

A Classification of Semiregular RDS's with $k = 12$

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1. Introduction

Definition 1.1. An incidence structure $(\mathbb{P}, \mathbb{B})$ is called a *square* (m, u, k, λ) -*divisible design* if the following conditions (i)-(iii) are satisfied.

- (i) $|\mathbb{P}| = |\mathbb{B}| = mu$.
- (ii) There exists a partition $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2 \cup \cdots \cup \mathbb{P}_m$ of $\mathbb{P}$ satisfying
 $|\mathbb{P}_1| = \cdots = |\mathbb{P}_m| = u$ and
 $[p, q] = \begin{cases} 0 & \text{if } p, q \in \mathbb{P}_i, \exists i, \\ \lambda & \text{otherwise.} \end{cases} \quad (p \neq q \in \mathbb{P}).$
- (iii) $|B| = k \quad (\forall B \in \mathbb{B})$.

The following hold.

$$k(k-1) = (m-1)u\lambda, \quad [p] = k \quad (\forall p \in \mathbb{P}) \quad (1)$$

$$k \geq u\lambda \quad (\text{Bose - Connor}[1]) \quad (2)$$

Let $p \in \mathbb{P}_1$ and $B \in \mathbb{B}$ and assume that an automorphism group G of $(\mathbb{P}, \mathbb{B})$ acts regularly on both $\mathbb{P}$ and $\mathbb{B}$. Set

$$D = \{x \in G \mid px \in B\} \quad \text{and} \quad U = \{x \in G \mid px \in \mathbb{P}_1\}.$$

Then $|D| = k$ and U is a subgroup of G of order u satisfying

$$DD^{(-1)} = k + \lambda(G - U). \quad (3)$$

The equation (3) is equivalent to the following.

$$|aD \cap bD| = \begin{cases} 0 & \text{if } aU = bU, \\ \lambda & \text{otherwise.} \end{cases} \quad (a \neq b \in G) \quad (4)$$

Definition 1.2. Let G be a group of order mu and U a subgroup of G of order u . A k -subset D is called a (m, u, k, λ) -*difference set* relative to U if D satisfies (3). D is also called a *relative difference set* (RDS) relative to U .

Conversely, given a (m, u, k, λ) -difference set D in G relative to U . Then we can show that $\text{dev}(D)$ is a square (m, u, k, λ) -divisible design, where

$$\text{dev}(D) := (G, \{Dx \mid x \in G\}).$$

Definition 1.3. A square (m, u, k, λ) -divisible design is said to be *symmetric* if its dual is also a square (m, u, k, λ) -divisible design. In other words, there is a partition $\mathbb{B} = \mathbb{B}_1 \cup \dots \cup \mathbb{B}_m$ of $\mathbb{B}$ satisfying

$$|B \cap C| = \begin{cases} 0 & \text{if } B, C \in \mathbb{B}_i, \exists i, \\ \lambda & \text{otherwise.} \end{cases} \quad (B \neq C \in \mathbb{B})$$

Result 1.4. (W. S. Connor [3]) Let $(\mathbb{P}, \mathbb{B})$ be a square (m, u, k, λ) -divisible design such that $k > u\lambda$. If $(k, \lambda) = 1$, then $(\mathbb{P}, \mathbb{B})$ is symmetric.

Remark 1.5. Let D be a (m, u, k, λ) -difference set in G relative to a subgroup U . If $DD^{(-1)} = D^{(-1)}D$, then $\text{dev}(D)$ is symmetric.

Result 1.6. (D. Jungnickel [9]) If $G \triangleright U$, then $DD^{(-1)} = D^{(-1)}D$.

Concerning this, we have the following results.

Proposition 1.7. $\text{dev}(D)$ is symmetric if and only if $D^{(-1)}D = u\lambda + \lambda(G - V)$ for a subgroup V of G .

