

ON SOME FUNCTIONAL EQUATION OF A PREHOMOGENEOUS SPACE OF SYMMETRIC MATRICES.

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Introduction

We gave an expository talk on the results of the paper of J. Sweet [5],[6]. Here we will give another proof of his results.

§1. Properties of Weil constants.

Let k be a local field with $\text{char. } k \neq 2$. We fix a non-trivial additive character ψ of k . When k non-archimedean, we denote by $\mathfrak{o}$ the maximal order of k . Let ϖ and q be a prime element of $\mathfrak{o}$ and the number of elements of the residue field of $\mathfrak{o}$. We put $c_k = [k^\times : (k^\times)^2] = 4|2|^{-1}$. Let $\langle \cdot, \cdot \rangle$ be the Hilbert symbol of k .

We define a Weil constant $\alpha(a)$ by the equation

$$\int_k \phi(x) \psi(ax^2) dx = \alpha(a) |2a|^{-\frac{1}{2}} \int_k \hat{\phi}(x) \psi(-\frac{1}{4a}x^2) dx,$$

$\phi \in \mathcal{S}(k)$. Here,

$$\hat{\phi}(x) = \int_k \phi(y) \psi(xy) dy$$

is the Fourier transform of ϕ . Then $|\alpha(a)| = 1$, $\alpha(-a) = \overline{\alpha(a)}$, and

$$\frac{\alpha(1)\alpha(ab)}{\alpha(a)\alpha(b)} = \langle a, b \rangle.$$

For a quadratic form Q over k , we denote by d_Q and ϵ_Q the determinant and the Minkowski-Hasse invariant of Q , respectively. When Q is equivalent to a diagonal form $\text{diag}(q_1, \dots, q_m)$, $\epsilon_Q = \prod_{i < j} \langle q_i, q_j \rangle$. We put $\Delta_Q = (-1)^{[\frac{m}{2}]} d_Q$. We think of d_Q and Δ_Q as elements of $k^\times / (k^\times)^2$. we put $\alpha_Q(a) = \prod_{i=1}^m \alpha(q_i a)$. It is easy to see $\alpha_Q(1) = \alpha(1)^{m-1} \alpha(d_Q) \epsilon_Q$.

Proposition 1. For $a \in k^\times$,

$$\sum_{x \in k^\times / (k^\times)^2} \alpha(x) \langle x, a \rangle = c_k^{\frac{1}{2}} \frac{\alpha(1)}{\alpha(a)}.$$

Proof. It is enough to show that

$$\sum_{x \in k^\times / (k^\times)^2} \alpha(x) = c_k^{\frac{1}{2}}.$$

When k is archimedean, this is clear. Let k be a non-archimedean local field. We may assume that ψ is of order 0. Let e be an integer such that $|2| = q^{-e}$. Let A be a complete representative of $\mathfrak{o}^\times / (\mathfrak{o}^\times)^2$. Then $A \cup \varpi A$ is a complete representative of $k^\times / (k^\times)^2$. Let ϕ be the characteristic function of $\mathfrak{o}$. When $x \in A \cup \varpi A$, the left hand of the equation

$$\int_k \phi(t) \psi(xt^2) dt = \alpha(x) |2x|^{-\frac{1}{2}} \int_k \hat{\phi}(t) \psi(-\frac{1}{4x} t^2) dt$$

is equal to 1. If x runs over A and y runs over $\mathfrak{o} - \{0\}$, then $-xy^2$ runs over $\bigcup_{r=0}^{\infty} \varpi^{2r} \mathfrak{o}^\times$ twice. If we put $t = -xy^2$, then $dt = |2y| dy$. Therefore

$$\sum_{x \in A} \overline{\alpha(x)} = |2|^{-\frac{1}{2}} \cdot 2|2|^{-1} \sum_{r=0}^{\infty} q^{-r} \int_{\mathfrak{o}^\times} \psi(\varpi^{2r-2e} t) dt.$$

Similarly,

$$\sum_{x \in \varpi A} \overline{\alpha(x)} = |2|^{-\frac{1}{2}} q^{\frac{1}{2}} \cdot 2|2|^{-1} \sum_{r=0}^{\infty} q^{-r} \int_{\mathfrak{o}^\times} \psi(\varpi^{2r-2e-1} t) dt.$$

Observe that $[\mathfrak{o}^\times : (\mathfrak{o}^\times)^2] = 2|2|^{-1}$ and

$$\int_{\mathfrak{o}^\times} \psi(\varpi^l t) dt = \begin{cases} 1 - q^{-1}, & \text{if } l \geq 0, \\ -q^{-1}, & \text{if } l = -1, \\ 0, & \text{if } l < -1. \end{cases}$$

It follows that

$$\begin{aligned} \sum_{x \in k^\times / (k^\times)^2} \alpha(x) &= \sum_{x \in A} \overline{\alpha(x)} + \sum_{x \in \varpi A} \overline{\alpha(x)} \\ &= 2q^{e/2} = 2|2|^{-\frac{1}{2}}. \end{aligned}$$

