

NORM INEQUALITIES EQUIVALENT TO LÖWNER-HEINZ THEOREM

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ABSTRACT. We give several norm inequalities equivalent to the famous Löwner-Heinz inequality.

THEOREM. If A and B are positive bounded linear operators on a Hilbert space, then the following properties hold and follow from each other.

(1) $A \geq B \geq 0$ ensures $A^s \geq B^s$ for any $1 \geq s \geq 0$.

(2) $\|AB\|^q \leq \|A^q B^q\|$ for any $q \geq 1$, namely $\|A^q B^q\|^{1/q} \leq \|A^p B^p\|^{1/p}$ for any $p \geq q > 0$,

that is, $f(p) = \|A^p B^p\|^{1/p}$ is an increasing function on p .

(3) $\|A^s B^s\| \leq \|AB\|^s$ for any $1 \geq s \geq 0$, namely $\|A^{1/s} B^{1/s}\|^s \leq \|A^{1/t} B^{1/t}\|^t$ for any $s \geq t > 0$,

that is, $g(s) = \|A^{1/s} B^{1/s}\|^s$ is a decreasing function on s .

(4) $\|AB\|^{(p+q)/2} \leq \|A^p B^p\|^{1/2} \|A^q B^q\|^{1/2}$ for any $p \geq 0, q \geq 0$ with $(p+q)/2 \geq 1$.

(5) $\|A^{st} B^{st}\|^2 \leq \|A^s B^s\|^{2st/(s+t)} \|A^t B^t\|^{2st/(s+t)}$ for any $s > 0, t > 0$ with $2st/(s+t) \leq 1$.

(6) $\|AB\|^{(p+q)/2} \leq \|A^p B^q\|^{1/2} \|A^q B^p\|^{1/2}$ for any $p \geq 0, q \geq 0$ with $(p+q)/2 \geq 1$.

(7) $\|A^{st} B^{st}\|^2 \leq \|A^s B^t\|^{2st/(s+t)} \|A^t B^s\|^{2st/(s+t)}$ for any $s > 0, t > 0$ with $2st/(s+t) \leq 1$.

We remark that (1) has been shown in [cf., [3][4][5][6] etc.] and (3) is shown in [2]. Here we state the following lemma.

Lemma. If A and B are positive bounded linear operator on a Hilbert space, then

$$\|A^{(s+t)/2} B^{(s+t)/2}\|^2 \leq \|B^t A^{s+t} B^s\| \text{ for any } s \geq 0 \text{ and } t \geq 0.$$

Proof of Lemma.

$$\begin{aligned}
 \|A^{(s+t)/2} B^{(s+t)/2}\|^2 &= \|B^{(s+t)/2} A^{s+t} B^{(s+t)/2}\| \\
 &= r(B^{(s+t)/2} A^{s+t} B^{(s+t)/2}) \quad (r(A) \text{ means the spectral radius of } A) \\
 &= r(B^t A^{s+t} B^s) \quad \text{since } r(AB) = r(BA) \text{ for any } A \text{ and } B \\
 &\leq \|B^t A^{s+t} B^s\|.
 \end{aligned}$$

Proof of Theorem.

Proof of (3). Here we give an alternative proof to (3). Put $D = \{s \in [0,1] ; \|A^s B^s\| \leq \|AB\|^s\}$. Then D is a closed set such that $0, 1 \in D$, so we have only to show that if $s, t \in D$, then $(s+t)/2 \in D$.

$$\begin{aligned}
 \|A^{(s+t)/2} B^{(s+t)/2}\|^2 &\leq \|B^t A^{s+t} B^s\| \quad \text{by Lemma} \\
 &\leq \|B^t A^t\| \|A^s B^s\| \\
 &\leq \|AB\|^t \|AB\|^s = \|AB\|^{s+t} \quad \text{since } s, t \in D,
 \end{aligned}$$

so that $\|A^{(s+t)/2} B^{(s+t)/2}\|^2 \leq \|AB\|^{s+t}$, that is, $(s+t)/2 \in D$, whence we have (3).

(2) $\longleftrightarrow$ (3). Its proof is obvious.

(3) $\longleftrightarrow$ (1). We may assume that A and B are invertible.

Assume (3). The condition (3) is equivalent to the following (8) by the homogeneity of norm

$$(8) \quad \|AB\| \leq 1 \text{ ensures } \|A^s B^s\| \leq 1 \text{ for any } 1 \geq s \geq 0.$$

By replacing A by $A^{-1/2}$ and also B by $B^{1/2}$ in (8), this condition (8) means that $A^{-1/2} B A^{-1/2} \leq 1$ ensures $A^{-s/2} B^s A^{-s/2} \leq 1$, that is, $A \geq B \geq 0$ ensures $A^s \geq B^s$ for any $1 \geq s \geq 0$, so we have (3) $\longrightarrow$ (1). Conversely assume (1).