Proof. Set $(\mathbb{P}, \mathbb{B}) = \text{dev}(D)$ and assume $(\mathbb{P}, \mathbb{B})$ is symmetric. Then, there exists a partition $\mathbb{B} = \mathbb{B}_1 \cup \dots \cup \mathbb{B}_{u\lambda}$ of $\mathbb{B}$ such that

$$|B \cap C| = \begin{cases} 0 & \text{if } B, C \in \mathbb{B}_i, \exists i, \\ \lambda & \text{otherwise.} \end{cases} \quad (B \neq C \in \mathbb{B}).$$

Set $\mathbb{B}_1 = \{Dg_1, Dg_2, \dots, Dg_u\}$, where $g_1 = 1$. As $Dd_i \cap Dg_j = \emptyset$ for any distinct $i, j \in \{1, 2, \dots, u\}$, for each $\mathbb{B}_k$ there is an element $g \in G$ so that $\mathbb{B}_k = \{Dg_1g, Dg_2g, \dots, Dg_u g\}$.

We note that

$$\begin{aligned} Dg_i \cap Dg_j &= \emptyset \quad (\Leftrightarrow \{(d_1, d_2) \mid d_1, d_2 \in D, d_1g_i = d_2g_j\} = \emptyset) \\ &\Leftrightarrow \{(d_1, d_2) \mid d_1, d_2 \in D, d_1^{-1}d_2 = g_i g_j^{-1}\} = \emptyset \quad (*) \end{aligned}$$

Set $V = \{g_1 (= 1), g_2, \dots, g_u\}$. Let $g_i, g_j \in V$. Then, by (*), $Dg_i g_j^{-1} \cap D = \emptyset$. Hence $Dg_i g_j^{-1} = Dg_k$ for some $g_k \in V$. Thus $g_i g_j^{-1} = g_k \in V$ and so V is a subgroup of G of order u . By (*), we have the lemma. $\square$

Corollary 1.8. Let D be an RDS. Then $D^{(-1)}$ is also an RDS if and only if $\text{dev}(D)$ is symmetric.

Definition 1.9. An RDS D is called *symmetric* if $\text{dev}(D)$ is symmetric, otherwise *non-symmetric*.

If the equality in (2) holds, then $k = m = u\lambda$ and so $(m, u, k, \lambda) = (u\lambda, u, u\lambda, \lambda)$.

Definition 1.10. A square (m, u, k, λ) -divisible design $(\mathbb{P}, \mathbb{B})$ is called a *transversal design* and denoted by $\text{TD}_\lambda(k; u)$ if $|B \cap \mathbb{P}_i| = 1$ for $\forall B \in \mathbb{B}$ and $\forall i \in \{1, 2, \dots, m\}$.

Therefore, a square (m, u, k, λ) -divisible design is a transversal design iff $k = m (= u\lambda)$.

$$\iff k = m (= u\lambda).$$

Definition 1.11. If $k = m = u\lambda$, then a (m, u, k, λ) -difference set D in a group G is said to be *semiregular*. Clearly $|G| = u^2\lambda$.

Remark 1.12. Under the above assumption, $DD^{(-1)} \neq D^{(-1)}D$ in general. However, every known transversal design obtained from semiregular RDS is symmetric.

In this talk we give examples of semiregular RDS's D which do not satisfy the condition of Proposition 1.7. Then it gives us examples so that $\text{dev}(D)$'s are non-symmetric, and consequently non-symmetric transversal designs.

2. Known non-normal semiregular RDS's

Let D be a $(u\lambda, u, u\lambda, \lambda)$ -difference set (i.e. semiregular RDS) in G relative to U . Then D is called *normal* and *non-normal* according as $U \triangleleft G$ and $U \not\triangleleft G$, respectively. We note that $\text{dev}(D)$ is symmetric for every normal RDS applying Jungnickel's result.

