

Continuous graphs and crossed products of Cuntz algebras.

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0 Introduction

In [Ka1, Ka2, Ka3], the author examined the structure of crossed products of Cuntz-algebras by so-called quasi-free actions of abelian groups. Recently, he introduced a new class of C^* -algebras which are arising from *continuous graphs* [Ka4]. These C^* -algebras are generalization of graph algebras [KPRR, KPR, FLR] and homeomorphism C^* -algebras [T3, T4]. The above crossed products are examples of C^* -algebras arising from continuous graphs. From this point of view, some of results in [Ka1] and [Ka3] can be considered as a continuous counterpart of ones in [BHRS] and [HS]. This observation is further studied in [Ka5] for more general settings.

In this short article, we give a definition of continuous graphs and C^* -algebras associated with them, and then discuss how the results in [Ka1] and [Ka3] can be interpreted in terms of continuous graphs.

1 C^* -algebras arising from continuous graphs

Definition 1.1 Let E^0 and E^1 be locally compact (Hausdorff) spaces. A map $d : E^1 \rightarrow E^0$ is said to be *locally homeomorphic* if for any $e \in E^1$, there exists a neighborhood U of e such that the restriction of d on U is a homeomorphism onto $d(U)$ and that $d(U)$ is a neighborhood of $d(e)$.

Every local homeomorphisms are continuous and open.

Definition 1.2 ([Ka4, Definition 2.1]) A *continuous graph* $E = (E^0, E^1, d, r)$ consists of two locally compact spaces E^0, E^1 , a local homeomorphism $d : E^1 \rightarrow E^0$, and a continuous map $r : E^1 \rightarrow E^0$.

Note that $d, r : E^1 \rightarrow E^0$ are not necessarily surjective nor injective. We think that E^0 is a set of vertices and E^1 is a set of edges and that an edge $e \in E^1$ is directed from its domain $d(e) \in E^0$ to its range $r(e) \in E^0$. From a homeomorphism σ on a locally compact

space X , we can define a continuous graph $E = (E^0, E^1, d, r)$ by $E^0 = E^1 = X$, $d = \text{id}$ and $r = \sigma$. In this sense, a continuous graph can be considered as a generalization of dynamical systems.

Let us denote by $C_d(E^1)$ the set of continuous functions ξ of E^1 such that $\langle \xi | \xi \rangle(v) = \sum_{e \in d^{-1}(v)} |\xi(e)|^2 < \infty$ for any $v \in E^0$ and $\langle \xi | \xi \rangle \in C_0(E^0)$. For $\xi, \eta \in C_d(E^1)$ and $f \in C_0(E^0)$, we define $\xi f \in C_d(E^1)$ and $\langle \xi | \eta \rangle \in C_0(E^0)$ by

$$\begin{aligned} (\xi f)(e) &= \xi(e)f(d(e)) \quad \text{for } e \in E^1, \\ \langle \xi | \eta \rangle(v) &= \sum_{e \in d^{-1}(v)} \overline{\xi(e)}\eta(e) \quad \text{for } v \in E^0. \end{aligned}$$

With these operations, $C_d(E^1)$ is a (right) Hilbert $C_0(E^0)$ -module ([Ka4, Proposition 1.10]). We define a left action π_r of $C_0(E^0)$ on $C_d(E^1)$ by $(\pi_r(f)\xi)(e) = f(r(e))\xi(e)$ for $e \in E^1$, $\xi \in C_d(E^1)$ and $f \in C_0(E^0)$. Thus we get a Hilbert $C_0(E^0)$ -bimodule $C_d(E^1)$.

Definition 1.3 Let $E = (E^0, E^1, d, r)$ be a continuous graph. A *Toeplitz E -pair* on a C^* -algebra A is a pair of maps $T = (T^0, T^1)$ where $T^0 : C_0(E^0) \rightarrow A$ is a $*$ -homomorphism and $T^1 : C_d(E^1) \rightarrow A$ is a linear map satisfying that

- (i) $T^1(\xi)^*T^1(\eta) = T^0(\langle \xi | \eta \rangle)$ for $\xi, \eta \in C_d(E^1)$,
- (ii) $T^0(f)T^1(\xi) = T^1(\pi_r(f)\xi)$ for $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$.

