RIMS Kôkyûroku Bessatsu
B19 (2010), 25–33

Kisin conjecture on the moduli spaces of finite flat models

By

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Abstract

We explain a relationship between a local universal deformation ring and a moduli space of finite flat models. We also give an outline of a proof of the Kisin conjecture on the connected components of the moduli space of finite flat models.

Introduction

Let K be a p-adic field for $p > 2$. We consider a two-dimensional continuous representation $V_{\mathbb{F}}$ of the absolute Galois group G_K over a finite field $\mathbb{F}$ of characteristic p. By a finite flat model of $V_{\mathbb{F}}$, we mean a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$, equipped with an action of $\mathbb{F}$, and an isomorphism $V_{\mathbb{F}} \xrightarrow{\sim} \mathcal{G}(\overline{K})$ that respects the action of G_K and $\mathbb{F}$. Then there exists a moduli space of finite flat models of $V_{\mathbb{F}}$, which is projective scheme over $\mathbb{F}$, and we denoted it by $\mathscr{GR}_{V_{\mathbb{F}},0}$. Let $\mathscr{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}$ be the closed subscheme of $\mathscr{GR}_{V_{\mathbb{F}},0}$ determined by the condition that the p-adic Hodge type is (1).

It is important to study the connected components of $\mathscr{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}$, since it gives us information of a deformation ring. The ordinary component of $\mathscr{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}$ was determined in [Kis], and Kisin conjectured that the non-ordinary component is connected. In this survey paper, we explain the relationship between $\mathscr{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}$ and a deformation ring. We also give an outline of a proof of the Kisin conjecture. The theory of the moduli space of finite flat models was established in [Kis], and the Kisin conjecture was proved by Kisin in [Kis] if K is totally ramified over $\mathbb{Q}_p$, by Gee in [Gee] if $V_{\mathbb{F}}$ is the trivial representation, and by the author in [Ima] for general K and $V_{\mathbb{F}}$.

Received March 26, 2009. Revised May 21, 2009; October 28, 2009.
2000 Mathematics Subject Classification(s): Primary: 11F80; Secondary: 14D20
Key Words: *Galois representation, finite flat group scheme, moduli space*
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Acknowledgment

The author would like to thank a referee for a careful reading of this paper and suggestions for improvements.

Notation

Throughout this paper, we use the following notation. Let $p > 2$ be a prime number. For a positive number m, the finite field of cardinality p^m is denoted by $\mathbb{F}_{p^m}$. Let k be the finite extension of $\mathbb{F}_p$ of cardinality $q = p^n$. For a ring R, the ring of Witt vectors over R with respect to p is denoted by $W(R)$. We put $K_0 = W(k)[1/p]$. Let K be a totally ramified extension of K_0 of degree e. The ring of integers of K is denoted by $\mathcal{O}_K$, and the absolute Galois group of K is denoted by G_K. Let $\mathbb{F}$ be a finite field of characteristic p. The formal power series ring of u over $\mathbb{F}$ is denoted by $\mathbb{F}[[u]]$, and its quotient field is denoted by $\mathbb{F}((u))$. Let v_u be the valuation of $\mathbb{F}((u))$ normalized by $v_u(u) = 1$. For a local ring A, the maximal ideal of A is denoted by $\mathfrak{m}_A$. For a topological space X, the set of connected components of X is denoted by $\pi_0(X)$.

§ 1. Deformation ring and moduli space of finite flat models

In this section, we explain the relationship between a deformation ring and a moduli space of finite flat models.

First, we are going to introduce a deformation ring. Let $V_{\mathbb{F}}$ be a two-dimensional continuous G_K-representation over $\mathbb{F}$ with a fixed ordered basis. A G_K-representation over a finite ring is said to be flat if and only if it is isomorphic to the generic fiber of a finite flat group scheme over $\mathcal{O}_K$ as a G_K-module. We assume that $V_{\mathbb{F}}$ is flat. Let $\mathfrak{AR}_{W(\mathbb{F})}$ be the category of Artin local finite $W(\mathbb{F})$-algebra A whose residue field is isomorphic to $\mathbb{F}$ as a $W(\mathbb{F})$-algebra. To define a deformation, we use a notion of groupoids. For the notion of groupoids, please consult [Kis, Appendix on groupoids]. The framed flat deformation $D^{\mathrm{fl},\square}_{V_{\mathbb{F}}}$ of $V_{\mathbb{F}}$ over $\mathfrak{AR}_{W(\mathbb{F})}$ is a groupoid $D^{\mathrm{fl},\square}_{V_{\mathbb{F}}}$ over $\mathfrak{AR}_{W(\mathbb{F})}$ determined as in the followings:

