

QUANTIFIER ELIMINATION IN ADELIC STRUCTURES OVER ALGEBRAICALLY CLOSED VALUED FIELDS

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ABSTRACT. Applying Weispfenning's fundamental work on boolean products, we deduce that the theory of adelic structures over algebraically closed valued fields in the language $\mathcal{L}_{AA}$ admits quantifier elimination and is complete.

To call a boolean product over algebraically closed valued fields “adelic” one needs to express the product formula in a first-order way. This is not achieved. Here, by an “adelic struture” we just mean a boolean product over algebraically closed valued fields.

We use the Basarab-Kuhlmann style language $\mathcal{L}_{RV}$ for algebraically closed valued fields of mixed characteristic introduced in [1]. The theory ACVF of algebraically closed valued fields (in any characteristics) admits quantifier elimination in $\mathcal{L}_{RV}$, a short proof of which may be found in [3].

Theorem. The theory ACVF admits quantifier elimination.

Next we describe an expanded language $\mathcal{L}_{BF}$ for boolean algebras, which includes the following:

- the language of boolean algebras $\mathcal{L}_{BA} = \{0, 1, \cap, \cup, \sim, \leq\}$;
- a set of unary relations $\{0 <_n : n \geq 1\}$;
- a unary relation $\mathcal{F}$;
- a constant $\mathbf{a}$.

The theory of infinite atomic boolean algebras with the distinguished set of finite elements in $\mathcal{L}_{BF}$ (hereafter abbreviated as IABF) states the following:

- the usual axioms for boolean algebras;
- for every $\xi > 0$ there is an atom η such that $\xi \geq \eta$;
- $\mathbf{a}$ is an atom;
- axioms for $\mathcal{F}$:
 - $\mathcal{F}(0)$ and $\neg\mathcal{F}(1)$;
 - $\mathcal{F}(\xi \cup \eta)$ if and only if $\mathcal{F}(\xi)$ and $\mathcal{F}(\eta)$;
 - if $\neg\mathcal{F}(\xi)$ then there is an η such that $\neg\mathcal{F}(\xi \cap \eta)$ and $\neg\mathcal{F}(\xi \sim \eta)$;
- $0 <_n \xi$ if and only if there are $\eta_1, \dots, \eta_n$ such that $0 \leq \eta_1 < \dots < \eta_n < \xi$ and $\mathcal{F}(\eta_i)$ for each $i \leq n$.

Theorem (Weispfenning [2], Part II, 1.4(ii, iii)). The theory IABF admits quantifier elimination and is complete.

So IABF axiomatizes the theory of powerset algebras of infinite set, where $\mathcal{F}$ ranges over finite subsets.

In order to formulate a first-order language for adelic structures over algebraically closed valued fields we treat $(\mathcal{L}_{RV}, \mathcal{L}_{BF})$ as a 2-sorted language (these sorts shall be called the first-order sort, or FO-sort for short, and the boolean algebra sort, or BA-sort for short) and further expand it as follows. For each n -ary relation symbol R (including equality and functions) in $\mathcal{L}_{RV}$ we add an n -ary function $\mathcal{V}_R$ from the FO-sort to the BA-sort. For example, if a, b are two $\mathcal{L}_{RV}$ -terms then $\mathcal{V}_=(a, b)$ is considered an $\mathcal{L}_{BF}$ -term. In fact, since the boolean value of each quantifier-free formula in $\mathcal{L}_{RV}$ is determined by the functions $\mathcal{V}_R$, for notational simplicity we may think of one function $\mathcal{V}$ that assigns a boolean value $\mathcal{V}\phi$ to each quantifier-free $\mathcal{L}_{RV}$ -formula ϕ . Let $\mathcal{L}_{AA}$ denote this expansion of $(\mathcal{L}_{RV}, \mathcal{L}_{BF})$.

For each $\forall\exists$ -formula ϕ in $\mathcal{L}_{RV}$ of the form $\forall\vec{x} \exists\vec{y} \psi(\vec{x}, \vec{y}, \vec{z})$ with ψ quantifier-free, let $\phi^\mathcal{V}$ be the $\mathcal{L}_{AA}$ -formula $\forall\vec{x} \exists\vec{y} \mathcal{V}(\psi(\vec{x}, \vec{y}, \vec{z})) = 1$. Now it is routine to check that ACVF is a $\forall\exists$ -theory in $\mathcal{L}_{RV}$. Let $\text{ACVF}^\mathcal{V} = \{\phi^\mathcal{V} : \phi \text{ is an axiom of ACVF}\}$.

The theory of adelic structures over algebraically closed valued field in $\mathcal{L}_{AA}$ (hereafter abbreviated as AACF) states the following:

- $ACVF^V$ and IABF for the corresponding sorts;
- $\mathcal{V}(\text{char } \mathbf{k} = p)$ is an atom for each prime number p ;
- $\mathbf{a} < \mathcal{V}(\text{char } \mathbf{k} > p)$ for each prime number p ;
- the axioms for abstract boolean products (hereafter abbreviated as ABP):
 - $\phi \leftrightarrow \mathcal{V}\phi = 1$ for each atomic formula ϕ in $\mathcal{L}_{RV}$;
 - $\mathcal{V}(x = y) = \mathcal{V}(y = x)$;
 - $\bigcap_{i=1}^n \mathcal{V}(x_i = y_i) \cap \mathcal{V}\phi(x_1, \dots, x_n) \leq \mathcal{V}\phi(y_1, \dots, y_n)$ for each atomic formula ϕ in $\mathcal{L}_{RV}$;
 - **Finitary gluing**: For all $x, y \in \text{FO}$ and $\alpha, \beta \in \text{BA}$, if $\alpha \cap \beta = 0$ and $\alpha \cup \beta = 1$ then there is a z of the first sort such that $\mathcal{V}(z = x) \geq \alpha$ and $\mathcal{V}(z = y) \geq \beta$.

Theorem. The theory AACF admits quantifier elimination in all sorts and is complete.

Proof. Quantifier elimination is immediate by the theorems above and [2, Part II, 3.7(ii)]. Completeness follows from the representation theorem [2, Part I, 3.27]. $\square$

REFERENCES

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