

# Global real analytic length parameters and angle parameters for Teichmüller spaces and the geometry of hyperbolic transformations

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## Abstract

We consider global real analytic parameters for Teichmüller spaces. First we show a parametrization of  $T(2, 0, 0)$  by seven length parameters and describe this parameter space. Next we show a similar result about angle parameters. These parametrizations are obtained from the geometry of Möbius transformations. We define the one-half power of a Möbius transformation and consider the geometry of hyperbolic transformations related to these parametrizations.

## 1 Introduction

A Fuchsian group  $G$  acting on the unit disk  $\mathbf{D}$  is of *type*  $(g, n, m; \nu_1, \nu_2, \dots, \nu_n)$ , if the quotient space  $\mathbf{D}/G$  is a Riemann surface of genus  $g$  with  $n$  branch points and punctures of orders  $\nu_1, \nu_2, \dots, \nu_n$  and  $m$  holes. This Riemann surface is also called of *type*  $(g, n, m; \nu_1, \nu_2, \dots, \nu_n)$ . Teichmüller space  $T(g, n, m; \nu_1, \nu_2, \dots, \nu_n)$  is the set of equivalence classes of marked Fuchsian groups of type  $(g, n, m; \nu_1, \nu_2, \dots, \nu_n)$  and a global real analytic manifold of dimension  $6g + 2n + 3m - 6$ . We abbreviate  $(g, n, m; \nu_1, \nu_2, \dots, \nu_n)$  and  $T(g, n, m; \nu_1, \nu_2, \dots, \nu_n)$  to  $(g, n, m)$  and  $T(g, n, m)$ , respectively. There are various methods parametrizing  $T(g, n, m)$ . We will parametrize  $T(g, n, m)$  by some lengths of closed geodesics and intersection angles between geodesics on a Riemann surface represented by a marked Fuchsian group. Such lengths and angles are called *length parameters* and *angle parameters*, respectively.

A hyperbolic Riemann surface  $R$  of type  $(g, n, m)$  is obtained by pasting sides of some geodesic polygon  $P$  in  $\mathbf{D}$  which may have vertices on the circle at infinity and the boundary of  $P$  can ever contain arcs of the circle at infinity. The uniformization theorem implies that  $P$  is a fundamental domain of a Fuchsian group representing  $R$ . Since a side of  $P$  corresponds to a geodesic on  $R$  and  $P$  is determined by the lengths of the sides and the interior angles of  $P$ ,  $R$  is parametrized real analytically by some lengths of geodesics on  $R$  and angle parameters of  $R$ . Constructing a special polygon, we can take such lengths from length parameters, that is, lengths of closed geodesics on  $R$ . Moreover, we can parametrize  $R$  by  $3g + n + 2m - 3$  length

parameters and  $3g + n + m - 3$  angle parameters of  $R$ . By this plan, the following classical result is obtained.

**Theorem 1.1** [5] *Teichmüller space  $T(g, n, m)$  has global real analytic parameters consisting of  $3g + n + 2m - 3$  length parameters and  $3g + n + m - 3$  angle parameters. These parameters correspond to some lengths of sides and interior angles of a geodesic polygon in  $\mathbf{D}$  which is a fundamental domain of a marked Fuchsian group of type  $(g, n, m)$ . The total number of these parameters is  $\dim(T(g, n, m))$  and this parameter space is  $\mathbf{R}_+^{3g+n+2m-3} \times (0, \pi)^{3g+n+m-3}$ .*

In Section 2, we define the one-half power of a Möbius transformation, since this is useful for the geometry of Möbius transformations.

We consider a parametrization of  $T(g, n, m)$  by only length parameters. It is well known that length parameters parametrize  $T(g, n, m)$  global real analytically (see for example, [3], [7], [9], [11] and [15]). Wolpert [20] and [21] announced that in the case of  $T(g, 0, 0)$ , the minimal number of these parameters is greater than  $\dim(T(g, 0, 0)) = 6g - 6$ . Recently, Schmutz [13] stated that this minimal number is  $6g - 5$ . In the same time, the author also obtained this result and this parameter space independently. In Section 3, we show the result in the case of  $T(2, 0, 0)$ .

In the hyperbolic plane, a triangle is determined by three lengths of sides or three interior angles. Thus lengths and angles have same significance in the hyperbolic geometry. In Section 4, we show a parametrization of  $T(2, 0, 0)$  by only angle parameters which is useful for some considerations.

These two parametrizations are obtained from the geometry of Möbius transformations. In Section 5, we show the geometry of hyperbolic transformations related to these parametrizations.