For $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$, the equation $T^1(\xi)T^0(f) = T^1(\xi f)$ holds automatically from the condition (i). For a Toeplitz E -pair $T = (T^0, T^1)$, we write $C^*(T)$ for denoting the C^* -algebra generated by the images of the maps T^0 and T^1 . We can define a $*$ -homomorphism $\Phi^1 : \mathcal{K}(C_d(E^1)) \rightarrow C^*(T)$ by $\Phi^1(\theta_{\xi, \eta}) = T^1(\xi)T^1(\eta)^*$ for $\xi, \eta \in C_d(E^1)$ where $\theta_{\xi, \eta} \in \mathcal{K}(C_d(E^1))$ is defined by $\theta_{\xi, \eta}(\zeta) = \xi\langle \eta | \zeta \rangle$ for $\zeta \in C_d(E^1)$.

Definition 1.4 Let $E = (E^0, E^1, d, r)$ be a continuous graph. We define three open subsets E_{sce}^0 , E_{fin}^0 and E_{rg}^0 of E^0 by $E_{\text{sce}}^0 = E^0 \setminus \overline{r(E^1)}$,

$E_{\text{fin}}^0 = \{v \in E^0 \mid \text{there exists a neighborhood } V \text{ of } v \text{ such that } r^{-1}(V) \subset E^1 \text{ is compact}\},$

and $E_{\text{rg}}^0 = E_{\text{fin}}^0 \setminus \overline{E_{\text{sce}}^0}$. We define two closed subsets E_{inf}^0 and E_{sg}^0 of E^0 by $E_{\text{inf}}^0 = E^0 \setminus E_{\text{fin}}^0$ and $E_{\text{sg}}^0 = E^0 \setminus E_{\text{rg}}^0$.

A vertex in E_{sce}^0 is called a *source*. When E is a discrete graph, E_{fin}^0 is the set of vertices which receive finitely many edges, while E_{inf}^0 is the set of vertices which receive infinitely many edges. A vertex in E_{rg}^0 is said to be *regular*, and a vertex in E_{sg}^0 is said to be *singular*. Clearly we have that $E_{\text{sce}}^0 \subset E_{\text{fin}}^0$ and $E_{\text{sg}}^0 = \overline{E_{\text{sce}}^0} \cup E_{\text{inf}}^0$. We have that $\ker \pi_r = C_0(E_{\text{sce}}^0)$ and $\pi_r^{-1}(\mathcal{K}(C_d(E^1))) = C_0(E_{\text{fin}}^0)$ ([Ka4, Proposition 1.24]). Hence the restriction of π_r on $C_0(E_{\text{rg}}^0)$ is an injection into $\mathcal{K}(C_d(E^1))$.

Definition 1.5 Let $E = (E^0, E^1, d, r)$ be a continuous graph. A Toeplitz E -pair $T = (T^0, T^1)$ is called a *Cuntz-Krieger E -pair* if $T^0(f) = \Phi^1(\pi_r(f))$ for any $f \in C_0(E_{\text{rg}}^0)$.

We denote by $\mathcal{O}(E)$ the universal C^* -algebra generated by a Cuntz-Krieger E -pair

When E is a discrete graph, $\mathcal{O}(E)$ is isomorphic to the graph algebra of the opposite graph of E . When a continuous graph E is defined by a homeomorphism σ on a locally compact space X , $\mathcal{O}(E)$ is isomorphic to the homeomorphism C^* -algebra $C_0(X) \rtimes_{\sigma} \mathbb{Z}$. We have that t^0 is injective ([Ka4, Proposition 3.7]). Let $\mathbb{T}$ be the group of complex numbers $z \in \mathbb{C}$ with $|z| = 1$. By the universality of $\mathcal{O}(E)$, there exists an action $\beta : \mathbb{T} \curvearrowright \mathcal{O}(E)$ defined by $\beta_z(t^0(f)) = t^0(f)$ and $\beta_z(t^1(\xi)) = zt^1(\xi)$ for $f \in C_0(E^0)$, $\xi \in C_d(E^1)$ and $z \in \mathbb{T}$. The action β is called the *gauge action*. The next theorem says that the injectivity of T^0 together with the existence of a gauge action implies the universality of T .

