

INNER RADIUS OF UNIVALENCE FOR A STRONGLY STARLIKE DOMAIN

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ABSTRACT. The inner radius of univalence of a domain D with Poincaré density ρ_D is the possible smallest number σ such that the condition $\|S_f\|_D = \sup_{w \in D} \rho_D(w)^{-2} |S_f(w)| \leq \sigma$ implies the univalence of f for a nonconstant meromorphic function f on D , where S_f is the Schwarzian derivative of f . In this note, we will give a lower estimate of the inner radius of univalence for strongly starlike domains of order α with a concrete bound in terms of the order α .

1. MAIN RESULT

For a constant $0 \leq \alpha \leq 1$, a holomorphic function f on the unit disk is called *strongly starlike of order α* if f satisfies the condition

$$(1) \quad \left| \arg \frac{zf'(z)}{f(z) - f(0)} \right| \leq \frac{\pi\alpha}{2} \quad (z \in \mathbb{D}^* = \mathbb{D} \setminus \{0\}).$$

Note that a strongly starlike function is starlike in the usual sense. Every strongly starlike function f of order $\alpha < 1$ is bounded. In fact, Brannan and Kirwan [1] showed that

$$(2) \quad |f(z) - f(0)| \leq |zf'(0)|M(\alpha) \quad (z \in \mathbb{D}).$$

Here $M(\alpha)$ is defined by

$$\begin{aligned} M(\alpha) &= \exp \left[\int_0^1 \left\{ \left(\frac{1+t}{1-t} \right)^\alpha - 1 \right\} \frac{dt}{t} \right] \\ (3) \quad &= \exp \left\{ 2\alpha \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+1-\alpha)} \right\} \\ &= \frac{1}{4} \exp \left\{ -\frac{\Gamma'((1-\alpha)/2)}{\Gamma((1-\alpha)/2)} - \gamma \right\}, \end{aligned}$$

where Γ is the Gamma function and γ is the Euler constant.

A proper subdomain D of the complex plane $\mathbb{C}$ is said to be *strongly starlike of order α* with respect to a point $w_0 \in D$ if D is simply connected and if the Riemann mapping function $f: \mathbb{D} \rightarrow D$ of D with $f(0) = w_0$ is strongly starlike of order α . A strongly starlike domain of order 1 is nothing but a usual starlike domain. In what follows, without any pain, we always assume that $w_0 = 0$.

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We now introduce a standard domain adapted to the strong starlikeness. For a constant α with $0 < \alpha < 1$, we denote by V_α the bounded domain enclosed by the logarithmic spirals $\gamma_\alpha = \{\exp((- \tan(\pi\alpha/2) + i)\theta); 0 \leq \theta \leq \pi\}$ and $\bar{\gamma}_\alpha = \{w; \bar{w} \in \gamma_\alpha\}$.

Let D be a proper subdomain of $\mathbb{C}$ containing the origin. It will be convenient to consider the periodic function $R = R_D : \mathbb{R} \rightarrow (0, +\infty]$ of period 2π defined by

$$R(\theta) = \sup\{r > 0; [0, re^{i\theta}] \subset D\},$$

where $[a, b]$ denotes the closed line segment joining points a and b in $\mathbb{C}$. Note that R is lower semi-continuous.

In the sequel, we will use the convention $a \cdot D = \{aw; w \in D\}$ for $a \in \mathbb{C}$ and a domain D . Also, set $D^\vee = I(\text{Ext } D)$, where $\text{Ext } D = \widehat{\mathbb{C}} \setminus \overline{D}$ and $I(z) = 1/z$.

The next result will be fundamental for our aim here, whose proof can be found in [6].

Theorem A. *Let D be a proper subdomain of $\mathbb{C}$ with $0 \in D$ satisfying the condition $\text{Int } \overline{D} = D$ and let α be a constant with $0 < \alpha < 1$. Then the following conditions are equivalent.*

- (a) *D is strongly starlike of order α with respect to the origin.*
- (b) *$D^\vee$ is strongly starlike of order α with respect to the origin.*
- (c) *For each point $w \in D$ we have $w \cdot V_\alpha \subset D$.*
- (d) *The radius function $R = R_D$ is absolutely continuous and satisfies $|R'/R| \leq \tan(\pi\alpha/2)$ a.e. in $\mathbb{R}$.*

Remark. The implication (a) $\Rightarrow$ (d) is essentially due to Fait, Krzyż and Zygmunt [2]. Actually, we will employ their idea which was used to show the quasiconformal extendability of strongly starlike functions.