Now we consider a relationship of the Weil constant and the Tate functional equation. Let ω be a unitary character of $k^\times$. For $\Phi \in \mathcal{S}(k)$, the functional equation is given by

$$\frac{\int_{k^\times} \hat{\Phi}(x) \omega^{-1}(x) |x|^{1-s} d^\times x}{L(1-s, \omega^{-1})} = \varepsilon(s, \omega, \psi) \frac{\int_{k^\times} \Phi(x) \omega(x) |x|^s d^\times x}{L(s, \omega)}, \quad 0 < \operatorname{Re}(s) < 1$$

and both sides are meromorphically continued to whole s -plane. We put

$$\varepsilon'(s, \omega, \psi) = \varepsilon(s, \omega, \psi) \frac{L(1-s, \omega^{-1})}{L(s, \omega)}.$$

Note that even if Φ is not a Schwartz function, the functional equation holds as long as both sides converges in some vertical strip in $\{0 < \operatorname{Re}(s) < 1\}$.

We put $\chi_\theta(x) = \langle \theta, x \rangle$ for $\theta \in k^\times$.

Now we prove the following proposition.

Proposition 1. For any character ω of $k^\times$,

$$\sum_{\theta \in k^\times / (k^\times)^2} \frac{\alpha(1)}{\alpha(\theta)} \varepsilon'(s, \omega \chi_\theta, \psi)^{-1} = 2 |2|^{-2s} \omega\left(\frac{1}{4}\right) \varepsilon'(2s, \omega^2, \psi)^{-1} \varepsilon'\left(s + \frac{1}{2}, \omega, \psi\right).$$

Proof. We reproduce a proof of Rallis and Schiffmann [3]. For $\varphi \in \mathcal{S}(k)$, we define a L^1 -function $M_\varphi \in L^1(k)$ by

$$M_\varphi(x) = \begin{cases} 2|x|^{\frac{1}{2}} [\varphi(\sqrt{x}) + \varphi(-\sqrt{x})] & x \in (k^\times)^2, \\ 0 & \text{otherwise.} \end{cases}$$

The for any bounded function f over k ,

$$\int_k M_\varphi(y) f(y) dy = \int_k \varphi(y) f(y^2) dy.$$

In particular,

$$\begin{aligned} \int_k M_\varphi(y) \psi(xy) dy &= \int_k \varphi(y) \psi(xy^2) dy \\ &= \alpha(x) |2x|^{-\frac{1}{2}} \int_k \hat{\varphi}(y) \psi\left(-\frac{1}{4x} y^2\right) dy \\ &= \alpha(x) |2x|^{-\frac{1}{2}} \int_k M_{\hat{\varphi}}(y) \psi\left(-\frac{1}{4x} y\right) dy. \end{aligned}$$

I.e.,

$$\overline{\alpha(x)} |2x|^{\frac{1}{2}} \widehat{M_\varphi}(x) = \widehat{M_{\hat{\varphi}}}\left(-\frac{1}{4x}\right).$$

Since the right hand side is a Fourier transform of a L^1 -function, $\widehat{M_\varphi}(x) |x|^{\frac{1}{2}}$ is bounded when $|x| \rightarrow \infty$. It follows that

$$\int_{k^\times} \widehat{M_\varphi}(x) \omega^{-1}(x) |x|^{\frac{1}{2}-s} d^\times x$$

is absolutely convergent for $\operatorname{Re}(s) < 1$. On the other hand

$$\int_{k^\times} M_\varphi(x) \omega(x) |x|^{s+\frac{1}{2}} d^\times x = \int_{k^\times} \varphi(x) \omega^2(x) |x|^{2s} d^\times x$$

is absolutely convergent for $\Re(s) > \frac{1}{2}$. Therefore we have a functional equation

$$\begin{aligned} \int_{k^\times} \widehat{M_\varphi}(x) \omega^{-1}(x) |x|^{\frac{1}{2}-s} d^\times x &= \varepsilon'\left(s + \frac{1}{2}, \omega, \psi\right) \int_{k^\times} M_\varphi(x) \omega(x) |x|^{s+\frac{1}{2}} d^\times x \\ &= \varepsilon'\left(s + \frac{1}{2}, \omega, \psi\right) \int_{k^\times} \varphi(x) \omega^2(x) |x|^{2s} d^\times x \\ &= \varepsilon'\left(s + \frac{1}{2}, \omega, \psi\right) \varepsilon'(2s, \omega^2, \psi)^{-1} \\ &\quad \times \int_{k^\times} \varphi(x) \omega^2(x) |x|^{2s} d^\times x \end{aligned}$$