Example 2.1. ([6], [7]) *The following are all known examples of non-normal semiregular RDS's.*

$$(i) \quad (u, \lambda) = (2, 2), G = \langle x, y \mid x^4 = y^2 = 1, y^{-1}xy = x^{-1} \rangle (\simeq D_8), \\ U = \langle y \rangle : (4, 2, 4, 2)\text{-DS}$$

$$(ii) \quad (u, \lambda) = (4, 4), G = \langle x, y \mid x^4 = y^2, y^4 = 1, y^{-1}xy = x^{-1} \rangle \\ (\simeq Q_{16}), \quad U = \langle y \rangle : (4, 4, 4, 1)\text{-DS}$$

$$(iii) \quad (u, \lambda) = (2, 8), G = \langle x, y \mid x^{16} = y^2 = 1, y^{-1}xy = x^7 \rangle \\ (\simeq SD_{32}), \quad U = \langle y \rangle : (16, 2, 16, 8)\text{-DS}$$

$$(iv) \quad (u, \lambda) = (2, 8), G = \langle x, y \mid x^{16} = y^2 = 1, y^{-1}xy = x^9 \rangle \\ (\simeq M_5(2)), \quad U = \langle y \rangle : (16, 2, 16, 8)\text{-DS}$$

$$(v) \quad \text{Let } A \text{ be a } (4m^2, 2m^2 - m, m^2 - m)\text{-difference set in a group } N. \text{ Assume } t \text{ is an automorphism of } N \text{ of order 2. Then } D = A \cup (N \setminus A^t)t \text{ is a non-normal } (4m^2, 2, 4m^2, 2m^2)\text{-DS in a group } N\langle t \rangle \text{ relative to } \langle t \rangle.$$

3. Semiregular RDS's with $|D| = 12$

From now on we assume that D is a semiregular RDS in a group G relative to a subgroup U with $|D| = 12$. Set $u = |U|$. Then $G = 12u$ and D is a $(12, u, 12, \lambda)$ -DS, where $u\lambda = 12$. Thus $|D| = 12$ and one of the following holds.

- (i) $(u, \lambda) = (2, 6), |G| = 24, |U| = 2$.
- (ii) $(u, \lambda) = (3, 4), |G| = 36, |U| = 3$.
- (iii) $(u, \lambda) = (4, 3), |G| = 48, |U| = 4$.
- (iv) $(u, \lambda) = (6, 2), |G| = 72, |U| = 6$.
- (v) $(u, \lambda) = (12, 1), |G| = 144, |U| = 12$.

Remark 3.1. Let D be a semiregular RDS in a group G relative to U and let s be an automorphism of G . Then D^s is also a semiregular RDS (with the same parameters as D) in G relative to U^s .

CASE $(u, \lambda) = (2, 6), |G| = 24, |U| = 2$

The following lemma holds.

Lemma 3.2. ([7]) *If there exists a $(2n, 2, 2n, n)$ -difference set in G relative to U such that $G = NU$ for a subgroup N of G of index 2, then $n^* = 2$.*

By Lemma 3.2 we have the following.

Lemma 3.3. *If $(u, \lambda) = (2, 6)$, then $[G, G] \geq U$.*

Lemma 3.4. N. Ito ([8]) *If a group G of order $4n (> 4)$ contains a normal $(2n, 2, 2n, n)$ -DS relative to U , then a Sylow 2-subgroup of G is neither cyclic nor dihedral.*

By Remark 3.1, Lemmas 3.3, 3.4, there are five possibilities.

- (1) $G = Q_8 \times \mathbb{Z}_3, U = Z(Q_8) \times 1$.
- (2) $G = Q_{24}, U = Z(Q_{24})$.
- (3) $G = \mathbb{Z}_2 \times A_4$ and there are three possibilities for $U (\simeq \mathbb{Z}_2)$.
- (4) $G = SL(2, 3), U = Z(SL(2, 3))$.
- (5) $G = S_4$ and there are two possibilities for $U (\simeq \mathbb{Z}_2)$.