## 2 Preliminaries

The group of Möbius transformations preserving  $\mathbf{D}$ ,  $M(\mathbf{D})$ , is the group of isometries of  $\mathbf{D}$  with respect to the Poincaré metric  $d$ . For distinct two points  $p_1$  and  $p_2$  in  $\overline{\mathbf{D}}$ , let  $L(p_1, p_2)$  be the full geodesic through  $p_1$  and  $p_2$  with the direction from  $p_1$  to  $p_2$ , where this direction is sometimes ignored. An elliptic element  $A \in M(\mathbf{D})$  has the sole fixed point in  $\mathbf{D}$ . We denote it by  $fp(A)$ . A hyperbolic element  $A \in M(\mathbf{D})$  has the *attracting fixed point*,  $q(A)$ , and the *repelling fixed point*,  $p(A)$ , which are characterized by  $q(A) = \lim_{n \rightarrow \infty} A^n(z)$  and  $p(A) = \lim_{n \rightarrow \infty} A^{-n}(z)$  for any  $z \in \mathbf{D}$ . The *axis* of  $A$ ,  $ax(A) = L(p(A), q(A))$ , and the *translation length* of  $A$ ,  $tl(A) = \inf\{d(z, A(z)) | z \in \mathbf{D}\}$ , are characterized by

$$ax(A) = \{z \in \mathbf{D} | d(z, A(z)) = tl(A)\},$$

$$\cosh \frac{tl(A)}{2} = \frac{|tr A|}{2}.$$

Let  $A$  be a hyperbolic element of a Fuchsian group  $G$  acting on  $\mathbf{D}$ . Then  $ax(A)$  projects on a closed geodesic on  $\mathbf{D}/G$  whose length is  $tl(A)$  and corresponds to  $|tr A|$  real analytically. To define a marked Fuchsian group, we give the following:

**Proposition 2.1** [5] *Let  $G$  be a Fuchsian group of type  $(g, 0, m)$ . Then  $G$  has a system of generators*

$$\Sigma = (A_1, B_1, \dots, A_g, B_g, E_1, \dots, E_m);$$

$$E_m E_{m-1} \cdots E_1 C_g C_{g-1} \cdots C_1 = \text{identity},$$

where  $A_j, B_j, C_j = [B_j, A_j] = B_j^{-1} A_j^{-1} B_j A_j$  ( $j = 1, \dots, g$ ) and  $E_k$  ( $k = 1, \dots, m$ ) are hyperbolic with axes illustrated as in Figure 2.1, and if  $g = 0$  (resp.  $m = 0$ ), then  $A_j, B_j$  and  $C_j$  (resp.  $E_k$ ) are omitted.

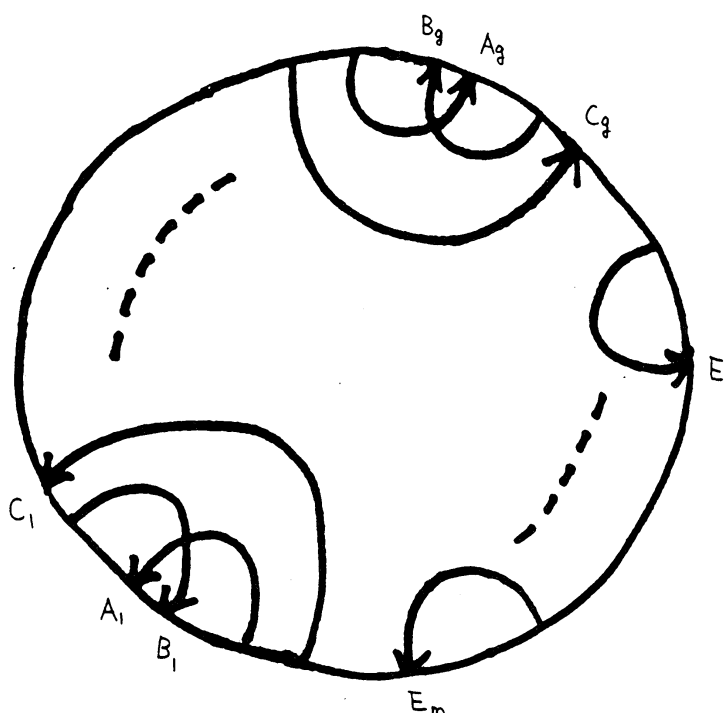


Figure 2.1.

A system  $\Sigma$  mentioned in Proposition 2.1 is a *canonical system of generators* of  $G$ . A pair of  $G$  and this system  $\Sigma$ ,  $(G, \Sigma)$ , is a *marked Fuchsian group*. Two marked Fuchsian groups  $(G_1, \Sigma_1)$  and  $(G_2, \Sigma_2)$  are *equivalent* if  $G_2 = hG_1h^{-1}$  and  $\Sigma_2 = h\Sigma_1h^{-1}$  for some  $h \in M(\mathbb{D})$ . Teichmüller space  $T(g, 0, m)$  is the set of equivalence classes of  $(G, \Sigma)$  of type  $(g, 0, m)$ . Similarly,  $T(g, n, m)$ ,  $n \neq 0$  are defined.