Theorem 1.6 ([Ka4, Theorem 4.5]) *For a continuous graph E and a Cuntz-Krieger E -pair T , the natural surjection $\mathcal{O}(E) \rightarrow C^*(T)$ is an isomorphism if and only if T^0 is injective and there exists an automorphism β'_z of $C^*(T)$ such that $\beta'_z(T^0(f)) = T^0(f)$ and $\beta'_z(T^1(\xi)) = zT^1(\xi)$ for every $z \in \mathbb{T}$.*

2 Invariant subsets of continuous graphs

We review definitions and results in [Ka5]. Let $E = (E^0, E^1, d, r)$ be a continuous graph.

Definition 2.1 A subset X^0 of E^0 is said to be *positively invariant* if $d(e) \in X^0$ implies $r(e) \in X^0$ for each $e \in E^1$, and to be *negatively invariant* if for $v \in X^0 \cap E^0_{\text{rg}}$, there exists $e \in E^1$ with $r(e) = v$ and $d(e) \in X^0$. A subset X^0 of E^0 is said to be *invariant* if X^0 is both positively and negatively invariant.

These terminologies coincides with the ordinal ones when continuous graphs are arising from dynamical systems. When E is a discrete graph, X^0 is positively invariant if and only if its complement is hereditary, and X^0 is negatively invariant if and only if its complement is saturated (cf. [BHRS]). For a closed positively invariant subset X^0 of E^0 , we set $X^1 = d^{-1}(X^0)$. Then $X = (X^0, X^1, d, r)$ is a continuous graph. A closed positively invariant set X^0 is invariant if and only if $X^0_{\text{sg}} \subset E^0_{\text{sg}} \cap X^0$.

Definition 2.2 A pair $\rho = (X^0, Z)$ of closed subsets of E^0 satisfying the following two conditions is called an *admissible pair*;

- (i) X^0 is invariant,
- (ii) $X^0_{\text{sg}} \subset Z \subset E^0_{\text{sg}} \cap X^0$.

Definition 2.3 For an admissible pair $\rho = (X^0, Z)$, we define a continuous graph $E_{\rho} = (E_{\rho}^0, E_{\rho}^1, d_{\rho}, r_{\rho})$ as follows. Set $Y_{\rho} = X^0_{\text{rg}} \cap Z$, $\partial Y_{\rho} = \overline{Y_{\rho}} \setminus Y_{\rho}$, and define

$$E_{\rho}^0 = X^0 \amalg_{\partial Y_{\rho}} \overline{Y_{\rho}}, \quad E_{\rho}^1 = X^1 \amalg_{d^{-1}(\partial Y_{\rho})} d^{-1}(\overline{Y_{\rho}}).$$

The domain map $d_{\rho} : E_{\rho}^1 \rightarrow E_{\rho}^0$ is defined from $d : X^1 \rightarrow X^0$ and $d : d^{-1}(\overline{Y_{\rho}}) \rightarrow \overline{Y_{\rho}}$. The range map $r_{\rho} : E_{\rho}^1 \rightarrow E_{\rho}^0$ is defined from $r : X^1 \rightarrow X^0$ and $r : d^{-1}(\overline{Y_{\rho}}) \rightarrow X^0$.

Note that for an admissible pair $\rho = (X^0, Z)$ with $Z = X_{\text{rg}}^0$, we have $E_\rho = X$. Define a C^* -subalgebra $\mathcal{F}^1 \subset \mathcal{O}(E)$ and a $*$ -homomorphism $\pi_0^1 : \mathcal{F}^1 \rightarrow C_0(E_{\text{sg}}^0)$ by

$$\mathcal{F}^1 = \{t^0(f) + \varphi^1(x) \mid f \in C_0(E^0), x \in \mathcal{K}(C_d(E^1))\},$$

and $\pi_0^1(t^0(f) + \varphi^1(x)) = f|_{E_{\text{sg}}^0}$. For an ideal I of $\mathcal{O}(E)$, we define closed subsets X_I^0 and Z_I of E^0 by

$$\begin{aligned} X_I^0 &= \{v \in E^0 \mid f(v) = 0 \text{ for all } f \in C_0(E^0) \text{ with } t^0(f) \in I\}, \\ Z_I &= \{v \in E_{\text{sg}}^0 \mid f(v) = 0 \text{ for all } f \in \pi_0^1(I \cap \mathcal{F}^1)\}. \end{aligned}$$

Proposition 2.4 *For an ideal I of $\mathcal{O}(E)$, the pair $\rho_I = (X_I^0, Z_I)$ is an admissible pair.*

By using Theorem 1.6, we can show the following.

