

On standing waves for nonlinear Schrödinger equations with potentials

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This is a joint work with Reika Fukuizumi (Tohoku University). We consider the instability of standing wave solution $u_\omega(t, x) = e^{i\omega t}\phi_\omega(x)$ for the nonlinear Schrödinger equation with potential $V(x)$:

$$iu_t = -\Delta u + V(x)u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R}^{1+n}. \quad (1)$$

We always assume $1 < p < \infty$ if $n = 1, 2$, and $1 < p < 1 + 4/(n-2)$ if $n \geq 3$. Moreover, we suppose that $\omega \in \mathbb{R}$ and $\phi_\omega(x)$ is a ground state for

$$-\Delta \phi + \omega \phi + V(x)\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^n. \quad (2)$$

In this note, under appropriate assumptions on $V(x)$, we will show that if $p > 1 + 4/n$, the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1) is unstable for sufficiently large $\omega > 0$. Before stating our result precisely, we recall some known results. First, we consider the case $V(x) \equiv 0$. For any $\omega > 0$, there exists a unique positive radial solution $\psi_\omega(x)$ of (2) with $V(x) \equiv 0$ in $H^1(\mathbb{R}^n)$, and the standing wave solution $e^{i\omega t}\psi_\omega(x)$ of (1) with $V(x) \equiv 0$ is stable for any $\omega > 0$ if $p < 1 + 4/n$, and unstable for any $\omega > 0$ if $p \geq 1 + 4/n$ (see, e.g., [1, 3, 6, 11, 12]). Meanwhile, when $-\Delta + V(x)$ has the first eigenvalue λ_1 , it is shown in [10, 4] using standard bifurcation theory that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1) is stable for $\omega > -\lambda_1$ sufficiently close to $-\lambda_1$, even if $p \geq 1 + 4/n$.

For potential $V(x)$, we assume

(V1) $V(x) \in C^2(\mathbb{R}^n, \mathbb{R})$, and there exist $m \geq 0$ and $C > 0$ such that

$$0 \leq V(x) \leq C(1 + |x|^m) \text{ on } \mathbb{R}^n, \text{ and}$$

(V2) $|x \cdot \nabla V(x)| + \left| \sum_{j,k=1}^n x_j x_k \partial_j \partial_k V(x) \right| \leq C(1 + V(x))$ on $\mathbb{R}^n$.

Example. (i) (Harmonic potentials) Let $c_1, \dots, c_n$ be positive constants.

Then $V_1(x) = \sum_{j=1}^n c_j x_j^2$ satisfies (V1) and (V2).

(ii) $V_2(x) = 1 + \sin x_1$ satisfies (V1), but does not satisfy (V2).

(iii) For $c \geq 0$, $V_1(x) + cV_2(x)$ satisfies (V1) and (V2).

(iv) (V1) and (V2) are satisfied if $V(x) \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfies

$$V(x) \geq 0, \quad |\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-|\alpha|} \quad (|\alpha| \leq 2).$$

We use the following notation.

$$\begin{aligned} X &:= \{v \in H^1(\mathbb{R}^n) : V(x)|v(x)|^2 \in L^1(\mathbb{R}^n)\}, \\ E(v) &:= \frac{1}{2} \|\nabla v\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^n} V(x)|v(x)|^2 dx - \frac{1}{p+1} \|v\|_{p+1}^{p+1}, \\ S_\omega(v) &:= E(v) + \frac{\omega}{2} \|v\|_2^2, \\ P(v) &:= \|\nabla v\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^n} x \cdot \nabla V(x)|v(x)|^2 dx - \frac{n(p-1)}{2(p+1)} \|v\|_{p+1}^{p+1}, \\ I_\omega(v) &:= \|\nabla v\|_2^2 + \omega \|v\|_2^2 + \int_{\mathbb{R}^n} V(x)|v(x)|^2 dx - \|v\|_{p+1}^{p+1}. \end{aligned}$$

We note that

$$P(v) = \partial_\lambda S_\omega(v^\lambda)|_{\lambda=1}, \quad I_\omega(v) = \partial_\lambda S_\omega(\lambda v)|_{\lambda=1},$$

where $v^\lambda(x) := \lambda^{n/2} v(\lambda x)$ for $\lambda > 0$.