for $\frac{1}{2} < \Re(s) < 1$. Note that

$$\begin{aligned}
& \int_{k^\times} \widehat{M_\varphi}(x) \omega^{-1}(x) |x|^{\frac{1}{2}-s} d^\times x \\
&= |2|^{-\frac{1}{2}} \int_{k^\times} \alpha(x) \widehat{M_{\hat{\varphi}}}(-\frac{1}{4x}) \omega^{-1}(x) |x|^{-s} d^\times x \\
&= |2|^{-\frac{1}{2}} \omega(-4) \int_{k^\times} \overline{\alpha(x)} \widehat{M_{\hat{\varphi}}}(x) \omega(x) |\frac{1}{4x}|^{-s} d^\times x \\
&= |2|^{2s-\frac{1}{2}} \omega(-4) \int_{k^\times} \overline{\alpha(x)} \widehat{M_{\hat{\varphi}}}(x) \omega(x) |x|^s d^\times x \\
&= |2|^{2s-\frac{1}{2}} \omega(-4) \sum_{\beta \in k^\times / (k^\times)^2} \overline{\alpha(\beta)} \int_{\beta \cdot (k^\times)^2} \widehat{M_{\hat{\varphi}}}(x) \omega(x) |x|^s d^\times x
\end{aligned}$$

Here,

$$\begin{aligned}
& \sum_{\beta \in k^\times / (k^\times)^2} \overline{\alpha(\beta)} \int_{\beta \cdot (k^\times)^2} \widehat{M_{\hat{\varphi}}}(x) \omega(x) |x|^s d^\times x \\
&= c_k^{-1} \sum_{\beta, \theta \in k^\times / (k^\times)^2} \overline{\alpha(\beta)} \chi_\beta(\theta) \int_{k^\times} \widehat{M_{\hat{\varphi}}}(x) \omega \chi_\theta(x) |x|^s d^\times x
\end{aligned}$$

Here, by Proposition 1, $\sum_{\beta \in k^\times / (k^\times)^2} \overline{\alpha(\beta)} \chi_\beta(\theta) = c_k^{\frac{1}{2}} \frac{\alpha(1)}{\alpha(\theta)} \chi_\theta(-1)$. Therefore the equation above is

$$\begin{aligned}
& 2^{-1} |2|^{\frac{1}{2}} \sum_{\theta \in k^\times / (k^\times)^2} \frac{\alpha(1)}{\alpha(\theta)} \chi_\theta(-1) \int_{k^\times} \widehat{M_{\hat{\varphi}}}(x) \omega \chi_\theta(x) |x|^s d^\times x \\
&= 2^{-1} |2|^{\frac{1}{2}} \sum_{\theta \in k^\times / (k^\times)^2} \frac{\alpha(1)}{\alpha(\theta)} \chi_\theta(-1) \varepsilon'(1-s, \omega^{-1} \chi_\theta, \psi) \int_{k^\times} M_{\hat{\varphi}}(x) \omega^{-1} \chi_\theta(x) |x|^{1-s} d^\times x \\
&= 2^{-1} |2|^{\frac{1}{2}} \omega(-1) \sum_{\theta \in k^\times / (k^\times)^2} \frac{\alpha(1)}{\alpha(\theta)} \varepsilon'(s, \omega \chi_\theta, \psi)^{-1} \int_{k^\times} \hat{\varphi}(x) \omega^{-2}(x) |x|^{1-2s} d^\times x
\end{aligned}$$

It follow that

$$\sum_{\theta \in k^\times / (k^\times)^2} \frac{\alpha(1)}{\alpha(\theta)} \varepsilon'(s, \omega \chi_\theta, \psi)^{-1} = 2 |2|^{-2s} \omega(\frac{1}{4}) \varepsilon'(2s, \omega^2, \psi)^{-1} \varepsilon'(s + \frac{1}{2}, \omega, \psi).$$

Corollary.

$$\varepsilon(\frac{1}{2}, \chi_\beta, \psi) = \frac{\alpha(1)}{\alpha(\beta)}$$

Proof. Put $\omega = \chi_\beta$ in Proposition 2 and consider the limit of the equation

$$\sum_{\theta \in k^\times / (k^\times)^2} \frac{\alpha(1)}{\alpha(\theta)} \frac{\varepsilon'(2s, 1, \psi)}{\varepsilon'(s, \chi_{\beta\theta}, \psi)} = 2 |2|^{-2s} \varepsilon'(s + \frac{1}{2}, \chi_\beta, \psi)$$

for $s \rightarrow 0$.

§§2. The space of symmetric matrices of rank n .

Let $V = \text{Sym}_n(k)$, $G = \text{GL}_n$. V is a prehomogeneous vector space under the action of an algebraic group G . The set of open orbits of V is denoted by $\mathcal{O} = \mathcal{O}_V$. If k is non-archimedean, $\mathcal{O}$ is parametrised by the determinant and the Minkowski-Hasse invariant.

For each $\eta \in \mathcal{O}$, we put $d_\eta = \det \eta \in k^\times / (k^\times)^2$, $\Delta_\eta = (-1)^{[\frac{n}{2}]} d_\eta$. We denote the Minkowski-Hasse invariant of η by ϵ_η .