Proposition 2.5 *For a gauge-invariant ideal I of $\mathcal{O}(E)$, there exists a natural isomorphism $\mathcal{O}(E)/I \cong \mathcal{O}(E_{\rho_I})$.*

From this proposition and some computation, we get the next theorem.

Theorem 2.6 *The map $I \mapsto \rho_I$ gives us an inclusion reversing one-to-one correspondence between the set of all gauge-invariant ideals and the set of all admissible pairs.*

This theorem is a continuous counterpart of [BHRS, Theorem 3.6]. It is known that gauge-invariant ideals of a homeomorphism C^* -algebra correspond bijectively to closed invariant subsets [T2, Theorem 2]. The next proposition is a generalization of this fact.

Proposition 2.7 *When a continuous graph E satisfies that $E_{\text{rg}}^0 = E^0$, the map $I \mapsto X_I^0$ gives an inclusion reversing one-to-one correspondence between the set of all gauge-invariant ideals and the set of closed invariant sets.*

Proof. For a closed invariant set X^0 , we have $X_{\text{sg}}^0 = E_{\text{sg}}^0 \cap X^0 = \emptyset$. Hence admissible pairs correspond bijectively to closed invariant subsets. Now the assertion follows from Theorem 2.6. $\blacksquare$

3 Free and topologically free continuous graphs

For $n = 2, 3, \dots$, we define a space E^n of *paths* with length n by

$$E^n = \{(e_n, \dots, e_2, e_1) \in E^1 \times \dots \times E^1 \times E^1 \mid d(e_{k+1}) = r(e_k) \ (1 \leq k \leq n-1)\}.$$

We define domain and range maps $d, r : E^n \rightarrow E^0$ by $d(e) = d(e_1)$ and $r(e) = r(e_n)$ for $e = (e_n, \dots, e_1) \in E^n$. A path $e = (e_n, \dots, e_1) \in E^n$ ($n \geq 1$) is called a *loop* if $r(e) = d(e)$, and the vertex $r(e) = d(e)$ is called the *base point* of the loop e . A loop $e = (e_n, \dots, e_1)$ is said to be *without entrances* if $r^{-1}(r(e_k)) = \{e_k\}$ for $k = 1, \dots, n$.

Definition 3.1 A continuous graph E is said to be *topologically free* if the set of base points of loops without entrances has an empty interior.

This generalizes topological freeness of ordinary dynamical systems and Condition L of graph algebras (see, for example, [T1] and [KPR]).

Theorem 3.2 ([Ka4, Theorem 5.12]) *If a continuous graph $E = (E^0, E^1, d, r)$ is topologically free, then the natural surjection $\mathcal{O}(E) \rightarrow C^*(T)$ is an isomorphism for all Cuntz-Krieger E -pair $T = (T^0, T^1)$ such that T^0 is injective.*

By the above theorem, we have the following (cf. Proposition 2.5).

Proposition 3.3 ([Ka5]) *Let I be an ideal of $\mathcal{O}(E)$. If a continuous graph E_{ρ_I} is topologically free, then I is gauge-invariant.*

We define a *positive orbit space* $\text{Orb}^+(v) \subset E^0$ of $v \in E^0$ by

$$\text{Orb}^+(v) = \{v\} \cup \{r(e) \in E^0 \mid e \in E^n \text{ with } d(e) = v \text{ } (n \geq 1)\}.$$

It is easy to see that a subset X^0 of E^0 is positively invariant if and only if $\text{Orb}^+(v) \subset X^0$ for all $v \in X^0$. For $v \in E^0$, we define $L(v) \subset E^0$ by

$$L(v) = \{v' \in \text{Orb}^+(v) \mid v \in \text{Orb}^+(v')\}.$$

Definition 3.4 For a positive integer n , we denote by $\text{Per}_n(E)$ the set of vertices v_1 satisfying the following three conditions;

- (i) $L(v_1)$ is a finite set $\{v_1, v_2, \dots, v_n\}$,
- (ii) $\{e \in E^1 \mid d(e), r(e) \in L(v_1)\} = \{e_1, e_2, \dots, e_n\}$ with $d(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for $i = 1, 2, \dots, n$ where $v_{n+1} = v_1$,
- (iii) v_1 is isolated in $\text{Orb}^+(v_1)$.

We set $\text{Per}(E) = \bigcup_{n=1}^{\infty} \text{Per}_n(E)$ and $\text{Aper}(E) = E^0 \setminus \text{Per}(E)$.