- For an object A in $\mathfrak{AR}_{W(\mathbb{F})}$, an object of $D^{\mathrm{fl},\square}_{V_{\mathbb{F}}}(A)$ is a triple (V_A, ψ, β), where V_A is a flat continuous G_K-representation that is a free A-module of rank 2 with an ordered basis β over A, and $\psi : V_A \otimes_A \mathbb{F} \xrightarrow{\sim} V_{\mathbb{F}}$ is an $\mathbb{F}$-linear G_K-isomorphism sending β to the fixed ordered basis of $V_{\mathbb{F}}$.

- A morphism $(V_A, \psi, \beta) \to (V_{A'}, \psi', \beta')$ covering a given morphism $A \to A'$ in $\mathfrak{AR}_{W(\mathbb{F})}$ is an equivalence class $[\alpha]$, where $\alpha : V_A \otimes_A A' \xrightarrow{\sim} V_{A'}$ is an A'-linear G_K-isomorphism that is compatible with the morphisms ψ, ψ' and sending β to β', and two morphisms are equivalent if they differ by an element of $A'^{\times}$.

Then the framed flat deformation $D_{V_{\mathbb{F}}}^{\mathrm{fl},\square}$ is pro-represented by a complete local $W(\mathbb{F})$-algebra $R_{V_{\mathbb{F}}}^{\mathrm{fl},\square}$.

We are going to define a deformation ring with the condition that the p-adic Hodge type $\mathbf{v}=(1)$, which is denoted by $R_{V_{\mathbb{F}}}^{\mathrm{fl},\square,\mathbf{v}}$. Let $\big(R_{V_{\mathbb{F}}}^{\mathrm{fl},\square}[1/p]\big)^{\mathbf{v}}$ be the quotient of $R_{V_{\mathbb{F}}}^{\mathrm{fl},\square}[1/p]$ corresponding to the connected components of $\operatorname{Spec} R_{V_{\mathbb{F}}}^{\mathrm{fl},\square}[1/p]$ whose closed points ξ satisfy the following:

> If V_ξ is the deformation corresponding to ξ, then $\mathrm{Fil}^0 D_{\mathrm{crys}}\big(V_\xi[1/p]\big)_K$ is free of rank 1 over $k(\xi)\otimes_{\mathbb{Q}_p} K$. Here, $k(\xi)$ is the residue field of ξ.

We note that $V_\xi[1/p]$ is Barsotti-Tate representation, since we are considering a flat deformation. Then we define $R_{V_{\mathbb{F}}}^{\mathrm{fl},\square,\mathbf{v}}$ by the image of $R_{V_{\mathbb{F}}}^{\mathrm{fl},\square}$ in $\big(R_{V_{\mathbb{F}}}^{\mathrm{fl},\square}[1/p]\big)^{\mathbf{v}}$.

The information of the connected components of $\operatorname{Spec} R_{V_{\mathbb{F}}}^{\mathrm{fl},\square,\mathbf{v}}[1/p]$ is very important for an application to a theorem comparing a deformation ring and a Hecke ring ([Kis, Theorem 3.4.11], [Ima, Theorem 3.1]). So we want to know $\pi_0(\operatorname{Spec} R_{V_{\mathbb{F}}}^{\mathrm{fl},\square,\mathbf{v}}[1/p])$.

Next, we are going to explain the Kisin module and the moduli space of finite flat models of $V_{\mathbb{F}}$. By a finite flat model of $V_{\mathbb{F}}$, we mean a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$, equipped with an action of $\mathbb{F}$, and an isomorphism $V_{\mathbb{F}} \xrightarrow{\sim} \mathcal{G}(\overline{K})$ that respects the action of G_K and $\mathbb{F}$.