By a computer search, we have the following.

Lemma 3.5. *Assume $(u, \lambda) = (2, 6)$. Then, a group G of order 24 contains a $(12, 2, 12, 6)$ -DS if and only if $G \simeq Q_8 \times \mathbb{Z}_3, Q_{24}$ or $SL(2, 3)$.*

CASE $(u, \lambda) = (3, 4), \quad |G| = 36, \quad |U| = 3.$

In this case $(m, u, k, \lambda) = (12, 3, 12, 4).$

Lemma 3.6. *Let G be a nonabelian group of order 36. Then there are eleven possibilities.*

- (i) $G \simeq D_{36}$, (ii) $G \simeq Q_{36}$, (iii) $G \simeq D_{18} \times \mathbb{Z}_2$,
- (iv) $G \simeq S_3 \times S_3$, (v) $G \simeq S_3 \times \mathbb{Z}_6$,
- (vi) $G \simeq (\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_3)) \times \mathbb{Z}_2, \quad |Z(G)| = 2,$
- (vii) $G \simeq (\mathbb{Z}_4 \times \mathbb{Z}_3) \times \mathbb{Z}_3, \quad |Z(G)| = 6,$
- (viii) $G \simeq \mathbb{Z}_4 \times (\mathbb{Z}_3 \times \mathbb{Z}_3)$, where an element of order 4 inverts $O_3(G)$,
- (ix) $G = \langle d \rangle \langle a, b \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_3 \times \mathbb{Z}_3), \quad a^3 = b^3 = d^4 = [a, b] = 1, a^d = b^{-1}, b^d = a,$
- (x) $G \simeq A_4 \times \mathbb{Z}_3$, (xi) $G \simeq \mathbb{Z}_9 \times (\mathbb{Z}_2 \times \mathbb{Z}_2).$

By a computer search we have the following.

Lemma 3.7. *Assume $(u, \lambda) = (3, 4)$. Then, a nonabelian group G of order 36 contains a $(12, 3, 12, 4)$ -DS if and only if*

$$G \simeq (\mathbb{Z}_4 \times \mathbb{Z}_3) \times \mathbb{Z}_3 \quad (|Z(G)| = 6, \quad U = O_3(Z(G))),$$

$$A_4 \times \mathbb{Z}_3 \quad (U = 1 \times \mathbb{Z}_3), \quad \text{or} \quad S_3 \times \mathbb{Z}_6.$$

The first and the second cases have been previously known. But the third is a new one and has unusual properties.

Example 3.8. Let $G = \langle a, b, c \mid a^3 = b^2 = c^6 = 1,$
 $b^{-1}ab = a^{-1}, ac = ca, bc = cb \rangle$
 and set $D = \{1, c, c^2, c^3, a, ac, b, a^2bc^5, abc^4, a^2bc, bc^4, abc\}.$

Then D is a non-symmetric $(12, 3, 12, 4)$ -DS relative to $U = \langle ac^2 \rangle \simeq \mathbb{Z}_3.$

Let $(\mathbb{P}, \mathbb{B}) (= \text{dev}(D))$ be the corresponding transversal design and let A be an incidence matrix of $(\mathbb{P}, \mathbb{B})$. Then

$$AA^T = \begin{bmatrix} 12I & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J \\ 4J & 12I & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J \\ 4J & 4J & 12I & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J \\ 4J & 4J & 4J & 12I & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J \\ 4J & 4J & 4J & 4J & 12I & 4J & 4J & 4J & 4J & 4J & 4J & 4J \\ 4J & 4J & 4J & 4J & 4J & 12I & 4J & 4J & 4J & 4J & 4J & 4J \\ 4J & 4J & 4J & 4J & 4J & 4J & 12I & 4J & 4J & 4J & 4J & 4J \\ 4J & 4J & 4J & 4J & 4J & 4J & 4J & 12I & 4J & 4J & 4J & 4J \\ 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 12I & 4J & 4J & 4J \\ 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 12I & 4J & 4J \\ 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 12I & 4J \\ 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 12I \end{bmatrix}$$