An element in $\text{Per}(E)$ is called a *periodic point* while an element in $\text{Aper}(E)$ is called an *aperiodic point*.

Definition 3.5 A continuous graph E is said to be *free* if $\text{Aper}(E) = E^0$.

This is a generalization of freeness of ordinary dynamical systems and Condition K of graph algebras (see, for example, [KPRR]).

Proposition 3.6 ([Ka5]) *A continuous graph E is free if and only if E_{ρ} is topologically free for every admissible pair ρ .*

In particular, free continuous graphs are topologically free.

Theorem 3.7 ([Ka5]) *If a continuous graph E is free, then every ideal is gauge-invariant. Hence the set of all ideals corresponds bijectively to the set of all admissible pairs by the map $I \mapsto \rho_I$.*

Proof. Clear from Proposition 3.6, Proposition 3.3 and Theorem 2.6. ■

4 Crossed products of Cuntz algebras

For $n = 2, 3, \dots, \infty$, the Cuntz algebra $\mathcal{O}_n$ is the universal C^* -algebra generated by n isometries $S_1, S_2, \dots, S_n$ (we also use this notation for $n = \infty$), satisfying

$$\begin{aligned} \sum_{i=1}^n S_i S_i^* &= 1 && \text{if } n < \infty, \\ S_i^* S_j &= 0 \quad (\text{for any } i, j \text{ with } i \neq j) && \text{if } n = \infty. \end{aligned}$$

We fix a locally compact abelian group G whose dual group is denoted by Γ . We always use $+$ for multiplicative operations of abelian groups except for $\mathbb{T}$. The pairing of $t \in G$ and $\gamma \in \Gamma$ is denoted by $\langle t | \gamma \rangle \in \mathbb{T}$.

Definition 4.1 Let $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \Gamma^n$ be given. We define the action $\alpha^\omega : G \curvearrowright \mathcal{O}_n$ by

$$\alpha_i^\omega(S_i) = \langle t | \omega_i \rangle S_i \quad (i = 1, 2, \dots, n, t \in G).$$

We recall some elementary facts on the crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ by the action α^ω , which was stated in [Ka1]. The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ has a C^* -subalgebra $\mathbb{C}1 \rtimes_{\alpha^\omega} G$, which is isomorphic to $C_0(\Gamma)$ via the Fourier transform. We denote by T^0 the isomorphism

$$T^0 : C_0(\Gamma) \rightarrow \mathbb{C}1 \rtimes_{\alpha^\omega} G \subset \mathcal{O}_n \rtimes_{\alpha^\omega} G.$$

The Cuntz algebra $\mathcal{O}_n$ is naturally embedded into the multiplier algebra $M(\mathcal{O}_n \rtimes_{\alpha^\omega} G)$ of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$. The crossed product $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ is generated as a C^* -algebra by

$$\{S_i T^0(f) \mid i \in \{1, \dots, n\}, f \in C_0(\Gamma)\}.$$

For $\gamma_0 \in \Gamma$, we define a (reverse) shift automorphism $\sigma_{\gamma_0} : C_0(\Gamma) \rightarrow C_0(\Gamma)$ by $(\sigma_{\gamma_0} f)(\gamma) = f(\gamma + \gamma_0)$. Then we have $T^0(f) S_i = S_i T^0(\sigma_{\omega_i} f)$ for all $f \in C_0(\Gamma)$ and $i \in \{1, \dots, n\}$. From the gauge action of $\mathcal{O}_n$, we can define an action $\beta : \mathbb{T} \curvearrowright \mathcal{O}_n \rtimes_{\alpha^\omega} G$ which is also called a gauge action. We have $\beta_z(T^0(f)) = T^0(f)$ and $\beta_z(S_i T^0(f)) = z S_i T^0(f)$ for $f \in C_0(\Gamma)$, $i \in \{1, \dots, n\}$, and $z \in \mathbb{T}$.

Definition 4.2 Let $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \Gamma^n$ be given. We define a continuous graph $E_\omega = (E_\omega^0, E_\omega^1, d_\omega, r_\omega)$ as follows. We set $E_\omega^0 = \Gamma$ and $E_\omega^1 = \coprod_{i=1}^n \Gamma_i$ where $\Gamma_i = \Gamma$ for $i = 1, 2, \dots, n$. The map $d_\omega : E_\omega^1 \rightarrow E_\omega^0$ is defined by identity maps on each Γ_i , and the map $r_\omega : E_\omega^1 \rightarrow E_\omega^0$ is defined by $r_\omega|_{\Gamma_i}(\gamma) = \gamma + \omega_i$ for $i = 1, 2, \dots, n$.