Assumption (A1). For any $u_0 \in X$, there exist $T = T(\|u_0\|_X) > 0$ and a unique solution $u(t) \in C([0, T], X)$ of (1) such that $u(0) = u_0$ and

$$E(u(t)) = E(u_0), \quad \|u(t)\|_2^2 = \|u_0\|_2^2, \quad t \in [0, T].$$

In addition, if $u_0 \in X$ satisfies $|x|u_0 \in L^2(\mathbb{R}^n)$, then we have

$$\frac{d^2}{dt^2} \|xu(t)\|_2^2 = 8P(u(t)), \quad t \in [0, T].$$

For sufficient conditions on $V(x)$ that (A1) holds, see, e.g., Section 9.2 in [2]. We remark that (A1) is satisfied for (iii) in Example above, and for $V(x) \in C^1(\mathbb{R}^n, \mathbb{R})$ such that

$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-|\alpha|} \quad (|\alpha| \leq 1).$$

Definition 1. We say that a standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1) is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $u_0 \in X$ satisfies

$$\inf_{\theta \in \mathbb{R}} \|u_0 - e^{i\theta}\phi_\omega\|_X < \delta,$$

then the solution $u(t)$ of (1) with $u(0) = u_0$ exists for all $t \geq 0$ and satisfies

$$\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}} \|u(t) - e^{i\theta}\phi_\omega\|_X < \varepsilon.$$

Otherwise, $e^{i\omega t}\phi_\omega(x)$ is said to be unstable.

Definition 2. $\mathcal{G}_\omega :=$ the set of minimizers for

$$\inf\{S_\omega(v) : v \in X \setminus \{0\}, I_\omega(v) = 0\}.$$

An element of $\mathcal{G}_\omega$ is called a ground state of (2).

Assumption (A2). There exists $\omega_0 \in (0, \infty)$ such that $\mathcal{G}_\omega$ is not empty for any $\omega \in (\omega_0, \infty)$.

If $V(x) \in C(\mathbb{R}^n, \mathbb{R})$ satisfies $\lim_{|x| \rightarrow \infty} V(x) = +\infty$, by the compactness of the embedding $X \subset L^q(\mathbb{R}^n)$ with $2 \leq q < 2n/(n-2)$, it is easy to see that (A2) is satisfied. However, for bounded $V(x)$, we may need some additional assumptions related to the concentration compactness principle (see [7, 8]).

Theorem 1. Assume (A1), (A2), (V1) and (V2). Let $p > 1 + 4/n$ and $\phi_\omega(x) \in \mathcal{G}_\omega$. Then there exists $\omega_* \in (\omega_0, \infty)$ such that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1) is unstable for any $\omega \in (\omega_*, \infty)$.

By the general theory in [6], under an assumption on the spectrum of a linearized operator, the standing wave solution $e^{i\omega_1 t}\phi_{\omega_1}(x)$ of (1) is stable (resp. unstable) if the function $\|\phi_\omega\|_2^2$ is strictly increasing (resp. decreasing) at $\omega = \omega_1$. In case of $V(x) \equiv 0$, by the scaling $\psi_\omega(x) = \omega^{1/(p-1)}\psi_1(\sqrt{\omega}x)$, it is easy to check the monotonicity of $\|\psi_\omega\|_2^2$. However, it seems difficult to check this property for general $V(x)$. So, for the proof of Theorem 1, we use the following sufficient condition for the instability, which is a modification of Theorem 3 in [9] (see also [4, 5, 11]).

Proposition 2. Assume (A1), (A2), (V1) and (V2), and let $\phi_\omega(x) \in \mathcal{G}_\omega$. If $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$, then the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1) is unstable. Here, $v^\lambda(x) := \lambda^{n/2}v(\lambda x)$ for $\lambda > 0$.

We note that $\|v^\lambda\|_2^2 = \|v\|_2^2$ and

$$\begin{aligned} E(v^\lambda) &= \frac{\lambda^2}{2} \|\nabla v\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^n} V\left(\frac{x}{\lambda}\right) |v(x)|^2 dx - \frac{\lambda^{n(p-1)/2}}{p+1} \|v\|_{p+1}^{p+1}, \\ \partial_\lambda^2 E(v^\lambda)|_{\lambda=1} &= \|\nabla v\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^n} \left\{ 2x \cdot \nabla V(x) + \sum_{j,k=1}^n x_j x_k \partial_j \partial_k V(x) \right\} |v(x)|^2 dx \\ &\quad - \frac{n(p-1)}{2(p+1)} \left\{ \frac{n(p-1)}{2} - 1 \right\} \|v\|_{p+1}^{p+1}. \end{aligned}$$

