

Renormalization for Near-Parabolic Fixed Points of Holomorphic Maps

Hiroyuki Inou and Mitsuhiro Shishikura

Consider a fixed point z_0 of a holomorphic function $f(z)$. We say z_0 is *parabolic* if its multiplier $\lambda = f'(z_0)$ is a root of unity. Here we consider the case $\lambda = 1$ and it is non-degenerate, i.e., $f''(z_0) \neq 0$. By a Möbius transformation sending z_0 to infinity, $f(z)$ is conjugate to $\tilde{f}(z) = z + 1 + O(1/z)$. There exist univalent maps $\Phi_{\pm} : \{z; \pm \operatorname{Re} z > L\} \rightarrow \mathbb{C}$ for sufficiently large L such that

$$\Phi_{\pm}(\tilde{f}(z)) = \Phi_{\pm}(z) + 1 \quad (1)$$

where both sides are defined. We call Φ_+ (resp. Φ_-) *attracting* (resp. *repelling*) *Fatou coordinate* for $\tilde{f}(z)$ (or $f(z)$). We can extend the domain of definition of Fatou coordinates (or its inverses) by the functional equation (1), and then $\Phi_{\pm}$ are defined on $W_{\pm} = \{z; \pm \operatorname{Im} z > M + |\operatorname{Re} z|\}$ for sufficiently large M . Therefore, $E_f = \Phi_+ \circ \Phi_-^{-1}$ is well-defined on $\Phi_-(W_{\pm})$. Since $E_f(z+1) = E_f(z) + 1$, it can be extended to $\{z; |\operatorname{Im} z| > L'\}$ for sufficiently large L' . It is called a *horn map*. Since $\Phi_+(z) + c_+$ and $\Phi_-(z) + c_-$ ($c_{\pm} \in \mathbb{C}$) are also Fatou coordinates for $\tilde{f}(z)$, we can replace $\Phi_{\pm}(z)$ by $\Phi_{\pm}(z) + c_{\pm}$ for some $c_{\pm}$ so that $E_f(z)$ is normalized as $E_f(z) = z + o(1)$ as $\operatorname{Im} z \rightarrow +\infty$.

Define $\Pi : \mathbb{C} \rightarrow \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ by $\Pi(z) = e^{2\pi iz}$. Then we can define a map $\hat{E}_f : \{0 < |z| < e^{-2\pi L'}\} \cup \{|z| > e^{2\pi L'}\} \rightarrow \mathbb{C}^*$ satisfying $\hat{E}_f \circ \Pi = \Pi \circ E_f$. We can extend $\hat{E}_f$ holomorphically at 0 and ∞ . They are fixed points of $\hat{E}_f$ and by the normalization above, 0 is a parabolic fixed point for $\hat{E}_f$ of multiplier 1.

When we perturb $f(z)$ in an appropriate direction, z_0 bifurcates to two fixed points and return maps near each of the fixed points can be defined (cf. Yoccoz renormalization for a holomorphic germ of an indifferent fixed point). Consider the case $f(z) = e^{2\pi i\alpha} z + O(z^2)$ is an perturbation of $f_0(z) = z + z^2 + O(z^3)$ and assume $\alpha \neq 0$, $|\arg \alpha| < \pi/4$. For f sufficiently close to f_0 , we can still define Fatou coordinates and the horn map E_f . Then the return map around a fixed point 0 can be written as $\operatorname{Return}(f)(z) = E_f(z) - 1/\alpha$. Taking a semiconjugacy by Π , we obtain a map $\widehat{\operatorname{Return}}(f)(z) = e^{-2\pi i/\alpha} \hat{E}_f(z)$.

Since $E_f(z)$ converges to $E_{f_0}(z)$ locally uniformly on an appropriate domain, horn maps play an important role in studying bifurcations at parabolic fixed points (e.g. linearizability at a fixed point, existence of a Julia set of positive measure, satellite renormalizations,...). Furthermore, if $\alpha = 1/(n - \alpha_1)$ for $n \in \mathbb{Z}$ sufficiently large, we have $(\widehat{\operatorname{Return}}(f))'(0) = \alpha_1$. If α_1 is also small, $|\arg \alpha_1| < \pi/4$ and $(\widehat{\operatorname{Return}}(f))''(0) \neq 0$, then we can again consider the return map $\widehat{\operatorname{Return}}^2(f)(z)$ for $\widehat{\operatorname{Return}}(f)(z)$.

Hence, for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ which has the continued fraction of the form

$$\alpha = \frac{1}{a_1 - \frac{1}{a_2 - \dots}}, \quad a_i > \exists N \gg 1 \quad (2)$$

and appropriate $f_0(z) = z + z^2 + O(z^3)$, we can consider a sequence of return maps $e^{2\pi i \alpha_n} f_n(z) = \widehat{\text{Return}}(e^{2\pi i \alpha_{n-1}} f_{n-1})$, where $\alpha_0 = \alpha$ and $\alpha_n \equiv -1/\alpha_{n-1} \pmod{\mathbb{Z}}$. To obtain such an infinite sequence $\{f_n\}$, we must have some a priori estimate for f_0 . Let us denote $\mathcal{R}_\alpha f = e^{-2\pi i/\alpha} \widehat{\text{Return}}(e^{2\pi i \alpha} f)$ if it is defined.

