

On the uniqueness theorem for nonlinear singular partial differential equations

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In this note, I will discuss the uniqueness of the solution of nonlinear singular partial differential equations

$$(t\partial/\partial t)^m u = F(t, x, \{(t\partial/\partial t)^j (\partial/\partial x)^\alpha u\}_{j+|\alpha|\leq m, j<m}).$$

Notations. $t \in \mathbf{C}$, $x = (x_1, \dots, x_n) \in \mathbf{C}^n$, $\mathbf{N} = \{0, 1, 2, \dots\}$, $\mathbf{N}^* = \{1, 2, \dots\}$, $m \in \mathbf{N}^*$, $j \in \mathbf{N}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$,

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

and $N = \#\{(j, \alpha) \in \mathbf{N} \times \mathbf{N}^n; j + |\alpha| \leq m \text{ and } j < m\}$. We denote by $\mathcal{R}(\mathbf{C} \setminus \{0\})$ the universal covering space of $\mathbf{C} \setminus \{0\}$.

§1. Equations and assumptions.

Let

$$t \in \mathbf{C}, \quad x = (x_1, \dots, x_n) \in \mathbf{C}^n, \quad Z = \{Z_{j,\alpha}\}_{j+|\alpha|\leq m, j<m} \in \mathbf{C}^N,$$

and let $F(t, x, Z)$ be a function in (t, x, Z) . In this note I will discuss the uniqueness of the solution of the following equation

$$(E) \quad \left(t \frac{\partial}{\partial t}\right)^m u = F\left(t, x, \left\{\left(t \frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{j+|\alpha|\leq m, j<m}\right)$$

with an unknown function $u = u(t, x)$.

The main assumptions are as follows:

- (A₁) $F(t, x, Z)$ is holomorphic in a neighborhood of $(t, x, Z) = (0, 0, 0)$;
- (A₂) $F(0, x, 0) \equiv 0$ near $x = 0$;
- (A₃) $\frac{\partial F}{\partial Z_{j,\alpha}}(0, x, 0) \equiv 0$ near $x = 0$, if $|\alpha| > 0$.

We denote by $\lambda_1(x), \dots, \lambda_m(x)$ the roots of the equation in λ :

$$\lambda^m - \sum_{j < m} \frac{\partial F}{\partial Z_{j,0}}(0, x, 0) \lambda^j = 0$$

and call them the characteristic exponents of (E).

Examples. The followings are typical examples of our equation:

- (1) $t \frac{\partial u}{\partial t} = \lambda u + u \left(\frac{\partial u}{\partial x} \right),$
- (2) $\left(t \frac{\partial}{\partial t} \right)^2 u = 3u \left(\frac{\partial^2 u}{\partial x^2} \right),$
- (3) $\left(t \frac{\partial}{\partial t} \right)^2 u + \left(t \frac{\partial}{\partial t} \right) u = (2u + x + 1) \left(\frac{\partial u}{\partial x} \right)^2.$

§2. Some results by Gérard-Tahara (1993).

Gérard-Tahara [2] proved the following result on the existence of holomorphic solutions.

Theorem 1 (holomorphic solutions). *Assume (A₁), (A₂) and (A₃). If $\lambda_i(0) \notin \{1, 2, 3, \dots\}$ for $i = 1, \dots, m$, the equation (E) has a unique holomorphic solution $u(t, x)$ near the origin of $\mathbf{C} \times \mathbf{C}^m$ satisfying $u(0, x) \equiv 0$.*

Moreover, about the uniqueness of the solution of (E), Gérard-Tahara [2] has proved Theorem 2 below.

Definition 1. We denote by $\mathcal{S}_+$ the set of functions $u(t, x)$ satisfying the following: $u(t, x)$ is a holomorphic function on $\{(t, x) \in \mathcal{R}(\mathbb{C} \setminus \{0\}) \times \mathbb{C}^n; 0 < |t| < \varepsilon, |\arg t| < \theta \text{ and } |x| \leq \delta\}$ for some $\varepsilon > 0, \theta > 0, \delta > 0$ and satisfies

$$\max_{|x| \leq \delta} |u(t, x)| = O(|t|^a) \quad (\text{as } t \rightarrow 0)$$

for some $a > 0$.

Theorem 2 (Uniqueness of the solution). Assume $(A_1), (A_2)$ and (A_3) . If

$$\operatorname{Re} \lambda_i(0) \leq 0 \quad (i = 1, \dots, m)$$

holds, the uniqueness of the solution of (E) is valid in $\mathcal{S}_+$.

Since in this case the equation (E) has a unique holomorphic solution, the above uniqueness theorem yields .

Corollary. Assume $(A_1), (A_2), (A_3)$ and

$$\operatorname{Re} \lambda_i(0) \leq 0 \quad (i = 1, \dots, m).$$

Then, if $u(t, x)$ is a solution of (E) belonging to $\mathcal{S}_+$, $u(t, x)$ is holomorphic in a neighborhood of the origin.

Thus, from the uniqueness theorem we can get the result on removable singularities of the solution of (E).

§3. New uniqueness theorem.

3.1. Class of solutions.

A function $\mu(t)$ on $(0, T)$ is called a *weight function* if it satisfies the following conditions $\mu_1) \sim \mu_4)$:

$$\mu_1) \quad \mu(t) \in C^0((0, T)),$$

$$\mu_2) \quad \mu(t) > 0 \text{ on } (0, T) \text{ and } \mu(t) \text{ is increasing in } t,$$

$$\mu_3) \quad \int_0^T \frac{\mu(s)}{s} ds < \infty,$$

$$\mu_4) \quad \mu(t + ct) = O(\mu(t)) \quad (\text{as } t \rightarrow +0) \text{ for some } c > 0.$$

By μ_2) and μ_3) the condition $\mu(t) \rightarrow 0$ (as $t \rightarrow +0$) is clear. The following functions are typical examples:

$$\mu(t) = t^a, \quad \frac{1}{(-\log t)^b}, \quad \frac{1}{(-\log t)(\log(-\log t))^c}$$

with $a > 0$, $b > 1$, $c > 1$.