Let $\mathfrak{S}=W(k)[[u]]$, and $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[1/u]$. We consider the action of ϕ on $\mathcal{O}_{\mathcal{E}}\otimes_{\mathbb{Z}_p}\mathbb{F}\cong k((u))\otimes_{\mathbb{F}_p}\mathbb{F}$ defined by p-th power on $k((u))$. Let $\Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}}\otimes_{\mathbb{Z}_p}\mathbb{F}}$ be the category of finite $\mathcal{O}_{\mathcal{E}}\otimes_{\mathbb{Z}_p}\mathbb{F}$-modules M with ϕ-semi-linear map $\phi: M\to M$ such that the induced linear map $\phi^*M\to M$ is bijective.

We choose a system $(\pi_m)_{m\geq 1}$ of elements in $\overline{K}$ such that $\pi_1^p=\pi$ and $\pi_{m+1}^p=\pi_m$ for $m\geq 1$, and put $K_\infty=\bigcup_{m\geq 1}K(\pi_m)$. Let $\mathrm{Rep}_{\mathbb{F}}(G_{K_\infty})$ be the category of finite-dimensional continuous G_{K_∞}-representations over $\mathbb{F}$.

Then the functor

$$T:\Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}}\otimes_{\mathbb{Z}_p}\mathbb{F}}\to \mathrm{Rep}_{\mathbb{F}}(G_{K_\infty});\ M\mapsto \big(\overline{k((u))}\otimes_{k((u))}M\big)^{\phi=1}$$

is an equivalence of abelian categories. We take $M_{\mathbb{F}}\in \Phi\mathrm{M}_{\mathcal{O}_{\mathcal{E}}\otimes_{\mathbb{Z}_p}\mathbb{F}}$ such that $T(M_{\mathbb{F}})$ is isomorphic to $V_{\mathbb{F}}(-1)|_{G_{K_\infty}}$. Here (-1) denotes the inverse of the Tate twist. Then $M_{\mathbb{F}}$ is a free $(\mathcal{O}_{\mathcal{E}}\otimes_{\mathbb{Z}_p}\mathbb{F})$-module of rank 2.

We put $\mathfrak{S}_{\mathbb{F}}=\mathfrak{S}\otimes_{\mathbb{Z}_p}\mathbb{F}$. Let $(\mathrm{Mod}/\mathfrak{S}_{\mathbb{F}})$ be the category of finite free $\mathfrak{S}_{\mathbb{F}}$-modules $\mathfrak{M}$ with ϕ-semi-linear map $\phi:\mathfrak{M}\to\mathfrak{M}$ such that the cokernel of the induced liner map $\phi^*\mathfrak{M}\to\mathfrak{M}$ is killed by u^e. An object of $(\mathrm{Mod}/\mathfrak{S}_{\mathbb{F}})$ is called a Kisin module with coefficients in $\mathbb{F}$. Let $(\mathbb{F}\text{-}\mathrm{Gr}/\mathcal{O}_K)$ be the category of finite flat group schemes over $\mathcal{O}_K$ with a structure of an $\mathbb{F}$-vector space.

Theorem 1.1. *There exists an equivalence of categories*

$$\mathrm{Gr}:(\mathrm{Mod}/\mathfrak{S}_{\mathbb{F}})\to(\mathbb{F}\text{-}\mathrm{Gr}/\mathcal{O}_K).$$

Proof. This follows from [Br, Théorèm 4.2.1.6] and [Kis, Lemma 1.2.5]. $\square$

Proposition 1.2 ([Kis, Proposition 1.1.13]). *For an object $\mathfrak{M}$ of $(\mathrm{Mod}/\mathfrak{S}_{\mathbb{F}})$, there exists a canonical isomorphism*

$$T(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M})(1) \xrightarrow{\sim} \mathrm{Gr}(\mathfrak{M})(\overline{K})|_{G_{K_\infty}}$$

as G_{K_∞}-representations. Here (1) denotes the Tate twist.

By this proposition, we see that a Kisin module which is a sublattice of $M_{\mathbb{F}}$ corresponds to a finite flat model of $V_{\mathbb{F}}$. Here and in the sequel, a sublattices means a finite free $\mathfrak{S}_{\mathbb{F}}$-submodule of M_F that spans $M_{\mathbb{F}}$ over $\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F}$. In the above, we have defined a Kisin module with coefficients in $\mathbb{F}$. More generally, we can define a Kisin module with coefficients in a $\mathbb{Z}_p$-algebra (cf. [Kis, (1.2)]). Using this general Kisin module, we can construct a moduli space of Kisin modules, which is denoted by $\mathscr{GR}_{V_{\mathbb{F}}}$ and projective over $\mathrm{Spec}\, R^{\mathrm{fl},\square}_{V_{\mathbb{F}}}$ (cf. [Kis, (2.1)]). The closed fiber of $\mathscr{GR}_{V_{\mathbb{F}}}$ over $\mathrm{Spec}\, R^{\mathrm{fl},\square}_{V_{\mathbb{F}}}$ is denoted by $\mathscr{GR}_{V_{\mathbb{F}},0}$. The scheme $\mathscr{GR}_{V_{\mathbb{F}},0}$ is a moduli space of finite flat models of $V_{\mathbb{F}}$ in the sense of the following proposition.