Let D be a subdomain of $\mathbb{C}$ with the hyperbolic metric $\rho_D(z)|dz|$ of constant curvature -4 . The *inner radius of univalence* of D , which will be denoted by $\sigma(D)$, is the possible maximal number σ for which the condition $\|S_f\|_D \leq \sigma$ implies the univalence of the nonconstant meromorphic function f on D , where S_f denotes the Schwarzian derivative $(f''/f')' - (f''/f')^2/2$ of f and $\|\varphi\|_D = \sup_{w \in D} \rho_D(w)^{-2} |\varphi(w)|$. Note that $\sigma(D)$ is Möbius invariant in the sense that $\sigma(L(D)) = \sigma(D)$ for a Möbius transformation L . In particular, $\sigma(D^\vee) = \sigma(\text{Ext } D)$. The reader may consult the textbook [4] by Lehto as a general reference for the inner radius of univalence and related notions. When D is simply connected, theorems of Ahlfors and Gehring imply that $\sigma(D) > 0$ if and only if D is a quasidisk, furthermore, $\sigma(D)$ is estimated from below by a positive constant $c(K)$ depending only on K for a K -quasidisk D . However, it is hard to give an explicit lower estimate of $\sigma(D)$ for a concrete quasidisk D in general. Our second result concerns the inner radius of univalence of strongly starlike domains.

Theorem 1. *A strongly starlike domain D of order α satisfies*

$$(4) \quad \sigma(D) \geq \frac{2}{M(\alpha)^2} \cdot \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)},$$

where $M(\alpha)$ is defined by (3).

Remarks. 1. When α tends to 0, the right-hand side above tends to 2. On the other hand, it is known that $\sigma(\mathbb{D}) = 2$. See also the final section.

2. By a result of Fait, Krzyż and Zygmunt [2], we know that a strongly starlike domain D of order α is a $K(\alpha)$ -quasidisk, where $(K(\alpha) - 1)/(K(\alpha) + 1) = \sin(\pi\alpha/2)$. Hence, as a corollary, we have $\sigma(D) \geq c(K(\alpha))$. So, the novelty of this theorem lies in the explicitness of the estimate.

3. From Theorem 1 we observe that $D^\vee$ is strongly starlike of order α under the assumption of Theorem 1. Hence, we obtain $\sigma(\text{Ext } D) \geq 2 \cos(\pi\alpha/2)/(1 + \sin(\pi\alpha/2))M(\alpha)^2$ simultaneously. We also note that the standard domain V_α has the property $\sigma(V_\alpha) = \sigma(\text{Ext } V_\alpha)$.

2. MAPPING FUNCTION OF V_α

Let $\mathcal{S}$ denote the set of holomorphic univalent functions on the unit disk $\mathbb{D}$ normalized by $f(0) = 0$ and $f'(0) = 1$. For $0 \leq \alpha \leq 1$, we define the function k_α in the class $\mathcal{S}$ by the relation

$$\frac{zk'_\alpha(z)}{k_\alpha(z)} = \left(\frac{1+z}{1-z} \right)^\alpha$$

on $\mathbb{D}$. More explicitly, k_α can be expressed by

$$k_\alpha(z) = z \exp \left[\int_0^z \left\{ \left(\frac{1+\zeta}{1-\zeta} \right)^\alpha - 1 \right\} \frac{d\zeta}{\zeta} \right].$$

This function is known to play a role of the usual Koebe function in the class of normalized strongly starlike functions of order α in many cases. Actually k_1 is nothing but the Koebe function.

Noting $k_\alpha(1) = M(\alpha)$, we consider the function

$$(5) \quad g_\alpha(z) = k_\alpha(z)/M(\alpha) = \exp \left[\int_1^z \left(\frac{1+\zeta}{1-\zeta} \right)^\alpha \frac{d\zeta}{\zeta} \right].$$

The following fact is useful to note. Although this result was stated in [6], we give a direct proof here.

Lemma 1. $g_\alpha(\mathbb{D}) = V_\alpha$ for $0 < \alpha < 1$.