$$\text{where, } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

However,

$$A^T A = \begin{bmatrix} 12 & 4 & 4 & 4 & \dots & \dots & \dots & 5 & 4 & 5 & 4 & 2 \\ 4 & 12 & 4 & 4 & \dots & \dots & \dots & 4 & 5 & 4 & 5 & 4 \\ 4 & 4 & 12 & 4 & \dots & \dots & \dots & 2 & 4 & 5 & 4 & 5 \\ 4 & 4 & 4 & 12 & \dots & \dots & \dots & 4 & 2 & 4 & 5 & 4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 4 & 5 & 4 & 2 & \dots & \dots & \dots & 4 & 12 & 4 & 4 & 4 \\ 5 & 4 & 5 & 4 & \dots & \dots & \dots & 4 & 4 & 12 & 4 & 4 \\ 4 & 5 & 4 & 5 & \dots & \dots & \dots & 4 & 4 & 4 & 12 & 4 \\ 2 & 4 & 5 & 4 & \dots & \dots & \dots & 4 & 4 & 4 & 4 & 12 \end{bmatrix}$$

An RDS D is called *symmetric* if $D^{(-1)}$ is also an RDS. Since AA^T has entries $2, 5 (\notin \{0, 4, 12\})$, the dual of $\text{dev}(D)$ is not a transversal design. Applying Proposition 1.7, $D^{(-1)}$ is a non-symmetric RDS. As far as I know, this is the only known non-symmetric RDS.

CASE $(u, \lambda) = (4, 3)$, $|G| = 48$, $|U| = 4$

In this case $(m, u, k, \lambda) = (12, 4, 12, 3)$.

We use the following two lemmas to settle the present case.

Lemma 3.9. *If $2 \mid u$ and $2 \nmid \lambda$, then U contains every involution of G .*

Lemma 3.10. (Elliott-Butson) *Let D be a $(u\lambda, u, u\lambda, \lambda)$ -DS in G relative to U . If U contains a normal subgroup N of G of order v , then DN/N is a $(u\lambda, u/v, u\lambda, v\lambda)$ -DS in G/N relative to U/N .*

Applying Lemmas 3.5 and 3.10, it suffices to check the following case.

$$G = \langle a, b, c \mid a^4 = b^4 = c^3 = 1, c^{-1}ac = ab, c^{-1}bc = ab^2 \rangle, \quad U = \langle a^2, b^2 \rangle$$

We have the following by a computer search.

Lemma 3.11. *There is no $(12, 4, 12, 3)$ -DS.*

CASE $(u, \lambda) = (6, 2)$, $|G| = 72$, $|U| = 6$.

In this case $(m, u, k, \lambda) = (12, 6, 12, 2)$.

Observation. For every known semiregular RDS, $|U|$ is a power of a prime.

The smallest undecided case is $u = 6$.

Lemma 3.12. *Let G be a nonabelian group of order 72. Then there are five possibilities.*

- (i) $G \simeq SL(2, 3) \times \mathbb{Z}_3$, (ii) $G \simeq A_4 \times \mathbb{Z}_6$, (iii) $G \simeq A_4 \times S_3$,
- (iv) $G \simeq \langle t \rangle (M \times T)$, $t^2 = 1$, $M \simeq A_4$, $T \simeq \mathbb{Z}_3$,
 $\langle t \rangle M \simeq S_4$, $\langle t \rangle T \simeq S_3$,
- (v) $G \triangleright Q$, $|Q| = 9$.

Applying Lemmas 3.5, 3.7 and 3.10, we have the following by a computer search.

Lemma 3.13. *There is no $(12, 6, 12, 2)$ -DS.*

CASE $(u, \lambda) = (12, 1)$, $|G| = 144$, $|U| = 12$.