Proposition 1.3 ([Kis, Corollary 2.1.13]). *For any finite extension $\mathbb{F}'$ of $\mathbb{F}$, there is a natural bijection between the set of isomorphism classes of finite flat models of $V_{\mathbb{F}'} = V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ and $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$.*

From now on, we assume $\mathbb{F}_q \subset \mathbb{F}$ and fix an embedding $k \hookrightarrow \mathbb{F}$. This assumption does not matter, since we may extend $\mathbb{F}$ to prove the main theorem. We consider the isomorphism

$$\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} \mathbb{F} \cong k((u)) \otimes_{\mathbb{F}_p} \mathbb{F} \xrightarrow{\sim} \prod_{\sigma \in \mathrm{Gal}(k/\mathbb{F}_p)} \mathbb{F}((u))\ ;\ \Big(\sum_i a_i u^i\Big) \otimes b \mapsto \Big(\sum_i \sigma(a_i) b u^i\Big)_\sigma$$

and let $\epsilon_\sigma \in k((u)) \otimes_{\mathbb{F}_p} \mathbb{F}$ be the primitive idempotent corresponding to σ. Take $\sigma_1, \cdots, \sigma_n \in \mathrm{Gal}(k/\mathbb{F}_p)$ such that $\sigma_{i+1} = \sigma_i \circ \phi^{-1}$. Here we regard ϕ as the p-th power Frobenius, and use the convention that $\sigma_{n+i} = \sigma_i$. In the sequel, we often use such conventions. Then we have $\phi(\epsilon_{\sigma_i}) = \epsilon_{\sigma_{i+1}}$, and $\phi : M_{\mathbb{F}} \to M_{\mathbb{F}}$ determines $\phi : \epsilon_{\sigma_i} M_{\mathbb{F}} \to \epsilon_{\sigma_{i+1}} M_{\mathbb{F}}$. For $(A_i)_{1 \le i \le n} \in GL_2\big(\mathbb{F}((u))\big)^n$, we write

$$M_{\mathbb{F}} \sim (A_1, A_2, \ldots, A_n) = (A_i)_i$$

if there is a basis $\{e_1^i, e_2^i\}$ of $\epsilon_{\sigma_i} M_{\mathbb{F}}$ over $\mathbb{F}((u))$ such that $\phi \begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} = A_i \begin{pmatrix} e_1^{i+1} \\ e_2^{i+1} \end{pmatrix}$. We use the same notation for any sublattice $\mathfrak{M}_{\mathbb{F}} \subset M_{\mathbb{F}}$ similarly.

Finally, for any sublattice $\mathfrak{M}_{\mathbb{F}} \subset M_{\mathbb{F}}$ with a chosen basis $\{e_1^i, e_2^i\}_{1 \le i \le n}$ and $B = (B_i)_{1 \le i \le n} \in GL_2\big(\mathbb{F}((u))\big)^n$, the module generated by the entries of $\left\langle B_i \begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} \right\rangle$ with the basis given by these entries is denoted by $B \cdot \mathfrak{M}_{\mathbb{F}}$. Note that $B \cdot \mathfrak{M}_{\mathbb{F}}$ depends on the choice of the basis of $\mathfrak{M}_{\mathbb{F}}$.

A closed subscheme $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}}} \subset \mathscr{GR}_{V_{\mathbb{F}}}$ is defined by the condition that p-adic Hodge type $\mathbf{v} = (1)$ as in [Kis, (2.4.2)]. The closed fiber of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}}}$ over $\operatorname{Spec} R^{\mathrm{fl},\square}_{V_{\mathbb{F}}}$ is denoted by $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$. The rational points of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ is characterized by the following Lemma.