For $\Phi \in \mathcal{S}(V)$, we put

$$\widehat{\Phi}(x) = \int_V \Phi(y) \psi(\text{tr } xy) dy.$$

Here the measure dx is the self dual measure for the Fourier transform, i.e., $dx = |2|^{\frac{n(n-1)}{4}} \prod_{i=1}^n dx_{ii} \prod_{i < j} dx_{ij}$.

The purpose of this section is to prove some local functional equations for the prehomogeneous vector space V .

Theorem 1. We put $\rho = \frac{n+1}{2}$. If $\Phi \in \mathcal{S}(V)$, $n = 2m + 1$, then

$$\begin{aligned} \int_V \Phi(X) \omega(\det X) |\det X|^{s-\rho} dX &= \epsilon'(s-m, \omega)^{-1} \prod_{r=1}^m \epsilon'(2s-2m-1+2r, \omega^2)^{-1} \\ &\times |2|^{-2ms + \frac{m(2m+1)}{2}} \omega\left(\frac{1}{4}\right)^m \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \\ &\times \sum_{\eta \in \mathcal{O}} \langle (-1)^m, d_\eta \rangle \epsilon_\eta \\ &\times \int_{V_\eta} \widehat{\Phi}(X) \omega^{-1}(\det X) |\det X|^{-s} dX. \end{aligned}$$

If $n = 2m$,

$$\begin{aligned} \int_V \Phi(X) \omega(\det X) |\det X|^{s-\rho} dX &= \epsilon'(s-m+\frac{1}{2}, \omega)^{-1} \prod_{r=1}^m \epsilon'(2s-2m+2r, \omega^2)^{-1} \\ &\times |2|^{-2ms + \frac{m(2m-1)}{2}} \omega\left(\frac{1}{4}\right)^m \\ &\times \sum_{\eta \in \mathcal{O}} \frac{\alpha(\Delta_\eta)}{\alpha(1)} \epsilon'(s+\frac{1}{2}, \omega \chi_{\Delta_\eta}) \\ &\times \int_{V_\eta} \widehat{\Phi}(X) \omega^{-1}(\det X) |\det X|^{-s} dX. \end{aligned}$$

Proof. From the prehomogeneity of V , there is a meromorphic function $c_\eta(\omega, s)$ such that

$$\int_V \Phi(X) \omega(\det X) |\det X|^{s-\rho} dX = \sum_{\eta \in \mathcal{O}} c_\eta(\omega, s) \int_{V_\eta} \widehat{\Phi}(X) \omega(\det X) |\det X|^{-s} dX.$$

We have to prove that if $n = 2m + 1$,

$$c_\eta(\omega, s) = \varepsilon'(s - m, \omega)^{-1} \prod_{r=1}^m \varepsilon'(2s - 2m - 1 + 2r, \omega^2)^{-1} \\ \times |2|^{-2ms + \frac{m(2m+1)}{2}} \omega\left(\frac{1}{4}\right)^m \langle (-1)^m \xi, d_\eta \rangle \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \epsilon_\eta,$$

and that if $n = 2m$,

$$c_\eta(\omega, s) = \varepsilon'(s - m + \frac{1}{2}, \omega)^{-1} \prod_{r=1}^m \varepsilon'(2s - 2m + 2r, \omega^2)^{-1} \\ \times |2|^{-2ms + \frac{m(2m-1)}{2}} \omega\left(\frac{1}{4}\right)^m \frac{\alpha(\Delta_\eta)}{\alpha(1)} \varepsilon'(s + \frac{1}{2}, \omega \chi_{\Delta_\eta}, \psi).$$

We proceed by induction on n . When $n = 1$, it is nothing but Tate's local functional equation for $\omega = \chi_\xi$: We will use this functional equation as the form

$$\int_{\rho(k^\times)^2} \Phi(x) \omega(x) |x|^s d^\times x \\ = c_k^{-1} \sum_{\beta, \theta \in k^\times / (k^\times)^2} \langle \theta, \rho\beta \rangle \varepsilon'(s, \omega \chi_\theta, \psi)^{-1} \int_{\beta(k^\times)^2} \widehat{\Phi}(x) \omega^{-1}(x) |x|^{1-s} d^\times x.$$

Now we assume $n \geq 1$. We assume the functional equation is true for n . We put $V = \text{Sym}_n(k)$, $V' = \text{Sym}_{n+1}(k)$, $\rho = \frac{n+1}{2}$, and $\rho' = \frac{n+2}{2}$. We denote the set of GL_n orbits of V by $\mathcal{O}$ and the set of GL_{n+1} orbits of V' by $\mathcal{O}'$.