In this case $(m, u, k, \lambda) = (12, 12, 12, 1)$.

By Lemma 1 of [2], the following holds.

Theorem 3.14. *Every transversal design with $\lambda = 1$ is symmetric.*

The above theorem implies that if there is a $(u, u, u, 1)$ -DS in a group G , then the corresponding transversal design can be extended to a projective plane of order u which admits G as a collineation group of order u^2 . Thus $u \neq 12$ by Baumert-Hall [4] and the following holds.

Lemma 3.15. *There is no $(12, 12, 12, 1)$ -DS.*

By Lemmas 3.5, 3.7, 3.11, 3.13 and 3.15, we have the following.

Theorem 3.16. *A group G contains a $(u\lambda, u, u\lambda, \lambda)$ -DS D with $|D| = 12$ if and only if G is isomorphic to one of the following.*

- (i) $(u, \lambda) = (2, 6)$, $G = \mathbb{Q}_8 \times \mathbb{Z}_3$, $U = Z(Q_8) \times 1 \simeq \mathbb{Z}_2$.
- (ii) $(u, \lambda) = (2, 6)$, $G = \mathbb{Q}_{24}$, $U = Z(Q_{24}) \simeq \mathbb{Z}_2$.
- (iii) $(u, \lambda) = (2, 6)$, $G = SL(2, 3)$, $U = Z(SL(2, 3)) \simeq \mathbb{Z}_2$.
- (iv) $(u, \lambda) = (3, 4)$, $G = S_3 \times \mathbb{Z}_6$, $U \simeq \mathbb{Z}_3$ (a non-symmetric RDS).
- (v) $(u, \lambda) = (3, 4)$, $G = (\mathbb{Z}_4 \rtimes \mathbb{Z}_3) \times \mathbb{Z}_3$, $|Z(G)| = 6$, $U = O_3(Z(G))$.
- (vi) $(u, \lambda) = (3, 4)$, $G = A_4 \times \mathbb{Z}_3$, $U = 1 \times \mathbb{Z}_3$.
- (vii) $(u, \lambda) = (3, 4)$, $G = \mathbb{Z}_6 \times \mathbb{Z}_6$, $U \simeq \mathbb{Z}_3$.

4. Construction of non-symmetric RDS's

In this section we show the following.

Theorem 4.1. *There exists a non-symmetric $(2^{2m}3^m, 3, 2^{2m}3^m, 2^{2m}3^{m-1})$ -
 $(m-1)$ times
 difference set in $(S_3 \times \mathbb{Z}_6) \times \overbrace{(\mathbb{Z}_6 \times \mathbb{Z}_2) \times \cdots \times (\mathbb{Z}_6 \times \mathbb{Z}_2)}^{(m-1)\text{times}}$ relative to $U \times 1 \times \cdots \times 1$,
 where D is a non-symmetric $(12, 3, 12, 4)$ -difference set in $S_3 \times \mathbb{Z}_6$ relative to its
 non-normal subgroup U of order 3 (see Example 3.8).*

Corollary 4.2. *There exists a non-symmetric $TD_{2^{2m}3^{m-1}}[2^{2m}3^m; 3]$ for every
 $m \in \mathbb{N}$.*

Example 4.3.

In order to prove Theorem 4.1, we need the following lemma.

Lemma 4.4. *Let $L = G \times H$, where G be a group of order $u^2\lambda$ and H is a
 group of order $u\mu$. Let D be a $(u\lambda, u, u\lambda, \lambda)$ DS in G relative to a subgroup U of
 G of order u and let C be a $(u\mu, u, u\mu, \mu)$ DS in $U \times H$ relative to U . Then*

(i) CD is a $(u^2\lambda\mu, u, u^2\lambda\mu, u\lambda\mu)$ DS in L relative to U .

(ii) CD is symmetric if and only if D is symmetric.