Lemma 1.4 ([Gee, Lemma 2.2]). *If $\mathbb{F}'$ is a finite extension of $\mathbb{F}$, the elements of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ naturally correspond to free $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}'$-submodules $\mathfrak{M}_{\mathbb{F}'} \subset M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ of rank 2 that satisfy the following:*

1. *$\mathfrak{M}_{\mathbb{F}'}$ is ϕ-stable.*
2. *For some (so any) choice of $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}'$-basis for $\mathfrak{M}_{\mathbb{F}'}$, and for each $\sigma \in \operatorname{Gal}(k/\mathbb{F}_p)$, the map*
$$\phi : \epsilon_{\sigma}\mathfrak{M}_{\mathbb{F}'} \to \epsilon_{\sigma\circ\phi^{-1}}\mathfrak{M}_{\mathbb{F}'}$$
has determinant αu^e for some $\alpha \in \mathbb{F}'[[u]]^{\times}$.

Then there is the following relation between the deformation ring $R^{\mathrm{fl},\square,\mathbf{v}}_{V_{\mathbb{F}}}$ and the moduli space $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$.

Proposition 1.5. *There exists a natural bijection*
$$\pi_0(\operatorname{Spec} R^{\mathrm{fl},\square,\mathbf{v}}_{V_{\mathbb{F}}}[1/p]) \stackrel{\sim}{=} \pi_0(\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}).$$

Proof. This follows from [Kis, Corollary 2.4.10], since $\mathscr{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0} = \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ by [Kis, Proposition 2.4.6] if the p-adic Hodge type $\mathbf{v} = (1)$. $\square$

So the problem has been reduced to study $\pi_0(\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0})$. The connected components $\mathscr{GR}^{\mathbf{v},\mathrm{ord}}_{V_{\mathbb{F}},0} \subset \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ is defined by the points corresponding to the ordinary finite flat group schemes. We can easily determine the set $\pi_0(\mathscr{GR}^{\mathbf{v},\mathrm{ord}}_{V_{\mathbb{F}},0})$ as in the following:

Proposition 1.6 ([Kis, Proposition 2.5.15]). *If $\mathscr{GR}^{\mathbf{v},\mathrm{ord}}_{V_{\mathbb{F}},0}$ is non-empty, then it consist of a single point, unless $V_{\mathbb{F}} \sim \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$ where χ_1 and χ_2 are unramified characters of G_K. In the latter case, we have the followings:*

1. *If $\chi_1 \neq \chi_2$, then $\mathscr{GR}^{\mathbf{v},\mathrm{ord}}_{V_{\mathbb{F}},0}$ consists of two points.*
2. *If $\chi_1 = \chi_2$, then $\mathscr{GR}^{\mathbf{v},\mathrm{ord}}_{V_{\mathbb{F}},0} \stackrel{\sim}{=} \mathbb{P}^1_{\mathbb{F}}$.*

Next, we consider the non-ordinary part. We put
$$\mathscr{GR}^{\mathbf{v},\mathrm{non\text{-}ord}}_{V_{\mathbb{F}},0} = \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0} \setminus \mathscr{GR}^{\mathbf{v},\mathrm{ord}}_{V_{\mathbb{F}},0}.$$

Then Kisin conjectured that $\mathscr{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}},0}$ is connected.

§ 2. Proof of Kisin conjecture

We use the following Lemma on the structure of $M_{\mathbb{F}}$.

Lemma 2.1 ([Ima, Lemma 1.2]). *Suppose $V_{\mathbb{F}}$ is absolutely irreducible and $\mathbb{F}_{q^2} \subset \mathbb{F}$. If $\mathbb{F}'$ is the quadratic extension of $\mathbb{F}$, then*

$$M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}' \sim \left(\begin{pmatrix} 0 & \alpha_1 \\ \alpha_1 u^s & 0 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \dots, \begin{pmatrix} \alpha_n & 0 \\ 0 & \alpha_n \end{pmatrix} \right)$$

for some $\alpha_i \in (\mathbb{F}')^{\times}$ and a positive integer s such that $(q+1) \nmid s$.

In fact, we prove that $\mathscr{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}},0}$ is rationally connected. To join two points by $\mathbb{P}^1_{\mathbb{F}}$, we use the following two Lemmas.