Since $P(\phi_\omega) = \partial_\lambda S_\omega(\phi_\omega^\lambda)|_{\lambda=1} = 0$, if we put

$$V^*(x) = 3x \cdot \nabla V(x) + \sum_{j,k=1}^n x_j x_k \partial_j \partial_k V(x),$$

then we have

$$\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} = \frac{1}{2} \int_{\mathbb{R}^n} V^*(x) |\phi_\omega(x)|^2 dx - \frac{n(p-1)}{2(p+1)} \left\{ \frac{n(p-1)}{2} - 2 \right\} \|\phi_\omega\|_{p+1}^{p+1}.$$

Thus, we see that the condition $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$ is equivalent to

$$\frac{\int_{\mathbb{R}^n} V^*(x) |\phi_\omega(x)|^2 dx}{\|\phi_\omega\|_{p+1}^{p+1}} < \frac{n(p-1)\{n(p-1) - 4\}}{2(p+1)}. \quad (3)$$

We remark that the right hand side of (3) is a positive constant by the assumption $p > 1 + 4/n$ in Theorem 1. By using the variational characterization of the ground state $\phi_\omega(x)$ of (2) and the rescaling (4) below, we will show that the left hand side of (3) converges to 0 as $\omega \rightarrow \infty$. For $\phi_\omega(x) \in \mathcal{G}_\omega$, we define $\tilde{\phi}_\omega(x)$ by

$$\phi_\omega(x) = \omega^{1/(p-1)} \tilde{\phi}_\omega(\sqrt{\omega}x), \quad \omega \in (\omega_0, \infty). \quad (4)$$

Then, $\tilde{\phi}_\omega(x)$ satisfies

$$-\Delta \phi + \phi + \omega^{-1} V\left(\frac{x}{\sqrt{\omega}}\right) \phi - |\phi|^{p-1} \phi = 0, \quad x \in \mathbb{R}^n.$$

Recall that $\psi_1(x)$ is the unique positive radial solution of (2) with $V(x) \equiv 0$ and $\omega = 1$ in $H^1(\mathbb{R}^n)$, and we put

$$\begin{aligned} \tilde{I}_\omega(v) &:= \|\nabla v\|_2^2 + \|v\|_2^2 + \omega^{-1} \int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right) |v(x)|^2 dx - \|v\|_{p+1}^{p+1}, \\ I_1^0(v) &:= \|\nabla v\|_2^2 + \|v\|_2^2 - \|v\|_{p+1}^{p+1}. \end{aligned}$$

Lemma 3. Assume (A2) and (V1). Then, we have

$$(i) \lim_{\omega \rightarrow \infty} \|\tilde{\phi}_\omega\|_{p+1}^{p+1} = \|\psi_1\|_{p+1}^{p+1}, \quad (ii) \lim_{\omega \rightarrow \infty} I_1^0(\tilde{\phi}_\omega) = 0, \quad (iii) \lim_{\omega \rightarrow \infty} \|\tilde{\phi}_\omega\|_{H^1}^2 = \|\psi_1\|_{H^1}^2.$$

Proof. First of all, we note that $\tilde{\phi}_\omega(x)$ is a minimizer of

$$\inf\{\|v\|_{p+1}^{p+1} : v \in X \setminus \{0\}, \tilde{I}_\omega(v) \leq 0\},$$

and $\psi_1(x)$ is a minimizer of

$$\inf\{\|v\|_{p+1}^{p+1} : v \in H^1(\mathbb{R}^n) \setminus \{0\}, I_1^0(v) \leq 0\}.$$

First, we show (i). Since $\tilde{I}_\omega(\tilde{\phi}_\omega) = 0$, we have $I_1^0(\tilde{\phi}_\omega) \leq 0$. Thus, we have $\|\psi_1\|_{p+1}^{p+1} \leq \|\tilde{\phi}_\omega\|_{p+1}^{p+1}$ for any $\omega \in (\omega_0, \infty)$. Moreover, for any $\mu > 1$, it follows from $I_1^0(\psi_1) = 0$ that

$$\mu^{-2} \tilde{I}_\omega(\mu\psi_1) = -(\mu^{p-1} - 1)\|\psi_1\|_{p+1}^{p+1} + \omega^{-1} \int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right) |\psi_1(x)|^2 dx.$$

Here, from the assumption (V1), we have

$$\left| \int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right) |\psi_1(x)|^2 dx \right| \leq C \int_{\mathbb{R}^n} (1 + \omega^{-m/2} |x|^m) |\psi_1(x)|^2 dx.$$

Since $\psi_1(x)$ has exponential decay at infinity, we have $(1 + |x|^m) |\psi_1(x)|^2 \in L^1(\mathbb{R}^n)$ and

$$\lim_{\omega \rightarrow \infty} \omega^{-1} \int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right) |\psi_1(x)|^2 dx = 0.$$

Thus, there exists $\omega(\mu) \in (\omega_0, \infty)$ such that $\tilde{I}_\omega(\mu\psi_1) < 0$ for any $\omega \in (\omega(\mu), \infty)$, so we have

$$(\|\psi_1\|_{p+1}^{p+1} \leq) \|\tilde{\phi}_\omega\|_{p+1}^{p+1} \leq \|\mu\psi_1\|_{p+1}^{p+1} = \mu^{p+1} \|\psi_1\|_{p+1}^{p+1}$$

for any $\omega \in (\omega(\mu), \infty)$. Since $\mu > 1$ is arbitrary, we conclude (i).