Proof. If we set $g_\alpha(e^{it}) = r(t)e^{i\Theta(t)} = R(\theta)e^{i\theta}$, then we have $e^{it}g'_\alpha(e^{it})/g_\alpha(e^{it}) = \Theta'(t) - ir'(t)/r(t)$. Since $\arg(zg'_\alpha(z)/g_\alpha(z)) = \pi\alpha/2$ for $z = e^{it}$ with $t \in (0, \pi)$, we obtain

$$\frac{R'(\theta)}{R(\theta)} = \frac{r'(t)}{r(t)\Theta'(t)} = -\tan \frac{\pi\alpha}{2},$$

which yields $\log R(\theta) = -\theta \tan(\pi\alpha/2)$ for $\theta = \Theta(t) \in (0, \pi)$. In the same way, we have $\log R(-\theta) = -\theta \tan(\pi\alpha/2)$ for $\theta \in (0, \pi)$. These imply that the radius function R of $g_\alpha(\mathbb{D})$ agrees with that of V_α , and hence $g_\alpha(\mathbb{D}) = V_\alpha$. $\square$

3. PROOF OF MAIN THEOREM

First, we recall the construction of a quasiconformal reflection in the boundary of a strongly starlike domain given by [2]. Let D be a strongly starlike domain of order $\alpha \in (0, 1)$ with respect to the origin and R be its radius function. Then we can take the quasiconformal reflection λ in ∂D defined by

$$\lambda(re^{i\theta}) = \frac{R(\theta)^2}{r} e^{i\theta}$$

for all $r > 0$ and $\theta \in \mathbb{R}$ following [2]. We then calculate

$$(6) \quad \partial\lambda = i \frac{RR'}{r^2} \quad \text{and} \quad \bar{\partial}\lambda = \frac{e^{2i\theta}}{r^2} (iRR' - R^2)$$

at $w = re^{i\theta}$.

Now we use the following estimate to prove our main result. This estimate is originally due to Lehto [3], however, the following more general form can be found in [5].

Theorem B. *Let D be a quasidisk with quasiconformal reflection λ in ∂D . Then the following inequality holds:*

$$(7) \quad \sigma(D) \geq \varepsilon(\lambda, D) := 2 \operatorname{ess.} \inf_{w \in D} \frac{|\bar{\partial}\lambda(w)| - |\partial\lambda(w)|}{|\lambda(w) - w|^2 \rho_D(w)^2}.$$

Let us return to our case. By (6) and Theorem 1 (d), we obtain the estimates

$$\begin{aligned} |\bar{\partial}\lambda(w)| - |\partial\lambda(w)| &= \frac{R^2}{r^2} \left(\sqrt{1 + |R'/R|^2} - |R'/R| \right) \\ &\geq \frac{R^2}{r^2} \left(\sqrt{1 + \tan^2(\pi\alpha/2)} - \tan(\pi\alpha/2) \right) \\ &= \frac{R^2}{r^2} \cdot \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)} \end{aligned}$$

and

$$|\lambda(w) - w| = \frac{R^2}{r} - r = \frac{R^2 - r^2}{r}$$

for almost all $w = re^{i\theta} \in D$.

Secondly, we estimate ρ_D from above. Fix $w = re^{i\theta}$ and set $R = R(\theta)$. If we think of the domain $W = w_0 \cdot V_\alpha$, where $w_0 = Re^{i\theta} \in \partial D$, from Theorem 1 (c), we have $W \subset D$. The monotonicity property of the Poincaré metric then implies $\rho_D(w) \leq \rho_W(w)$. Now we write $\tau_\alpha = \rho_{V_\alpha}$. Then $\rho_W(w) = \tau_\alpha(w/w_0)/|w_0| = \tau_\alpha(r/R)/R$. Consequently, we have $\rho_D(w) \leq \tau_\alpha(r/R)/R$.

Summarizing the above, we have the estimate

$$\begin{aligned} \frac{|\bar{\partial}\lambda(w)| - |\partial\lambda(w)|}{|\lambda(w) - w|^2 \rho_D(w)^2} &\geq \frac{R^2}{r^2} \cdot \left(\frac{r}{R^2 - r^2} \right)^2 \cdot \frac{R^2}{\tau_\alpha(r/R)^2} \cdot \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)} \\ &= \frac{1}{(1 - (r/R)^2)^2 \tau(r/R)^2} \cdot \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)}. \end{aligned}$$

Hence,

$$\varepsilon(\lambda, D) \geq \frac{2}{\sup_{0 < u < 1} (1 - u^2)^2 \tau_\alpha(u)^2} \cdot \frac{\cos(\pi\alpha/2)}{1 + \sin(\pi\alpha/2)}.$$

So, if we can show the following lemma, the proof of our main theorem will be finished.