Lemma 2.2 ([Gee, Lemma 2.4]). *Suppose $x_0, x_1 \in \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$ correspond to objects $\mathfrak{M}_{0,\mathbb{F}}, \mathfrak{M}_{1,\mathbb{F}}$ of $(\mathrm{Mod}/\mathfrak{S}_{\mathbb{F}})$ respectively. Let $N = (N_i)_{1 \leq i \leq n}$ be a nilpotent element of $M_2\big(\mathbb{F}((u))\big)^n$ such that $\mathfrak{M}_{1,\mathbb{F}} = (1+N) \cdot \mathfrak{M}_{0,\mathbb{F}}$ with a basis of $\mathfrak{M}_{0,\mathbb{F}}$, and $A = (A_i)_{1 \leq i \leq n}$ be an element of $GL_2\big(\mathbb{F}((u))\big)^n$ such that $\mathfrak{M}_{0,\mathbb{F}} \sim A$ for the same basis of $\mathfrak{M}_{0,\mathbb{F}}$. If $\phi(N_i) A_i N_{i+1} \in M_2\big(\mathbb{F}[[u]]\big)$ for all i, then there is a morphism $\mathbb{P}^1_{\mathbb{F}} \to \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ sending 0 to x_0 and 1 to x_1.*

Proof. We put $\mathfrak{M}_{t,\mathbb{F}} = (1+tN) \cdot \mathfrak{M}_{0,\mathbb{F}}$. Then we have

$$\begin{aligned} \mathfrak{M}_{t,\mathbb{F}} \sim & \big(\phi(1+tN_i) A_i (1+tN_{i+1})^{-1}\big)_i \\ = & \Big(A_i + t\big(\phi(N_i)A_i - A_i N_{i+1}\big) - t^2 \phi(N_i) A_i N_{i+1}\Big)_i . \end{aligned}$$

The ϕ-stability of $\mathfrak{M}_{1,\mathbb{F}}$ ensures that $\big(\phi(N_i)A_i - A_i N_{i+1} - \phi(N_i)A_i N_{i+1}\big) \in M_2(\mathbb{F}[[u]])$. So we get $\big(\phi(N_i)A_i - A_i N_{i+1}\big) \in M_2(\mathbb{F}[[u]])$ by $\phi(N_i)A_i N_{i+1} \in M_2(\mathbb{F}[[u]])$. Then $\mathfrak{M}_{t,\mathbb{F}}$ is ϕ-stable, and a parameterization $t \mapsto \mathfrak{M}_{t,\mathbb{F}}$ gives a morphism $\mathbb{A}^1_{\mathbb{F}} \to \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ sending 0 to x_0 and 1 to x_1. By the properness of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$, this morphism extends to $\mathbb{P}^1_{\mathbb{F}} \to \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$. □

Lemma 2.3 ([Ima, Lemma 2.3]). *Suppose $n \geq 2$ and that $x \in \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$ corresponds to a object $\mathfrak{M}_{\mathbb{F}}$ of $(\mathrm{Mod}/\mathfrak{S}_{\mathbb{F}})$. Fix a basis of $\mathfrak{M}_{\mathbb{F}}$ over $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$. Consider $U^{(i)} = (U^{(i)}_j)_{1 \leq j \leq n} \in GL_2\big(\mathbb{F}((u))\big)^n$ such that $U^{(i)}_i = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ and $U^{(i)}_j = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for all $j \neq i$. If $U^{(i)} \cdot \mathfrak{M}_{\mathbb{F}}$ is ϕ-stable, it corresponds to a point $x' \in \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$, and x' lies on the same connected component of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ as x.*

Proof. First, $U^{(i)} \cdot \mathfrak{M}_{\mathbb{F}}$ corresponds to a point $x' \in \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$, since it satisfies the conditions of Lemma 1.4.

Next, we consider $N^{(i)} = (N_j^{(i)})_{1\le j\le n} \in M_2\big(\mathbb{F}((u))\big)^n$ such that

$$N_i^{(i)} = \begin{pmatrix} 1 & -u \\ u^{-1} & -1 \end{pmatrix} \text{ and } N_j^{(i)} = 0 \text{ for all } j \neq i.$$

Then $U^{(i)} \cdot \mathfrak{M}_{\mathbb{F}} = (1 + N^{(i)}) \cdot \mathfrak{M}_{\mathbb{F}}$, since $\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2u \end{pmatrix} \begin{pmatrix} 2 & -u \\ u^{-1} & 0 \end{pmatrix}$. We have $\phi(N_j^{(i)})A_jN_{j+1}^{(i)} = 0$ for any $A = (A_j) \in GL_2\big(\mathbb{F}((u))\big)^n$, since $n \geq 2$. So we can apply Lemma 2.2. $\square$

Theorem 2.4 (Kisin conjecture). *$\mathscr{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}},0}$ is connected.*

Proof. If $n = 1$, this was proved in [Kis]. Here, we give an outline of a proof in the case $n \geq 2$ and $V_{\mathbb{F}}$ is absolutely irreducible.