We may assume $\Phi \begin{pmatrix} T & y \\ \imath y & x \end{pmatrix} = \phi_1(T) \phi_2(y) \phi_3(x)$ for some $\phi_1 \in \mathcal{S}(V)$, $\phi_2 \in \mathcal{S}(k^n)$, $\phi_3 \in \mathcal{S}(k)$. Note that $\widehat{\Phi} \begin{pmatrix} T & y \\ \imath y & x \end{pmatrix} = |2|^{\frac{n}{2}} \hat{\phi}_1(T) \hat{\phi}_2(2y) \hat{\phi}_3(x)$.

$$\int_{T' \in V'} \Phi(T') \omega(\det T') |\det T'|^{s-\rho'} dT' \\ = \int_{T \in V} \int_{y \in k^n} \int_{x \in k} \Phi \begin{pmatrix} (T + yx^{-1} \imath y) & y \\ \imath y & x \end{pmatrix} \omega(x \det T) |x \det T|^{s-\rho'} dx dy dT \\ = \sum_{\eta \in \mathcal{O}} c_\eta(\omega, s - \frac{1}{2}) \int_{T \in V_\eta} \int_{y \in k^n} \int_{x \in k} \hat{\phi}_1(T) \psi(-x^{-1} \imath y T y) \phi_2(y) \phi_3(x) \\ \times \omega^{-1}(\det T) \omega(x) |x|^{s-\rho'} |\det T|^{-s+\frac{1}{2}} dx dy dT \\ = \sum_{\eta \in \mathcal{O}} c_\eta(\omega, s - \frac{1}{2}) \int_{T \in V_\eta} \int_{y \in k^n} \int_{x \in k} \\ \times \overline{\alpha_\eta(x)} |2^n x^{-n} \det T|^{-\frac{1}{2}} \hat{\phi}_1(T) \hat{\phi}_2(y) \phi_3(x) \\ \times \psi\left(\frac{1}{4} x \imath y T^{-1} y\right) \omega^{-1}(\det T) |\det T|^{-s+\frac{1}{2}} \omega(x) |x|^{s-\rho'} dx dy dT$$

$$\begin{aligned}
&= \sum_{\eta \in \mathcal{O}} \sum_{\rho \in k^\times / (k^\times)^2} c_\eta(\omega, s - \frac{1}{2}) \overline{\alpha_\eta(\rho)} \int_{T \in V_\eta} \int_{y \in k^n} \int_{x \in \rho(k^\times)^2} |2|^{\frac{n}{2}} \hat{\phi}_1(T) \hat{\phi}_2(2y) \phi_3(x) \\
&\quad \times \psi(x {}^t y T^{-1} y) \omega(\det T) |\det T|^{-s} \omega(x) |x|^{s-1} dx dy dT \\
&= \sum_{\eta \in \mathcal{O}} \sum_{\rho, \theta, \beta \in k^\times / (k^\times)^2} c_k^{-1} c_\eta(\omega, s - \frac{1}{2}) \overline{\alpha_\eta(\rho)} \langle \theta, \rho \beta \rangle \varepsilon'(s, \omega \chi_\theta)^{-1} \\
&\quad \times \int_{T \in V_\eta} \int_{y \in k^n} \int_{x \in \rho(k^\times)^2} |2|^{\frac{n}{2}} \hat{\phi}_1(T) \hat{\phi}_2(2y) \hat{\phi}_3(x + {}^t y T^{-1} y) \\
&\quad \times \omega^{-1}(x \det T) |x \det T|^{-s} dx dy dT
\end{aligned}$$

Therefore if we put $\eta \oplus \beta = \begin{pmatrix} \eta & \\ & \beta \end{pmatrix}$, then

$$c_{\eta \oplus \beta}(\omega, s) = c_k^{-1} c_\eta(\omega, s - \frac{1}{2}) \sum_{\rho, \theta} \overline{\alpha_\eta(\rho)} \langle \theta, \rho \beta \rangle \varepsilon'(s, \omega \chi_\theta)^{-1}$$

