r-1} \rightarrow 0$$

we prove the following.

Theorem 3. *One has an exact sequence*

$$0 \rightarrow \nu_{1,X|D}^r \rightarrow \Omega_{X|D}^r \xrightarrow{1-C^{-1}} \Omega_{X|D}^r / d\Omega_{X|D}^{r-1} \rightarrow 0 ,$$

where $\Omega_{X|D}^r = \Omega_X^r(\log D_{red})(-D)$.

An important tool for the duality is the introduction of Γ -filtered rings A for a not necessarily totally ordered abelian group Γ , given by collections of subgroups A^γ with $A^{\gamma'} \subset A^\gamma$ for $\gamma \leq \gamma'$ (descending filtration!) and $A^\gamma \dots A^{\gamma'} \subset A^{\gamma+\gamma'}$.

For a Γ -filtered ring A we consider an associated filtration on the Witt rings which is inspired by a filtration introduced by Kato and Brylinski: We say that a Witt vector $a = (a_0, a_1, a_2, \dots)$ is in $W(A)^\gamma$ if $a_i \in A^{p^i \gamma}$ for all $i \geq 0$.

Moreover, with this we define filtered de Rham-Witt sheaves, by using the universal definition of Hesselholt and Madsen [1] for $p \neq 2$, and Costeanu [3] for $p = 2$.

In our situation we start with the descending filtration on $j_*\mathcal{O}_U$, where for a divisor $D = \sum_\lambda n_\lambda D_\lambda$ with $n_\lambda \in \mathbb{Z}$ we define

$$f^D \mathcal{O}_U := \mathcal{O}_X(-D),$$

and the associated filtered de Rham-Witt complex

$$f^D W_m \Omega_U^r \subset j_* W_m \Omega_U^r.$$

Then we get

Theorem 4. *There is a perfect pairing between a discrete group and a profinite group*

$$H^i(U, \nu_{m,U}^r) \times \varprojlim_D H^{d+1-i}(X, \nu_{m,X|D}^{d-r}) \longrightarrow H^{d+1}(X, \nu_{m,X}^d) \cong \mathbb{Z}/p^m \mathbb{Z}$$

where the inverse limit is over the divisors (with multiplicities) D with support in D_{red} (compare Lemma 1).

The proof is in two steps. First of all, the pairings give a commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & \varprojlim_D H^{d+1-i}(X, \nu_{m-1,X|D/p}^{d-r}) & \longrightarrow & \varprojlim_D H^{d+1-i}(X, \nu_{m,X|D}^{d-r}) & \longrightarrow & \varprojlim_D H^{d+1-i}(X, \nu_{1,X|D}^{d-r}) \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & H^i(U, \nu_{m-1,U}^r)^\vee & \longrightarrow & H^i(U, \nu_{m,U}^r)^\vee & \longrightarrow & H^i(U, \nu_1^r)^\vee \longrightarrow \dots \end{array}$$

where the first row is induced by Theorem 1, and its exactness is due to the fact that the inverse limit is exact for projective systems of finite groups; and where the second row comes from the classical exact sequence

$$0 \rightarrow \nu_{1,U}^r \rightarrow \nu_{m,U}^r \rightarrow \nu_{m-1,U}^r \rightarrow 0.$$

so that its exactness is clear. Using this commutative diagram and induction on m , we reduce our question to the case $m = 1$. For $m = 1$ we use Theorem 2 to replace $\nu_{1,X|D}^r$ in the derived category by the two-term complex

$$\mathcal{F}^\bullet = [\Omega_{X|-D+D_{red}}^r \xrightarrow{1-C^{-1}} \Omega_{X|-D+D_{red}}^r / d\Omega_{X|-D+D_{red}}^{r-1}],$$

and similarly one can show that, in the derived category, $\nu_{1,U}^{d-r}$ is isomorphic to the direct limit (with respect to D) of two-term complexes

$$\mathcal{G}^\bullet = [Z\Omega_{X|D}^{d-r} \xrightarrow{1-C} \Omega_{X|D}^{d-r}],$$

where $Z\Omega_{X|D}^{d-r} = Z\Omega_X^{d-r} \cap \Omega_{X|D}^{d-r}$.

Finally $\nu_{1,X}^d$ is isomorphic to the two-term complex

$$\mathcal{H}^\bullet = [\Omega_X^d \xrightarrow{1-C} \Omega_X^d].$$

Now we use Milne's method of pairings of two-term complexes to see that one has a non-degenerate pairing

$$\mathcal{F}^\bullet \times \mathcal{G}^\bullet \longrightarrow \mathcal{H}^\bullet$$

which reduces the pairing to a duality in coherent $\mathcal{O}_X$ -sheaves, which also works in étale cohomology, and therefore also gives perfect pairings in étale cohomology

$$H^{d+1-i}(X, \mathcal{F}^\bullet) \times H^i(X, \mathcal{G}^\bullet) \longrightarrow H^{d+1}(X, \mathcal{H}^\bullet) \cong \mathbb{Z}/p^m\mathbb{Z}.$$

Passing to the inductive limit over D on the first terms and the inverse limit on D on the right term we obtain the wanted pairing in Theorem 3, because the left limit gives $H^i(U, \nu_{1,U}^r)$. For $i = 1$ and $r = 0$ Theorem 3 now gives a continuous isomorphism

$$\varprojlim_D H^d(X, \nu_{m,X|D}^d) \longrightarrow H^1(U, \mathbb{Z}/p^m)^\vee \cong \pi_1^{ab}(U)/p^m$$

which gives a canonical ramification filtration of the abelianized fundamental group on the right. A quotient is ramified of order D if it factors through $H^{d+1-i}(X, \nu_{1,X|D}^{d-r})$.

References

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