Each $v \in E_\omega^0$ receives and emits n -edges. It is easy to see that $E_\omega^0 = (E_\omega^0)_{\text{rg}}$ if $n < \infty$, and $E_\omega^0 = (E_\omega^0)_{\text{inf}}$ if $n = \infty$. Since d_ω is defined by identity maps, we have

$$C_{d_\omega}(E_\omega) = \bigoplus_{i=1}^n C_0(\Gamma_i),$$

where $C_0(\Gamma_i) = C_0(\Gamma)$ has natural Hilbert $C_0(\Gamma)$ -module structure. The left action $\pi_{r_\omega} : C_0(\Gamma) \rightarrow \mathcal{L}(C_{d_\omega}(E_\omega))$ satisfies

$$\pi_{r_\omega}(f)(\xi_1, \xi_2, \dots, \xi_n) = (\sigma_{\omega_1}(f)\xi_1, \sigma_{\omega_2}(f)\xi_2, \dots, \sigma_{\omega_n}(f)\xi_n) \in \bigoplus_{i=1}^n C_0(\Gamma_i),$$

for $f \in C_0(\Gamma)$ and $(\xi_1, \xi_2, \dots, \xi_n) \in \bigoplus_{i=1}^n C_0(\Gamma_i)$.

We have a $*$ -homomorphism $T^0 : C_0(\Gamma) \rightarrow \mathcal{O}_n \rtimes_{\alpha^\omega} G$. We define a linear map $T^1 : \bigoplus_{i=1}^n C_0(\Gamma_i) \rightarrow \mathcal{O}_n \rtimes_{\alpha^\omega} G$ by

$$T^1(\xi_1, \xi_2, \dots, \xi_n) = \sum_{i=1}^n S_i T^0(\xi_i) \in \mathcal{O}_n \rtimes_{\alpha^\omega} G$$

for $(\xi_1, \xi_2, \dots, \xi_n) \in \bigoplus_{i=1}^n C_0(\Gamma_i)$.

Proposition 4.3 *The pair $T = (T^0, T^1)$ is a Cuntz-Krieger E_ω -pair, and this induces an isomorphism $\mathcal{O}(E_\omega) \cong \mathcal{O}_n \rtimes_{\alpha^\omega} G$.*

Proof. It is not difficult to see that T is a Toeplitz E_ω -pair. When $n = \infty$, T is a Cuntz-Krieger E_ω -pair because $C_0((E_\omega^0)_{\text{rg}}) = 0$. When $n < \infty$, we have $C_0((E_\omega^0)_{\text{rg}}) = C_0(\Gamma)$. For $f \in C_0(\Gamma)$, we see that

$$\pi_{r_\omega}(f) = \sum_{i=1}^n \theta_{\xi_i, \eta_i}$$

where $\xi_i, \eta_i \in C_0(\Gamma_i)$ satisfies that $\xi_i \bar{\eta}_i = \sigma_{\omega_i}(f)$ for $i = 1, 2, \dots, n$. We have

$$\begin{aligned} \Phi^1(\pi_{r_\omega}(f)) &= \sum_{i=1}^n T^1(\xi_i) T^1(\eta_i)^* = \sum_{i=1}^n S_i T^0(\xi_i) T^0(\eta_i)^* S_i^* \\ &= \sum_{i=1}^n S_i T^0(\sigma_{\omega_i}(f)) S_i^* = \sum_{i=1}^n T^0(f) S_i S_i^* = T^0(f). \end{aligned}$$

Hence T is a Cuntz-Krieger E_ω -pair. By definition, T^0 is injective, and the gauge action on $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ satisfies the condition of Theorem 1.6. Hence the natural surjection $\mathcal{O}(E_\omega) \rightarrow \mathcal{O}_n \rtimes_{\alpha^\omega} G$ is an isomorphism. $\blacksquare$

5 Ideal structures of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ ($n < \infty$)

In this section, we discuss the ideal structure of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ in the case that $n < \infty$. Let n be an integer greater than 1, and take $\omega \in \Gamma^n$. In [Ka1], we introduced the following notion.