Let $\mathbb{F}'$ be a finite extension of $\mathbb{F}$. Suppose $x_1, x_2 \in \mathscr{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ correspond to objects $\mathfrak{M}_{1,\mathbb{F}'}, \mathfrak{M}_{2,\mathbb{F}'}$ of $(\mathrm{Mod}/\mathfrak{S}_{\mathbb{F}'})$ respectively. We are going to show that x_1 and x_2 are joined by $\mathbb{P}^1_{\mathbb{F}}$'s.

If $e < p - 1$, then $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ is one point by [Ray, Theorem 3.3.3]. So we may assume $e \geq p - 1$. Furthermore, extending $\mathbb{F}'$ and replacing $V_{\mathbb{F}}$ by $V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$, we may assume $\mathbb{F} = \mathbb{F}' \supset \mathbb{F}_{q^2}$.

We construct some explicit point of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$. By using Lemma 2.1, we can prove that there exists a basis of $M_{\mathbb{F}}$ such that

$$M_{\mathbb{F}} \sim \left(\alpha_1 \begin{pmatrix} 0 & u^{s_1} \\ u^{t_1} & 0 \end{pmatrix}, \alpha_2 \begin{pmatrix} u^{s_2} & 0 \\ 0 & u^{t_2} \end{pmatrix}, \ldots, \alpha_n \begin{pmatrix} u^{s_n} & 0 \\ 0 & u^{t_n} \end{pmatrix}\right)$$

after replacing the field $\mathbb{F}$ by the quadratic extension. Here $\alpha_i \in \mathbb{F}$, $0 \leq s_i, t_i \leq e$, $s_i + t_i = e$ and $|s_i - t_i| \leq p + 1$ for all i. Let $\mathfrak{M}_{0,\mathbb{F}}$ be the sublattice of $M_{\mathbb{F}}$ defined by this basis. We take a point $x_0 \in \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$ corresponding to $\mathfrak{M}_{\mathbb{F}}$.

We are going to prove that x_0 and x_1 lie on the same connected component. We can prove that x_0 and x_2 lie on the same connected component by the same argument.

By the Iwasawa decomposition and the determinant conditions, we can take $B = (B_i)_{1\le i\le n} \in GL_2\big(\mathbb{F}((u))\big)^n$ such that $\mathfrak{M}_{1,\mathbb{F}} = B \cdot \mathfrak{M}_{0,\mathbb{F}}$ and $B_i = \begin{pmatrix} u^{-a_i} & v_i \\ 0 & u^{a_i} \end{pmatrix}$ for $a_i \in \mathbb{Z}$ and $v_i \in \mathbb{F}((u))$. Then we put $r_i = v_u(v_i)$. Now we have

$$\phi(B_1) \begin{pmatrix} 0 & u^{s_1} \\ u^{t_1} & 0 \end{pmatrix} B_2^{-1} = \begin{pmatrix} \phi(v_1)u^{t_1+a_2} & u^{s_1-pa_1-a_2} - \phi(v_1)v_2u^{t_1} \\ u^{t_1+pa_1+a_2} & -v_2u^{t_1+pa_1} \end{pmatrix},$$

$$\phi(B_i) \begin{pmatrix} u^{s_i} & 0 \\ 0 & u^{t_i} \end{pmatrix} B_{i+1}^{-1} = \begin{pmatrix} u^{s_i-pa_i+a_{i+1}} & \phi(v_i)u^{t_i-a_{i+1}} - v_{i+1}u^{s_i-pa_i} \\ 0 & u^{t_i+pa_i-a_{i+1}} \end{pmatrix}$$

for $2 \leq i \leq n$. On the right-hand sides, every component of the matrices is integral since $\mathfrak{M}_{1,\mathbb{F}}$ is ϕ-stable.