Lemma 2. *The Poincaré density τ_α of V_α satisfies*

$$\sup_{0 < u < 1} (1 - u^2) \tau_\alpha(u) = M(\alpha).$$

Proof. Since $g_\alpha : \mathbb{D} \rightarrow V_\alpha$ is biholomorphic by Lemma 1, we have $(1 - |z|^2)^{-1} = \tau_\alpha(g_\alpha(z)) |g'_\alpha(z)|$ for $z \in \mathbb{D}$. Note here that $u = u(x) = g_\alpha(x) > 0$ and $g'_\alpha(x) > 0$ for positive x . If we set

$$Q(x) = (1 - u(x)^2) \tau_\alpha(u(x)) = \frac{1 - u(x)^2}{(1 - x^2) u'(x)}$$

for $x \in (0, 1)$, we have only to show that Q is non-increasing in the interval $(0, 1)$ because $\lim_{x \rightarrow 0} Q(x) = \tau_\alpha(0) = 1/|g'_\alpha(0)| = M(\alpha)$.

Since $xu'/u = \{(1+x)/(1-x)\}^\alpha$, we have the expression

$$Q = \frac{1 - u^2}{1 - x^2} \cdot \frac{x}{u} \cdot \left(\frac{1 - x}{1 + x} \right)^\alpha = \frac{x}{u} \cdot \frac{1 - u^2}{(1 + x)^{1+\alpha} (1 - x)^{1-\alpha}}.$$

Taking the logarithmic derivative, we obtain

$$x \frac{Q'}{Q} = 1 - \frac{xu'}{u} \cdot \frac{1 + u^2}{1 - u^2} + \frac{2x(x - \alpha)}{1 - x^2} = \frac{1 - 2\alpha x + x^2}{1 - x^2} - \frac{1 + u^2}{1 - u^2} \left(\frac{1 + x}{1 - x} \right)^\alpha.$$

Therefore, $Q' \geq 0$ if and only if

$$\frac{1 - u^2}{1 + u^2} \leq \left(\frac{1 + x}{1 - x} \right)^\alpha \cdot \frac{1 - x^2}{1 - 2\alpha x + x^2} = \frac{(1 + x)^{1+\alpha} (1 - x)^{1-\alpha}}{1 - 2\alpha x + x^2} =: P(x).$$

By representation (5), we see

$$\frac{1 - u^2}{1 + u^2} = \tanh \left[- \int_1^x \left(\frac{1+t}{1-t} \right)^\alpha \frac{dt}{t} \right].$$

Hence, the assertion $Q' \geq 0$ on the interval $(0, 1)$ is further equivalent to the validity of the statement that

$$\int_x^1 \left(\frac{1+t}{1-t} \right)^\alpha \frac{dt}{t} \leq \operatorname{arctanh} P(x)$$

holds whenever $P(x) < 1$.

We now investigate the behaviour of the function P on $(0, 1)$. Since

$$\frac{P'(x)}{P(x)} = \frac{4(x - \alpha)(\alpha x - 1)}{(1 - x^2)(1 - 2\alpha x + x^2)},$$

P is increasing in $(0, \alpha)$ and decreasing in $(\alpha, 1)$. Noting $P(0) = 1$ and $P(1) = 0$, we observe that $P(x) > 1$ for $x \in (0, \beta)$ and that $0 < P(x) < 1$ for $x \in (\beta, 1)$ for some number β between α and 1. Here, we use the following elementary fact.

Lemma 3. Let S and T be continuous functions on the interval $(\beta, 1]$ that are positive, have continuous integrable derivatives on $(\beta, 1)$ and satisfy $S(1) = T(1) = 0$ and $S'(x) \leq T'(x)$ for $x \in (\beta, 1)$. Then $S(x) \geq T(x)$ for $x \in (\beta, 1]$.

Thus it is enough to show the inequality

$$\begin{aligned} -\frac{1}{x} \left(\frac{1+x}{1-x} \right)^\alpha &\geq \frac{d}{dx} \operatorname{arctanh} P(x) = \frac{P'(x)}{1-P(x)^2} \\ &= \frac{-4(x-\alpha)(1-\alpha x)}{(1-2\alpha x+x^2)^2 - (1+x)^{2+2\alpha}(1-x)^{2-2\alpha}} \left(\frac{1+x}{1-x} \right)^\alpha \end{aligned}$$

for $x \in (\beta, 1)$. This inequality is equivalent to

$$\begin{aligned} (1-2\alpha x+x^2)^2 - (1+x)^{2+2\alpha}(1-x)^{2-2\alpha} &\leq 4x(x-\alpha)(1-\alpha x) \\ \Leftrightarrow (1+x)^{2+2\alpha}(1-x)^{2-2\alpha} &\geq (1-2\alpha x+x^2)^2 - 4x(x-\alpha)(1-\alpha x) = (1-x^2)^2 \\ \Leftrightarrow \left(\frac{1+x}{1-x} \right)^{2\alpha} &\geq 1. \end{aligned}$$

The last inequality is certainly valid for $x \in (0, 1)$. So, now the proof is complete. $\square$

Remark. We can see from the proof that $\varepsilon(\lambda, V_\alpha) = 2 \cos(\pi\alpha/2)/(1 + \sin(\pi\alpha/2))M(\alpha)^2$ holds, where λ is the quasiconformal reflection constructed for V_α as above.