Definition 5.1 ([Ka1, Definition 3.2]) A subset X^0 of Γ is called ω -invariant if X^0 is a closed set satisfying the following two conditions:

- (i) For any $\gamma \in X^0$ and any $i \in \{1, 2, \dots, n\}$, we have $\gamma + \omega_i \in X^0$.
- (ii) For any $\gamma \in X^0$, there exists $i \in \{1, 2, \dots, n\}$ such that $\gamma - \omega_i \in X^0$.

The condition (i) above corresponds to positive invariance of $X^0 \subset \Gamma = E^0$, and the condition (ii) corresponds to negative invariance of X^0 . Hence X^0 is an ω -invariant set

if and only if X^0 is a closed invariant set of the continuous graph E_ω . For an ideal I of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$, we define $X_I^0 \subset \Gamma$ by

$$X_I^0 = \{\gamma \in \Gamma \mid f(\gamma) = 0 \text{ for all } f \in C_0(\Gamma) \text{ with } T^0(f) \in I\}.$$

Then X_I^0 is an ω -invariant subset of Γ ([Ka1, Proposition 3.3]). The following is the one of main results in [Ka1].

Theorem 5.2 ([Ka1, Theorem 3.14]) *The correspondence $I \mapsto X_I^0$ gives an inclusion reversing bijection between the set of gauge-invariant ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ and the set of ω -invariant subsets of Γ .*

Proof. This follows from Theorem 2.6 and Proposition 2.7. ■

Definition 5.3 ([Ka1, Definition 4.2]) An ω -invariant subset X of Γ is said to be *bad* if there exists $\gamma_0 \in X$ such that there is only one element $i_0 \in \{1, 2, \dots, n\}$ with $\gamma_0 - \omega_{i_0} \in X$, and this element i_0 satisfies that $m\omega_{i_0} = 0$ for some positive integer m . An ω -invariant subset X of Γ is said to be *good* if X is not bad.

Lemma 5.4 *An ω -invariant subset X^0 is good if and only if the continuous graph $X = (X^0, X^1, d, r)$ is topologically free.*

Proof. If an ω -invariant subset X^0 is bad, then there exists $\gamma_0 \in X^0$ satisfying that there is only one element $i_0 \in \{1, 2, \dots, n\}$ with $\gamma_0 - \omega_{i_0} \in X^0$ and $m\omega_{i_0} = 0$ for some positive integer m . Let $V = X^0 \setminus \bigcup_{i \neq i_0} X^0 + \omega_i$. The set V is an open subset of X^0 and it is not empty because $\gamma_0 \in V$. All $\gamma \in V$ is a base point of a loop

$$\gamma \xrightarrow{\omega_{i_0}} \gamma + \omega_{i_0} \rightarrow \dots \rightarrow \gamma + m\omega_{i_0} = \gamma$$

which has no entrances in the continuous graph X . Hence the continuous graph X is not topologically free. Conversely if the continuous graph X is not topologically free, then a base point γ of a loop without entrances satisfies that there is only one element $i_0 \in \{1, 2, \dots, n\}$ with $\gamma - \omega_{i_0} \in X^0$, and for some positive integer m we have $m\omega_{i_0} = 0$. Hence X^0 is bad. ■

Proposition 5.5 ([Ka1, Theorem 4.5]) *Let I be an ideal of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ such that X_I^0 is good. Then I is gauge-invariant.*

Proof. Combine Proposition 3.3 and Lemma 5.4. ■

An element $\omega \in \Gamma^n$ is said to satisfy *Condition 5.1* if for each $i \in \{1, 2, \dots, n\}$, one of the following two conditions is satisfied ([Ka1]):

- (i) For any positive integer k , $k\omega_i \neq 0$.
- (ii) There exists $j \neq i$ such that $-\omega_j$ is in the closed semigroup generated by $\omega_1, \dots, \omega_n$ and $-\omega_i$.

It is not difficult to see that Condition 5.1 is exactly same as the condition that a continuous graph E_ω is free. Hence from Theorem 3.7, we get the following.

Proposition 5.6 ([Ka1, Theorem 5.2]) *When ω satisfies Condition 5.1, all ideals are gauge-invariant and there is a one-to-one correspondence between the set of ideals of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ and the set of ω -invariant subsets of Γ .*

6 Ideal structures of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$

In [Ka3], we discussed, among others, the ideal structure of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$. The argument there was analogous to the case that $n < \infty$ done in [Ka1]. However we need to change some details, for example, the definition of ω -invariant sets. Take $\omega = (\omega_1, \omega_2, \dots) \in \Gamma^\infty$ and fix it.