(1) is equivalent to the following (9)

$$(9) \quad \|A^{-1/2} B^{1/2}\| \leq 1 \text{ ensures } \|A^{-s/2} B^{s/2}\| \leq 1 \text{ for any } 1 \geq s \geq 0.$$

By replacing A by A^{-2} and also B by B^2 in (9), so we have (8) and this condition (8) is equivalent to (3), so we have (1) $\longrightarrow$ (3).

Whence we have (1) $\longleftrightarrow$ (3).

(2) $\longrightarrow$ (4) and (6). Assume (2). Then

$$\begin{aligned} \|AB\|^{p+q} &\leq \|A^{(p+q)/2} B^{(p+q)/2}\|^2 \quad \text{by (2) since } (p+q)/2 \geq 1 \\ &\leq \|B^p A^{p+q} B^q\| \quad \text{by Lemma} \\ &\leq \|B^p A^p\| \|A^q B^q\| \quad \text{or } \|B^p A^q\| \|A^p B^q\| \end{aligned}$$

whence we have (4) and (6).

(4) or (6) $\longrightarrow$ (2). Put $p = q$ in (4) or (6), then we have (2).

(4) $\longleftrightarrow$ (5) and (6) $\longleftrightarrow$ (7). Put $s = 1/p$ and $t = 1/q$ in (4) and (6) and also replace A by A^{st} and B by B^{st} , then we have (5) and (7) and the reverse implications are obvious.

Whence the proof of Theorem is complete.

REMARK. Related to (2) it is easily verified that $\|AB\|^q \leq \|A^q B^q\|$ does not always hold for $1 > q > 0$. Related to this result we would like to remark the following result. Put $h(p) = \|A^p B^p\| / \|AB\|^p$ for any $p \geq 0$. (4) asserts that $h(p)h(q) \geq 1$ for any $p \geq 0$, $q \geq 0$ with $p+q \geq 2$. Put $A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $h(1/2) = \sqrt{5}/34^{1/4} < 1$ and $h(3/2) = \sqrt{233}/34^{3/4} > 1$ but $h(1/2)h(3/2) = \sqrt{1165}/34 > 1$.

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Addendum. Recently we have the following results in [8] as an application of [7].

THEOREM 1. If A and B are arbitrary bounded linear operators on a Hilbert space, then the following properties hold and follow from each other.

- (1) $A \geq B \geq 0$ ensures $A^s \geq B^s$ for any $1 \geq s \geq 0$.
- (2) $\|AB\|^q \leq \| |A|^q |B^*|^q \|$ for any $q \geq 1$, namely $\| |A|^q |B^*|^q \|^{1/q} \leq \| |A|^p |B^*|^p \|^{1/p}$ for any $p \geq q > 0$, that is, $f(p) = \| |A|^p |B^*|^p \|^{1/p}$ is an increasing function on p .
- (3) $\| |A|^s |B^*|^s \| \leq \|AB\|^s$ for any $1 \geq s \geq 0$, namely $\| |A|^{1/s} |B^*|^{1/s} \| \leq \| |A|^{1/t} |B^*|^{1/t} \|$ for any $s \geq t > 0$, that is, $g(s) = \| |A|^{1/s} |B^*|^{1/s} \|$ is a decreasing function on s .
- (4) $\|AB\|^{(p+q)/2} \leq \| |A|^p |B^*|^p \|^{1/2} \| |A|^q |B^*|^q \|^{1/2}$ for any $p \geq 0, q \geq 0$ with $(p+q)/2 \geq 1$.
- (5) $\|AB\|^{(p+q)/2} \leq \| |A|^p |B^*|^q \|^{1/2} \| |A|^q |B^*|^p \|^{1/2}$ for any $p \geq 0, q \geq 0$ with $(p+q)/2 \geq 1$.

Definition 1. An operator T is said to be perinormal if

$$(T^*T)^n \leq T^{*n}T^n$$

holds for every natural number n . Our new class of perinormal operators occupies the place shown in the following schema and the inclusions are all proper.

$$\begin{array}{c} \text{Normal} \subsetneq \text{Quasinormal} \subsetneq \text{Heminormal} \\ \subsetneq \text{Perinormal} \subsetneq \text{Normaloid} \end{array}$$

Theorem 2. If A and B^* are perinormal, then (*) holds;

- (*) { the following (1) and (2) hold and follow from each other.
- (1) $\|AB\|^n \leq \|A^n B^n\|$ for every natural number n .
 - (2) $\|AB\|^{n+m} \leq \|A^n B^m\| \|A^m B^n\|$ for every natural number n and m .

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