4. UPPER ESTIMATE OF $\sigma(V_\alpha)$

In this section, we give a rough upper estimate of $\sigma(V_\alpha)$ in order to examine how good our estimate (4) is.

Theorem 2. For $0 < \alpha < 1$, we have $\sigma(V_\alpha) \leq 2(1-\alpha)^2$.

Proof. We consider the holomorphic function $f(w) = \log(1-w)$ on the domain $\mathbb{C} \setminus [1, +\infty)$. Although f is univalent, $f(V_\alpha)$ has an outward pointing cusp. So, $f(V_\alpha)$ is not a quasidisk. On the other hand, for a quasidisk D , if $\|S_f\|_D < \sigma(D)$, we know that $f(D)$ is also a quasidisk (see [4, p. 120]). Hence, we conclude $\sigma(V_\alpha) \leq \|S_f\|_{V_\alpha}$ for the above f .

Now we estimate $\|S_f\|_{V_\alpha}$. First note that $V_\alpha \subset W := \{w; |\arg(1-w)| < (1-\alpha)\pi/2\}$. By the monotonicity of the Poincaré metric, we have $\|S_f\|_{V_\alpha} \leq \|S_f\|_W$. Since $(1-w)^{1/(1-\alpha)}$ maps W conformally onto the right half plane, we compute

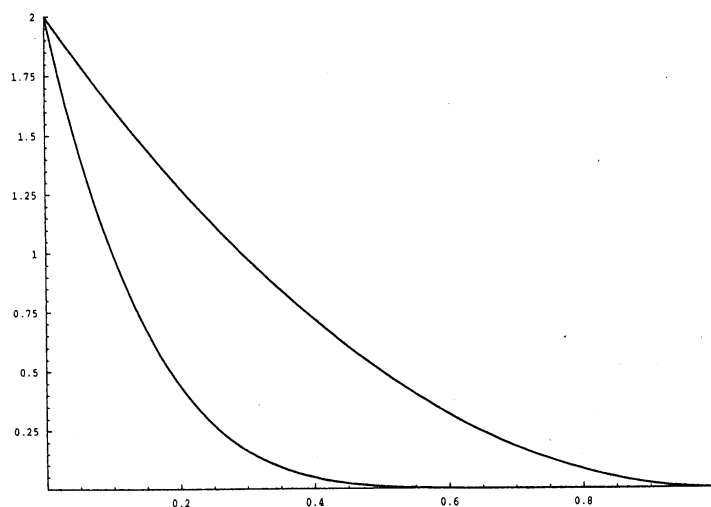
$$\rho_W(w) = \frac{|1-w|^{\alpha/(1-\alpha)}}{2(1-\alpha)\operatorname{Re}[(1-w)^{1/(1-\alpha)}]}.$$

On the other hand, $S_f(w) = 1/2(1-w)^2$. Thus, we calculate

$$\|S_f\|_W = \sup_{w \in W} 2(1-\alpha)^2 \operatorname{Re} \left[\frac{(1-w)^{1/(1-\alpha)}}{|1-w|^{1/(1-\alpha)}} \right] = 2(1-\alpha)^2.$$

Now the proof is completed. $\square$

Finally, we exhibit the graphs of the function $2 \cos(\pi\alpha/2)/(1 + \sin(\pi\alpha/2))M(\alpha)^2$ and $2(1-\alpha)^2$ below.



REFERENCES

1. D. A. Brannan and W. E. Kirwan, *On some classes of bounded univalent functions*, J. London Math. Soc. (2) **1** (1969), 431–443.
2. M. Fait, J. G. Krzyż, and J. Zygmunt, *Explicit quasiconformal extensions for some classes of univalent functions*, Comment. Math. Helv. **51** (1976), 279–285.
3. O. Lehto, *Remarks on Nehari's theorem about the Schwarzian derivative and schlicht functions*, J. Analyse Math. **36** (1979), 184–190.
4. ———, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, 1987.
5. T. Sugawa, *A remark on the Ahlfors-Lehto univalence criterion*, preprint, 2000.
6. ———, *A self-duality of strong starlikeness*, preprint, 2000.

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