Proof. Let $c_1, c_2 \in C$ and $d_1, d_2 \in D$ and assume $c_1d_1 = c_2d_2$. Then $c_1^{-1}c_2 = d_1d_2^{-1} \in UH \cap G = U$. Thus $d_1 = d_2$ and so $c_1 = c_2$. Therefore CD is a subset of L .

By assumption, the following hold.

$$DD^{(-1)} = u\lambda + \lambda(G - U) \quad (5)$$

$$CC^{(-1)} = u\mu + \mu(UH - U) \quad (6)$$

$$G = UD, \quad UC = UH \quad (7)$$

Hence $(CD)(CD)^{(-1)} = C(DD^{(-1)})C^{(-1)} = C(u\lambda + \lambda(G - U))C^{(-1)}$
 $= u\lambda CC^{(-1)} + \lambda CGC^{(-1)} - \lambda CUC^{(-1)}$ As $C, U \subset UH \triangleright U$, $CU = UC$. Similarly
 $GC = CG$. It follows that $(CD)(CD)^{(-1)} = u\lambda(u\mu + \mu(UH - U)) + \lambda G C C^{(-1)} - \lambda U C C^{(-1)}$
 $= u^2\mu\lambda + u\mu\lambda UH - u\mu\lambda U + \lambda G(u\mu + \mu UH - \mu U) - \lambda U(u\mu + \mu UH - \mu U)$
 $= u^2\mu\lambda + u\mu\lambda(L - U)$. Thus we have (i).

Since $UH \triangleright U$, $C^{(-1)}C = CC^{(-1)}$. Hence $(CD)^{(-1)}CD = D^{(-1)}(CC^{(-1)})D = D^{(-1)}(u\mu + \mu UH - \mu U)D$. By (7), the following holds.

$$(CD)^{(-1)}CD = u\mu D^{(-1)}D + u\mu\lambda L - u\mu\lambda G \quad (8)$$

Assume CD is symmetric. Then $(CD)^{(-1)}CD = u^2\mu\lambda + u\mu\lambda(L - V)$ for a subgroup V of L of order u . By (8), $u\mu D^{(-1)}D - u\mu\lambda G = u^2\mu\lambda - u\mu\lambda V$. Thus $D^{(-1)}D = u\lambda + \lambda(G - V)$. In particular, V is a subgroup of G of order u and so D is symmetric. Conversely, assume D is symmetric. Then $D^{(-1)}D = u\lambda + \lambda(G - V)$ for a subgroup V of G of order u . Then, by (8), $(CD)^{(-1)}CD = u\mu(u\lambda + \lambda(G - V)) + u\mu\lambda L - u\mu\lambda G = u^2\mu\lambda + u\mu\lambda(L - V)$. Therefore CD is symmetric. Thus we have (ii). $\square$

We note that Lemma 4.4(i) is a modification of Result 2.4 of [11], where N is assumed to be normal in G .

Proof of Theorem

Let D be a non-symmetric $(12, 3, 12, 4)$ DS in $M = S_3 \times \mathbb{Z}_6$ relative to a non-normal subgroup U of M (see Example 3.8). Let $H = \langle a \rangle \times \langle b \rangle (\simeq \mathbb{Z}_6 \times \mathbb{Z}_3)$. We note that an abelian group $H \times \langle c \rangle (\simeq \mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ contains $(12, 3, 12, 4)$ DS, say $\{1, a, a^2, a^3, a^4c, a^5c, bc^2, ab, a^2bc, a^3bc^2, a^4bc, a^5b\}$. Set $G = H \times M$ and choose $\langle c \rangle$ as a non-normal subgroup of M . Then, applying Lemma 4.4, $H \times M$ contains a non-symmetric $(2^2 3^2 \cdot 4, 3, 2^2 3^2 \cdot 2^2 3 \cdot 4)$ DS in G relative to $1 \times U (\simeq \mathbb{Z}_3)$ as D is non-symmetric. Repeating this procedure again and again we have the theorem.

References

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