If $n = 2m$ is even,

$$\begin{aligned}
&c_k^{-1} \sum_{\rho, \theta} \overline{\alpha_\eta(\rho)} \langle \theta, \rho \beta \rangle \varepsilon'(s, \omega \chi_\theta)^{-1} \\
&= \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_\eta \rangle \frac{\alpha(1)}{\alpha(\Delta_\eta)} \epsilon_\eta c_k^{-1} \sum_{\rho, \theta} \langle \Delta_\eta, \rho \rangle \langle \theta, \rho \beta \rangle \varepsilon'(s, \omega \chi_\theta)^{-1} \\
&= \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_\eta \rangle \frac{\alpha(1)}{\alpha(\Delta_\eta)} \epsilon_\eta \langle \Delta_\eta, \beta \rangle \varepsilon'(s, \omega \chi_{\Delta_\eta})^{-1} \\
&= \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_{\eta \oplus \beta} \rangle \frac{\alpha(1)}{\alpha(\Delta_\eta)} \epsilon_{\eta \oplus \beta} \varepsilon'(s, \omega \chi_{\Delta_\eta})^{-1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
c_{\eta \oplus \beta}(\omega, s) &= c_\eta(\omega, s - \frac{1}{2}) \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_{\eta \oplus \beta} \rangle \frac{\alpha(1)}{\alpha(\Delta_\eta)} \epsilon_{\eta \oplus \beta} \varepsilon'(s, \omega \chi_{\Delta_\eta})^{-1} \\
&= \varepsilon'(s - m, \omega)^{-1} \prod_{r=1}^m \varepsilon'(2s - 2m - 1 + 2r, \omega^2)^{-1} \\
&\quad \times |2|^{-2ms + \frac{m(2m+1)}{2}} \omega(\frac{1}{4})^m \frac{\alpha(\Delta_\eta)}{\alpha(1)} \varepsilon'(s, \omega \chi_{\Delta_\eta}) \\
&\quad \times \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_{\eta \oplus \beta} \rangle \frac{\alpha(1)}{\alpha(\Delta_\eta)} \epsilon_{\eta \oplus \beta} \varepsilon'(s, \omega \chi_{\Delta_\eta})^{-1} \\
&= \varepsilon'(s - m, \omega)^{-1} \prod_{r=1}^m \varepsilon'(2s - 2m - 1 + 2r, \omega^2)^{-1} \\
&\quad \times |2|^{-2ms + \frac{m(2m+1)}{2}} \omega(\frac{1}{4})^m \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_{\eta \oplus \beta} \rangle \epsilon_{\eta \oplus \beta}
\end{aligned}$$

On the other hand, if $n = 2m + 1$ is odd,

$$c_k^{-1} \sum_{\rho, \theta} \overline{\alpha_\eta(\rho)} \langle \theta, \rho \beta \rangle \varepsilon'(s, \omega \chi_\theta)^{-1}$$

$$\begin{aligned}
&= \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_\eta \rangle \frac{\alpha(1)}{\alpha(\Delta_\eta)} \epsilon_\eta c_k^{-1} \sum_{\rho, \theta} \overline{\alpha(\rho)} \langle \Delta_\eta, \rho \rangle \langle \theta, \rho \beta \rangle \epsilon'(s, \omega \chi_\theta)^{-1} \\
&= \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_\eta \rangle \frac{\alpha(1)}{\alpha(\Delta_\eta)} \epsilon_\eta c_k^{-1} \sum_{\rho, \theta} \overline{\alpha(\rho)} \langle \Delta_\eta \theta, \rho \rangle \langle \theta, \beta \rangle \epsilon'(s, \omega \chi_\theta)^{-1} \\
&= \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_\eta \rangle \overline{\alpha(\Delta_\eta)} \epsilon_\eta c_k^{-\frac{1}{2}} \sum_{\theta} \alpha(\Delta_\eta \theta) \langle \theta, \beta \rangle \epsilon'(s, \omega \chi_\theta)^{-1} \\
&= \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_\eta \rangle \overline{\alpha(\Delta_\eta)} \epsilon_\eta c_k^{-\frac{1}{2}} \sum_{\theta} \alpha(-\beta \theta) \langle \Delta_\eta \theta, \beta \rangle \epsilon'(s, \omega \chi_{-\Delta_\eta \beta \theta})^{-1} \\
&= \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_\eta \rangle \overline{\alpha(\Delta_\eta) \alpha(\beta)} \langle \Delta_\eta, \beta \rangle \epsilon_\eta c_k^{-\frac{1}{2}} \sum_{\theta} \frac{\alpha(1)}{\alpha(\theta)} \epsilon'(s, \omega \chi_{-\Delta_\eta \beta \theta})^{-1} \\
&= |2|^{-2s+\frac{1}{2}} \omega(\frac{1}{4}) \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_\eta \rangle \frac{\alpha(\Delta_\eta \oplus \beta)}{\alpha(1)} \epsilon_\eta \epsilon'(2s, \omega^2)^{-1} \epsilon'(s + \frac{1}{2}, \omega \chi_{\Delta_\eta \oplus \beta})
\end{aligned}$$