Our aim here is to define a space of holomorphic maps which is invariant by $\mathcal{R}_\alpha$ and to obtain contraction property of $\mathcal{R}_\alpha$ on it. Since $\mathcal{R}_\alpha \rightarrow \mathcal{R}_0$ when $\alpha \rightarrow 0$, we first study the operator $\mathcal{R} = \mathcal{R}_0$. We call $f \rightsquigarrow \mathcal{R}(f)$ *parabolic renormalization*. Let

$$\mathcal{F}_0 = \left\{ f : U_f \rightarrow \mathbb{C} \left| \begin{array}{l} 0 \in U_f: \text{ open and connected in } \mathbb{C}, f: \text{ holomorphic in } U_f, \\ f(0) = 0, f'(0) = 1, f : U_f \setminus \{0\} \rightarrow \mathbb{C}^* \text{ is a branched covering} \\ \text{with a unique critical value, all critical points have local degree 2} \end{array} \right. \right\}.$$

We can define $\mathcal{R}$ on $\mathcal{F}_0$ and the following theorem is known:

Theorem 1. (i) $\mathcal{R}(\mathcal{F}_0) \subset \mathcal{F}_0$.

(ii) Let $f_{\text{Koebe}} = z/(1-z)^2$ and $f_\star = \mathcal{R}(f_{\text{Koebe}})$. Then $f_\star$ is defined on $\mathbb{D}$ and $f_\star \in \mathcal{F}_0$. Any $f \in \mathcal{R}(\mathcal{F}_0)$ can be written as $f = f_\star \circ \phi^{-1}$ where $\phi : \mathbb{D} \rightarrow U_f$ a conformal map with $\phi(0) = 0$, $\phi'(0) = 1$.

Hence there exists a bijection between $\mathcal{R}(\mathcal{F}_0)$ and $\mathcal{S} = \{\phi : \mathbb{D} \rightarrow \mathbb{C}; \text{ univalent, holomorphic, } \phi(0) = 0, \phi'(0) = 1\}$. We consider a topology on $\mathcal{R}(\mathcal{F}_0)$ which is induced from local uniform convergence topology on $\mathcal{S}$ by this bijection.

However, considering this space is not sufficient to study $\mathcal{R}_\alpha$ for $\alpha \neq 0$ (e.g. $\mathcal{R}_\alpha f$ have infinitely many critical values). Therefore, we need to relax covering property and we obtain the following:

Main Theorem 1. Let $P = z(1-z)^2$. There exist simply connected domains $V, V' \subset \mathbb{C}$ such that V contains the fixed point 0 and the critical point $-1/3$ for P , $V \Subset V'$, V is a quasidisk and

$$\mathcal{F}_1 = \left\{ f = P \circ \phi^{-1} \left| \begin{array}{l} \phi : V \rightarrow \mathbb{C} \text{ is conformal, } \phi(0) = 0, \phi'(0) = 1, \\ \text{and has a quasiconformal extension to } \mathbb{C} \end{array} \right. \right\}.$$

satisfies the following:

- (i) $\mathcal{F}_0 \setminus \{\text{quadratic polynomial}\} \hookrightarrow \mathcal{F}_1$. In particular, $\mathcal{R}_0^n(z + z^2) \in \mathcal{F}_1$ ($n \geq 1$).
- (ii) $\mathcal{R}_0 f \in \mathcal{F}_1$ for $f \in \mathcal{F}_1$. If $\mathcal{R}_0 f = P \circ \psi^{-1}$, then ψ can be extended conformally on V' .
- (iii) $f \rightsquigarrow \mathcal{R}_0$ is "holomorphic".
- (iv) There exists $\alpha_0 > 0$ such that $\mathcal{R}_\alpha$ ($0 < \alpha < \alpha_0$) also satisfies (ii) and (iii).

Main Theorem 2. There exists a metric d on $\mathcal{F}_1$ such that $\mathcal{R}_\alpha$ ($0 \leq \alpha < \alpha_0$) is a uniform contraction with respect to d .

Corollary 2. For α satisfying (2) and $f = f_0 \in \mathcal{F}_1$, the sequence of return maps $\{e^{2\pi i \alpha_n} f_n = \widehat{\text{Return}}^n(e^{2\pi i \alpha} f)\}$ ($\alpha_n \equiv -1/\alpha_{n-1} \pmod{\mathbb{Z}}$ and $f_n = \mathcal{R}_{\alpha_{n-1}} f_{n-1}$ ($n = 1, 2, \dots$)) is defined and $f_n \in \mathcal{F}_1$. It is also true for $f(z) = z + z^2$.

Corollary 3. For α in Corollary 2, $g(z) = e^{2\pi i \alpha} z + z^2$ does not have dense critical orbit in its Julia set.