Definition 2. For $a > 0$, we denote by $\mathcal{S}_a(\mu(t))$ the set of functions $u(t, x)$ satisfying the following: $u(t, x)$ is a holomorphic function on $\{(t, x) \in \mathcal{R}(\mathbf{C} \setminus \{0\}) \times \mathbf{C}^n; 0 < |t| < \varepsilon, |\arg t| < \theta \text{ and } |x| \leq \delta\}$ for some $\varepsilon > 0$, $\theta > 0$, $\delta > 0$ and satisfies

$$\max_{|x| \leq \delta} |u(t, x)| = O(\mu(|t|)^a) \quad (\text{as } t \rightarrow 0).$$

Remark 1.

$$\mathcal{S}_+ = \bigcup_{a>0} \mathcal{S}_a(\mu(t) \equiv t).$$

Note that $\mu(t) \equiv t$ is a weight function.

3.2. A conjecture.

About the uniqueness of the solution of (E) in $\mathcal{S}_a(\mu(t))$, I have one conjecture:

Conjecture. Assume (A_1) , (A_2) and (A_3) . Let $\mu(t)$ be a weight function. If

$$\operatorname{Re} \lambda_i(0) \leq 0 \quad (i = 1, \dots, m)$$

holds, the uniqueness of the solution of (E) is valid in $\mathcal{S}_m(\mu(t))$.

In the case $m = 1$ this is already proved (see [1]). But in the case $m \geq 2$ this is still open. In the next section I will report a weaker result.

3.3. A weaker result.

Theorem A ([5]). Assume (A_1) , (A_2) and (A_3) . Let $\mu(t)$ be a weight function. If

$$\operatorname{Re} \lambda_i(x) \leq 0 \quad (i = 1, \dots, m)$$

holds in a neighborhood of $x = 0$, the uniqueness of the solution of (E) is valid in $\mathcal{S}_m(\mu(t))$.

Example 1. Let us consider

$$(e_1) \quad \left(t \frac{\partial}{\partial t}\right)^2 u = 3u \left(\frac{\partial^2 u}{\partial x^2}\right)$$

where $(t, x) \in \mathcal{C}^2$. The characteristic exponents are $\lambda_1 = 0$ and $\lambda_2 = 0$. In this case we have:

1) $u(t, x) \equiv 0$ is the unique holomorphic solution of (e_1) under the condition $u(0, x) \equiv 0$.

2) By Theorem A we see that the uniqueness of the solution of (e_1) is valid in $\mathcal{S}_2(\mu(t))$ for any weight function $\mu(t)$.

3) Since $\mathcal{S}_a(\mu(t)) \subset \mathcal{S}_2(\mu(t))$ holds for $a \geq 2$, the uniqueness of the solution of (e_1) is valid in $\mathcal{S}_a(\mu(t))$ for any $a \geq 2$ and any weight function $\mu(t)$.

4) Note that (e_1) has a family of non-trivial solutions

$$u(t, x) = \frac{x^2 + \alpha x + \beta}{(C - \log t)^2} \quad (\alpha, \beta, C \in \mathcal{C}).$$

This implies that if $0 < a < 2$ the uniqueness is not valid in $\mathcal{S}_a(\mu(t))$ for $\mu(t) = 1/(-\log t)^c$ with $1 < c \leq 2/a$.

3.4. Another uniqueness theorem.

In case $\mathcal{S}_a(\mu(t))$ with $a < m$, what happen ? About this we have:

Theorem B ([4],[5]). *If for some p with $0 \leq p \leq m - 1$ the characteristic exponents of (E) satisfy*

$$\begin{cases} \operatorname{Re} \lambda_i(x) \leq 0 & \text{for } i = 1, \dots, p, \\ \operatorname{Re} \lambda_i(0) < 0 & \text{for } i = p + 1, \dots, m \end{cases}$$

in a neighborhood of $x = 0$ and if $a > p$, then the uniqueness of the solution of (E) is valid in $\mathcal{S}_a(\mu(t))$.

Example 2. Let us consider

$$(e_2) \quad \left(t \frac{\partial}{\partial t}\right)^2 u + \left(t \frac{\partial}{\partial t}\right) u = (2u + x + 1) \left(\frac{\partial u}{\partial x}\right)^2$$

where $(t, x) \in \mathbf{C}^2$. The characteristic exponents are $\lambda_1 = 0$ and $\lambda_2 = -1$. In this case we have:

1) $u(t, x) \equiv 0$ is the unique holomorphic solution of (e_2) under the condition $u(0, x) \equiv 0$.

2) By Theorem B we see that if $a > 1$ the uniqueness of the solution of (e_2) is valid in $\mathcal{S}_a(\mu(t))$ for any weight function $\mu(t)$.

3) Note that (e_2) has a family of non-trivial solutions

$$u(t, x) = \frac{x + 1}{(C - \log t)} \quad (C \in \mathbf{C}).$$

This implies that if $0 < a < 1$ the uniqueness is not valid in $\mathcal{S}_a(\mu(t))$ for $\mu(t) = 1/(-\log t)^c$ with $1 < c \leq 1/a$.

4) In the case $a = 1$ it is still unknown whether the uniqueness of the solution of (e_2) is valid in $\mathcal{S}_1(\mu(t))$ for any weight function or not.

References

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