It follows that

$$\begin{aligned}
c_{\eta \oplus \beta}(\omega, s) &= c_\eta(\omega, s - \frac{1}{2}) \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_\eta \rangle \epsilon_\eta \\
&\quad \times |2|^{-2s+\frac{1}{2}} \omega(\frac{1}{4}) \frac{\alpha(\Delta_\eta \oplus \beta)}{\alpha(1)} \epsilon(2s, \omega^2)^{-1} \epsilon'(s + \frac{1}{2}, \omega \chi_{\Delta_\eta \oplus \beta}) \\
&= \epsilon'(s - m - \frac{1}{2}, \omega)^{-1} \prod_{r=1}^m \epsilon'(2s - 2m - 2 + 2r, \omega^2)^{-1} \\
&\quad \times |2|^{-2m(s-\frac{1}{2})+\frac{m(2m+1)}{2}-2s+\frac{1}{2}} \omega(\frac{1}{4})^{m+1} \frac{\alpha(\Delta_\eta \oplus \beta)}{\alpha(1)} \\
&\quad \times \epsilon(2s, \omega^2)^{-1} \epsilon'(s + \frac{1}{2}, \omega \chi_{\Delta_\eta \oplus \beta}) \\
&= \epsilon'(s - m - \frac{1}{2}, \omega)^{-1} \prod_{r=1}^m \epsilon'(2s - 2m - 2 + 2r, \omega^2)^{-1} \\
&\quad \times |2|^{-2(m+1)s+\frac{(m+1)(2m+1)}{2}} \omega(\frac{1}{4})^{m+1} \frac{\alpha(\Delta_\eta \oplus \beta)}{\alpha(1)} \epsilon'(s + \frac{1}{2}, \omega \chi_{\Delta_\eta \oplus \beta}).
\end{aligned}$$

Theorem 1 is thus proved.

Next we prove

Theorem 2.

If $n = 2m + 1$,

$$\begin{aligned}
\int_V \Phi(X) \omega(\det X) \epsilon_X |\det X|^{s-\rho} dX &= \epsilon'(s, \omega \chi_{(-1)^m})^{-1} \prod_{r=1}^m \epsilon'(2s - 2m - 2 + 2r, \omega^2)^{-1} \\
&\quad \times |2|^{-2ms+\frac{m(2m+3)}{2}} \omega(\frac{1}{4})^m \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \\
&\quad \times \int_V \widehat{\Phi}(X) \omega^{-1} \chi_{(-1)^m}(\det X) |\det X|^{-s} dX
\end{aligned}$$

If $n = 2m$,

$$\begin{aligned}
& \int_V \Phi(X) \omega(\det X) \epsilon_X |\det X|^{s-\rho} dX \\
&= \prod_{r=1}^m \varepsilon'(2s - 2m - 1 + 2r, \omega^2)^{-1} |2|^{-2ms + \frac{m(2m+1)}{2}} \omega\left(\frac{1}{4}\right)^m \frac{\alpha((-1)^m)}{\alpha(1)} \\
&\quad \times \sum_{\eta \in \mathcal{O}} \frac{\alpha(\Delta_\eta)}{\alpha(1)} \int_{V_\eta} \widehat{\Phi}(X) \omega^{-1} \chi_{(-1)^{m+1}}(\det X) \epsilon_X |\det X|^{-s} dX
\end{aligned}$$

Proof.

In fact, when n is odd, this is equivalent to Theorem 1.

As in the proof of Theorem 1, one can see there is a function $c'_\eta(\xi, s)$ such that

$$\int_V \Phi(X) \omega(\det X) \epsilon_X |\det X|^{s-\rho} dX = \sum_{\eta \in \mathcal{O}} c'_\eta(\omega, s) \int_{V_\eta} \widehat{\Phi}(X) \omega^{-1}(\det X) |\det X|^{-s} dX.$$

We have to prove that if $n = 2m$,

$$\begin{aligned}
c'_\eta(\omega, s) &= \prod_{r=1}^m \varepsilon'(2s - 2m - 1 + 2r, \omega^2)^{-1} \\
&\quad \times |2|^{-2ms + \frac{m(2m+1)}{2}} \omega\left(\frac{1}{4}\right)^m \frac{\alpha(\Delta_\eta)}{\alpha(1)} \epsilon_\eta
\end{aligned}$$

We have already proved that if $n = 2m + 1$, then

$$\begin{aligned}
c'_\eta(\omega, s) &= \varepsilon'(s, \omega \chi_{(-1)^m})^{-1} \prod_{r=1}^m \varepsilon'(2s - 2m - 2 + 2r, \omega^2)^{-1} \\
&\quad \times |2|^{-2ms + \frac{m(2m+3)}{2}} \omega\left(\frac{1}{4}\right)^m \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_\eta \rangle.
\end{aligned}$$

As in the proof of Theorem 1, one can prove

$$c'_{\eta \oplus \beta}(\omega, s) = c_k^{-1} \sum_{\rho, \theta} c'_\eta(\omega \chi_\rho, s - \frac{1}{2}) \overline{\alpha_\eta(\rho)} \langle \rho, d_\eta \rangle \langle \theta, \rho \beta \rangle \varepsilon'(x, \omega \chi_\theta)^{-1}$$