Next, we show (ii). Since $I_1^0(\tilde{\phi}_\omega) \leq 0$, for any $\omega \in (\omega_0, \infty)$ there exists $\mu(\omega) \in (0, 1]$ such that $I_1^0(\mu(\omega)\tilde{\phi}_\omega) = 0$. Thus, we have

$$\|\psi_1\|_{p+1}^{p+1} \leq \|\mu(\omega)\tilde{\phi}_\omega\|_{p+1}^{p+1} = \mu(\omega)^{p+1} \|\tilde{\phi}_\omega\|_{p+1}^{p+1},$$

which implies $\|\psi_1\|_{p+1}/\|\tilde{\phi}_\omega\|_{p+1} \leq \mu(\omega) \leq 1$, and from (i) we have $\lim_{\omega \rightarrow \infty} \mu(\omega) = 1$.

Moreover, from $I_1^0(\mu(\omega)\tilde{\phi}_\omega) = 0$, we have

$$I_1^0(\tilde{\phi}_\omega) = (\mu(\omega)^{p-1} - 1) \|\tilde{\phi}_\omega\|_{p+1}^{p+1}.$$

Hence, again from (i), we conclude (ii).

Since $I_1^0(\psi_1) = 0$, (iii) follows from (i) and (ii) immediately. $\square$

Proof of Theorem 1. As stated above, we have only to show that the left hand side of (3) converges to 0 as $\omega \rightarrow \infty$. Since we have

$$\frac{\int_{\mathbb{R}^n} V^*(x) |\phi_\omega(x)|^2 dx}{\|\phi_\omega\|_{p+1}^{p+1}} = \frac{\omega^{-1} \int_{\mathbb{R}^n} V^*(x/\sqrt{\omega}) |\tilde{\phi}_\omega(x)|^2 dx}{\|\tilde{\phi}_\omega\|_{p+1}^{p+1}},$$

by Lemma 3 (i), it suffices to prove

$$\lim_{\omega \rightarrow \infty} \omega^{-1} \int_{\mathbb{R}^n} V^*\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\phi}_\omega(x)|^2 dx = 0. \quad (5)$$

From $\tilde{I}_\omega(\tilde{\phi}_\omega) = 0$ and Lemma 3 (ii), we have

$$\lim_{\omega \rightarrow \infty} \omega^{-1} \int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\phi}_\omega(x)|^2 dx = - \lim_{\omega \rightarrow \infty} I_1^0(\tilde{\phi}_\omega) = 0.$$

By the assumption (V2), we have $|V^*(x)| \leq C(1 + V(x))$ on $\mathbb{R}^n$, and by Lemma 3 (iii) we obtain (5). $\square$

Remark. Let $\phi_\omega(x) \in \mathcal{G}_\omega$ and we assume (without loss of generality) that $\phi_\omega(x)$ is positive in $\mathbb{R}^n$. By Lemma 3 and the concentration compactness principle, we see that there exist a subsequence $\{\phi_{\omega_j}(x)\}$ of $\{\phi_\omega(x)\}$ and a sequence $\{y_j\} \subset \mathbb{R}^n$ such that

$$\lim_{j \rightarrow \infty} \|\phi_{\omega_j} - \psi_1(\cdot + y_j)\|_{H^1} = 0 \quad (6)$$

(see Theorem III.1 in [8]). Although (6) may give some information on the asymptotic behavior of $\phi_\omega(x) \in \mathcal{G}_\omega$ as $\omega \rightarrow \infty$, we did not use (6) in the proof of Theorem 1 directly. We also note that Lemma 3 holds for any p such that $1 < p < \infty$ if $n = 1, 2$, and $1 < p < 1 + 4/(n - 2)$ if $n \geq 3$. Finally, we remark that, in the case $p = 1 + 4/n$, it follows from (6) that

$$\lim_{\omega \rightarrow \infty} \|\phi_\omega\|_2^2 = \|\psi_1\|_2^2.$$

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