Definition 6.1 ([Ka3, Definition 3.3]) A subset X^0 of Γ is called ω -invariant if X^0 is a closed set with $X^0 + \omega_i \subset X^0$ for any positive integer i .

An ω -invariant set is same as a closed positively invariant set in the continuous graph E_ω . However, note that every positively invariant subsets of E_ω are invariant because $(E_\omega^0)_{\text{rg}} = \emptyset$. Hence we see that ω -invariant sets are same as closed invariant sets. For an ω -invariant set X^0 , we define a closed set H_{X^0} by

$$H_{X^0} = \overline{X^0 \setminus \bigcup_{i=1}^{\infty} (X^0 + \omega_i)} \cup \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} (X^0 + \omega_i)} \subset X^0.$$

Definition 6.2 ([Ka3, Definition 3.4]) A pair $\tilde{X} = (X^0, X^\infty)$ of subsets of Γ is called ω -invariant if X^0 is an ω -invariant set, and X^∞ is a closed set satisfying $H_{X^0} \subset X^\infty \subset X^0$.

It is not difficult to see that

$$X_{\text{sce}}^0 = X^0 \setminus \overline{\bigcup_{i=1}^{\infty} (X^0 + \omega_i)}, \quad X_{\text{inf}}^0 = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} (X^0 + \omega_i)},$$

and $H_{X^0} = \overline{X_{\text{sce}}^0} \cup X_{\text{inf}}^0 = X_{\text{sg}}^0$. From this fact, we see that the definition of ω -invariant pairs is same as the one of admissible pairs. For an ideal I of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ and $n \in \mathbb{N}$, we define the closed subset X_I^n of Γ by

$$X_I^n = \{\gamma \in \Gamma \mid f(\gamma) = 0 \text{ for all } f \in C_0(\Gamma) \text{ with } P_n T^0(f) \in I\},$$

where $P_0 = 1$ and $P_n = 1 - \sum_{i=1}^n S_i S_i^* \in \mathcal{O}_\infty$. Clearly, the definition of $X_I^0 \subset \Gamma$ is same as in Section 2. Set $X_I^\infty = \bigcap_{n=0}^{\infty} X_I^n$. The pair $\tilde{X}_I = (X_I^0, X_I^\infty)$ is ω -invariant ([Ka3, Proposition 3.5]). We can see that $X_I^\infty = Z_I$. Hence Theorem 2.6 gives the following.

Theorem 6.3 ([Ka3, Theorem 3.16]) *The correspondence $I \mapsto \tilde{X}_I$ gives a bijection between the set of gauge-invariant ideals of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ and the set of ω -invariant pairs.*

An element $\omega \in \Gamma^\infty$ is said to satisfy Condition 5.1 if for each $i \in \mathbb{Z}_+$, one of the following two conditions is satisfied:

- (i) For any positive integer k , $k\omega_i \neq 0$.
- (ii) For $k = 1, 2, \dots$, there exist positive integers $i_{1,k}, \dots, i_{n_k,k}$ ($n_k \geq 1$) with $i_{1,k} \neq i$ and $\lim_{k \rightarrow \infty} \sum_{j=1}^{n_k} \omega_{i_{j,k}} = 0$.

Similarly as in the case of $n < \infty$, we see that Condition 5.1 is exactly same as the condition that a continuous graph E_ω is free. Hence from Theorem 3.7, we get the following.

Theorem 6.4 ([Ka3, Theorem 5.3]) *Suppose that ω satisfies Condition 5.1. Then all ideal of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ is gauge-invariant. Hence there exists a one-to-one correspondence between the set of ideals of $\mathcal{O}_\infty \rtimes_{\alpha^\omega} G$ and the set of ω -invariant pairs of subsets of Γ .*

7 Primitive ideal spaces

In [Ka1] and [Ka3], we studied the ideal structures of $\mathcal{O}_n \rtimes_{\alpha^\omega} G$ by using primitive ideal spaces when ω does not satisfy Condition 5.1. These works can be considered as continuous counterparts of [HS]. So far, the author has not succeeded in generalizing these results to more general continuous graphs which are not free. Note that a continuous graph E_ω defined here is a special kind of continuous graph which satisfies that every vertices receive and emit same number of edges in the same way.

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