If the rank of η is $2m + 1$, then

$$\begin{aligned}
& c'_{\eta \oplus \beta}(\omega, s) \\
&= c_k^{-1} \sum_{\rho, \theta \in k^\times / (k^\times)^2} c'_\eta(\omega \chi_\rho, s - \frac{1}{2}) \overline{\alpha_\eta(\rho)} \langle \rho, d_\eta \rangle \langle \theta, \rho \beta \rangle \varepsilon'(s, \omega \chi_\theta)^{-1} \\
&= \prod_{r=1}^m \varepsilon'(2s - 2m - 2 + 2r, \omega^2)^{-1} |2|^{-2ms + \frac{m(2m+3)}{2}} \omega\left(\frac{1}{4}\right)^m \langle -1, -1 \rangle^{\frac{m(m+1)}{2}} \langle (-1)^m, d_\eta \rangle \\
&\quad \times c_k^{-1} \sum_{\rho, \theta \in k^\times / (k^\times)^2} \varepsilon'(s - \frac{1}{2}, \omega \chi_{(-1)^m \rho})^{-1} \overline{\alpha_\eta(\rho)} \langle \rho, d_\eta \rangle \langle \theta, \rho \beta \rangle \varepsilon'(s, \omega \chi_\theta)^{-1}
\end{aligned}$$

$$\begin{aligned}
&= \prod_{r=1}^m \varepsilon'(2s - 2m - 2 + 2r, \omega^2)^{-1} |2|^{-2ms + \frac{m(2m+3)}{2}} \omega\left(\frac{1}{4}\right)^m \frac{\alpha(1)}{\alpha(\Delta_\eta)} \epsilon_\eta \\
&\quad \times c_k^{-1} \sum_{\rho, \theta \in k^\times / (k^\times)^2} \overline{\alpha(\rho)} \langle \rho, (-1)^m \rangle \langle \theta, \rho\beta \rangle \varepsilon'\left(s - \frac{1}{2}, \omega\chi_{(-1)^m \rho}\right)^{-1} \varepsilon'(s, \omega\chi_\theta)^{-1}
\end{aligned}$$

Here,

$$\begin{aligned}
&c_k^{-1} \frac{\alpha(1)}{\alpha(\Delta_\eta)} \epsilon_\eta \sum_{\rho, \theta \in k^\times / (k^\times)^2} \overline{\alpha(\rho)} \langle \rho, (-1)^m \theta \rangle \langle \theta, \beta \rangle \varepsilon'\left(s - \frac{1}{2}, \omega\chi_{(-1)^m \rho}\right)^{-1} \varepsilon'(s, \omega\chi_\theta)^{-1} \\
&= c_k^{-1} \frac{\alpha(1)}{\alpha(\Delta_\eta)} \epsilon_\eta \sum_{\rho, \theta \in k^\times / (k^\times)^2} \frac{\alpha((-1)^m \theta)}{\alpha(1)\alpha((-1)^m \theta \rho)} \langle \theta, \beta \rangle \varepsilon'\left(s - \frac{1}{2}, \omega\chi_{(-1)^m \rho}\right)^{-1} \varepsilon'(s, \omega\chi_\theta)^{-1} \\
&= c_k^{-1} \frac{\alpha(1)}{\alpha(\Delta_\eta)} \epsilon_\eta \sum_{\rho, \theta \in k^\times / (k^\times)^2} \frac{\alpha((-1)^m \theta)}{\alpha(1)\alpha(\rho)} \langle \theta, \beta \rangle \varepsilon'\left(s - \frac{1}{2}, \omega\chi_{\theta \rho}\right)^{-1} \varepsilon'(s, \omega\chi_\theta)^{-1} \\
&= c_k^{-1} 2|2|^{-2s+1} \omega\left(\frac{1}{4}\right) \varepsilon'(2s - 1, \omega^2) \frac{\alpha(1)}{\alpha(\Delta_\eta)} \epsilon_\eta \sum_{\theta \in k^\times / (k^\times)^2} \frac{\alpha((-1)^m \theta)}{\alpha(1)^2} \langle \theta, \beta \rangle \\
&= 2^{-1} |2|^{-2s+2} \omega\left(\frac{1}{4}\right) \varepsilon'(2s - 1, \omega^2) \frac{\alpha(1)}{\alpha(\Delta_\eta)} \epsilon_\eta \sum_{\theta \in k^\times / (k^\times)^2} \frac{\alpha(\theta)}{\alpha(1)^2} \langle (-1)^m \theta, \beta \rangle \\
&= c_k^{\frac{1}{2}} 2^{-1} |2|^{-2s+2} \omega\left(\frac{1}{4}\right) \varepsilon'(2s - 1, \omega^2) \overline{\alpha(\Delta_\eta) \alpha(\beta)} \epsilon_\eta \langle (-1)^m, \beta \rangle \\
&= |2|^{-2s+\frac{3}{2}} \omega\left(\frac{1}{4}\right) \varepsilon'(2s - 1, \omega^2) \overline{\alpha(\Delta_\eta \beta) \alpha(1)} \epsilon_\eta \langle d_\eta, \beta \rangle \\
&= |2|^{-2s+\frac{3}{2}} \omega\left(\frac{1}{4}\right) \varepsilon'(2s - 1, \omega^2) \frac{\alpha(\Delta_\eta \oplus \beta)}{\alpha(1)} \epsilon_{\eta \oplus \beta}
\end{aligned}$$

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