First, we consider the case $t_1 + pa_1 + a_2 \leq e$. In this case, if we put $\mathfrak{M}_{3,\mathbb{F}} = \left(\begin{pmatrix} u^{-a_i} & 0 \\ 0 & u^{a_i} \end{pmatrix}\right)_i \cdot \mathfrak{M}_{0,\mathbb{F}}$, then

$$\mathfrak{M}_{3,\mathbb{F}} \sim \left(\alpha_1 \begin{pmatrix} 0 & u^{s_1-pa_1-a_2} \\ u^{t_1+pa_1+a_2} & 0 \end{pmatrix}, \alpha_2 \begin{pmatrix} u^{s_2-pa_2+a_3} & 0 \\ 0 & u^{t_2+pa_2-a_3} \end{pmatrix}, \ldots, \alpha_n \begin{pmatrix} u^{s_n-pa_n+a_1} & 0 \\ 0 & u^{t_n+pa_n-a_1} \end{pmatrix}\right)$$

and $\mathfrak{M}_{1,\mathbb{F}} = \left(\begin{pmatrix} 1 & v_i u^{-a_i} \\ 0 & 1 \end{pmatrix}\right)_i \cdot \mathfrak{M}_{3,\mathbb{F}}$. Note that $\mathfrak{M}_{3,\mathbb{F}}$ satisfies the conditions of Lemma 1.4, and let x_3 be the point of $\mathscr{G}\mathscr{R}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$ corresponding to $\mathfrak{M}_{3,\mathbb{F}}$. If we put $N_i = \begin{pmatrix} 0 & v_i u^{-a_i} \\ 0 & 0 \end{pmatrix}$, then

$$\phi(N_1) \begin{pmatrix} 0 & u^{s_1-pa_1-a_2} \\ u^{t_1+pa_1+a_2} & 0 \end{pmatrix} N_2 = \begin{pmatrix} 0 & \phi(v_1)v_2u^{t_1} \\ 0 & 0 \end{pmatrix},$$

$$\phi(N_i) \begin{pmatrix} u^{s_i-pa_i+a_{i+1}} & 0 \\ 0 & u^{t_i+pa_i-a_{i+1}} \end{pmatrix} N_{i+1} = 0$$

for $2 \leq i \leq n$. Here we have $v_u\bigl(\phi(v_1)v_2u^{t_1}\bigr) \geq 0$, since $s_1 - pa_1 - a_2 \geq 0$ and $v_u\bigl(u^{s_1-pa_1-a_2} - \phi(v_1)v_2u^{t_1}\bigr) \geq 0$. Hence x_1 and x_3 lie on the same connected component by Lemma 2.2.

Further, we can prove that x_0 and x_3 are joined by $\mathbb{P}^1_{\mathbb{F}}$'s by using Lemma 2.3. Hence x_0 and x_1 lie on the same connected component in the case $t_1 + pa_1 + a_2 \leq e$.

Next, we treat the case $t_1 + pa_1 + a_2 > e$. We consider the following operations:

$$a_i \rightsquigarrow a_i - 1,\ v_i \rightsquigarrow uv_i,\ \text{if it preserves the } \phi\text{-stability of } B \cdot \mathfrak{M}_{0,\mathbb{F}}.$$

These operations replace x_1 by a point that lies on the same connected component as x_1 by Lemma 2.3. We prove that we can continue these operations until we get to the situation where $t_1 + pa_1 + a_2 \leq e$. In other words, we reduce the problem to the case $t_1 + pa_1 + a_2 \leq e$. If we can continue the operations endlessly, we get to the situation where $t_1 + pa_1 + a_2 \leq e$, since the conditions $s_i - pa_i + a_{i+1} \geq 0$ for $2 \leq i \leq n$ exclude that both a_1 and a_2 remain bounded below. Suppose that we cannot continue the operations and $t_1 + pa_1 + a_2 > e$. The condition that we cannot continue the operations is equivalent to the following condition:

$$s_n - pa_n + a_1 = 0 \text{ or } r_2 + t_1 + pa_1 \leq p - 1,$$
$$pr_1 + t_1 + a_2 = 0 \text{ or } t_2 + pa_2 - a_3 \leq p - 1,$$
$$s_{i-1} - pa_{i-1} + a_i = 0 \text{ or } t_i + pa_i - a_{i+1} \leq p - 1 \text{ for each } 3 \leq i \leq n.$$

From these conditions, we can make a contradiction by elementary arguments. This completes the proof. □

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