One of the matrix representations of a Möbius transformation  $A$  is denoted by  $\tilde{A}$ . For two Möbius transformations  $A$  and  $B$ ,  $\text{tr}[\tilde{B}, \tilde{A}]$  is invariant under the

choice of matrix representations. The following equations of commutator traces of  $X, Y, Z = (YX)^{-1} \in SL(2, \mathbb{C})$  are useful: for  $\epsilon, \eta \in \{\pm 1\}$ ,

$$\begin{aligned} \text{tr}[X, Y] &= \text{tr}[X^\epsilon, Y^\eta] = \text{tr}[Y^\epsilon, X^\eta] \\ &= \text{tr}[Y^\epsilon, Z^\eta] = \text{tr}[Z^\epsilon, Y^\eta] \\ &= \text{tr}[Z^\epsilon, X^\eta] = \text{tr}[X^\epsilon, Z^\eta], \end{aligned}$$

Finally, we define the one-half power of  $A \in M(\mathbb{D})$ .

**Definition and Proposition 2.2** *Let  $A \in M(\mathbb{D})$  be hyperbolic or parabolic. Then there is a unique  $X \in M(\mathbb{D})$  satisfying  $X^2 = A$ .  $X$  is called the one-half power of  $A$  and denoted by  $A^{1/2}$ . If  $\tilde{A}$  is the matrix representation of  $A$  with negative trace (resp. positive trace), then the matrix representations of  $A^{1/2}$  are*

$$\frac{\pm 1}{\sqrt{|\text{tr} \tilde{A}| + 2}}(\tilde{A} - I) \quad (\text{resp. } \frac{\pm 1}{\sqrt{|\text{tr} \tilde{A}| + 2}}(\tilde{A} + I)).$$

Thus

$$|\text{tr} A^{1/2}| = \sqrt{|\text{tr} \tilde{A}| + 2}.$$

Since  $(A^{1/2})^{-1} = (A^{-1})^{1/2}$ , these are denoted by  $A^{-1/2}$ .

### 3 Global real analytic length parameters

**Theorem 3.1**  *$T(2, 0, 0)$  is parametrized global real analytically by seven length parameters which correspond to the absolute values of traces of the following hyperbolic elements of a marked Fuchsian group:*

$$A_1, B_1, B_1 A_1,$$

$$A_2, B_2, B_2 A_2 A_1, B_2 A_2 B_1^{-1}.$$

Thus these length parameters are lengths of simple closed geodesics on a Riemann surface represented by a marked Fuchsian group. This parameter space is defined by

$$x_j, y_j, z_1, u, v > 2; \quad j = 1, 2,$$

$$x_1^2 + y_1^2 + z_1^2 - x_1 y_1 z_1 = x_2^2 + y_2^2 + |\text{tr} B_2 A_2|^2 - x_2 y_2 |\text{tr} B_2 A_2| < 0,$$

$$\begin{aligned} |\text{tr} B_2 A_2| &= \frac{1}{z_1^2 - 4} (z_1 \sqrt{x_1 y_1 z_1 - (x_1^2 + y_1^2 + z_1^2)} + 4 \sqrt{u v z_1 - (u^2 + v^2 + z_1^2)} + 4 \\ &\quad + 2(x_1 u + y_1 v) - z_1(y_1 u + x_1 v)) > 2, \end{aligned}$$

where  $x_j = |\text{tr} A_j|$ ,  $y_j = |\text{tr} B_j|$  ( $j = 1, 2$ ),  $z_1 = |\text{tr} B_1 A_1|$ ,  $u = |\text{tr} B_2 A_2 A_1|$  and  $v = |\text{tr} B_2 A_2 B_1^{-1}|$ .

**Remark 3.2** [12] In the case of  $T(g, n, m)$ ,  $m \neq 0$ , the minimal number of global real analytic length parameters is  $\dim(T(g, n, m))$ .

## 4 Global real analytic angle parameters

Let  $\Sigma_{(2,0,0)} = (A_1, B_1, A_2, B_2)$  be a canonical system of generators of type  $(2, 0, 0)$ . Let  $p_j$  be the intersection of  $ax(A_j)$  and  $ax(B_j)$  for  $j = 1, 2$ . By Poincaré's polygon theorem, the geodesic decagon  $P$  with vertices

$$A_1^{-1}B_1^{-1}(p_1), A_1^{-1}(p_1), p_1, B_1^{-1}(p_1), B_1^{-1}A_1^{-1}(p_1),$$

$$A_2^{-1}B_2^{-1}(p_2), A_2^{-1}(p_2), p_2, B_2^{-1}(p_2), B_2^{-1}A_2^{-1}(p_2),$$

is a fundamental domain of the Fuchsian group generated by  $\Sigma_{(2,0,0)}$  (see Figure 4.1).

The axes of  $A_j$ ,  $B_j$  and  $B_jA_j$  determine a triangle  $T_j$  with vertices  $p_j$ ,  $A_j^{-1/2}(p_j)$  and  $B_j^{1/2}(p_j)$  (see Lemma 5.2). Let  $\theta(A_j)$ ,  $\theta(B_j)$  and  $\theta(B_jA_j)$  be three interior angles of  $T_j$ . We can show that  $ax(C_1)$  and the segment  $[B_1^{-1}A_1^{-1}(p_1), A_2^{-1}B_2^{-1}(p_2)]$  intersect. Let  $\mu$  be the intersection angle between them.

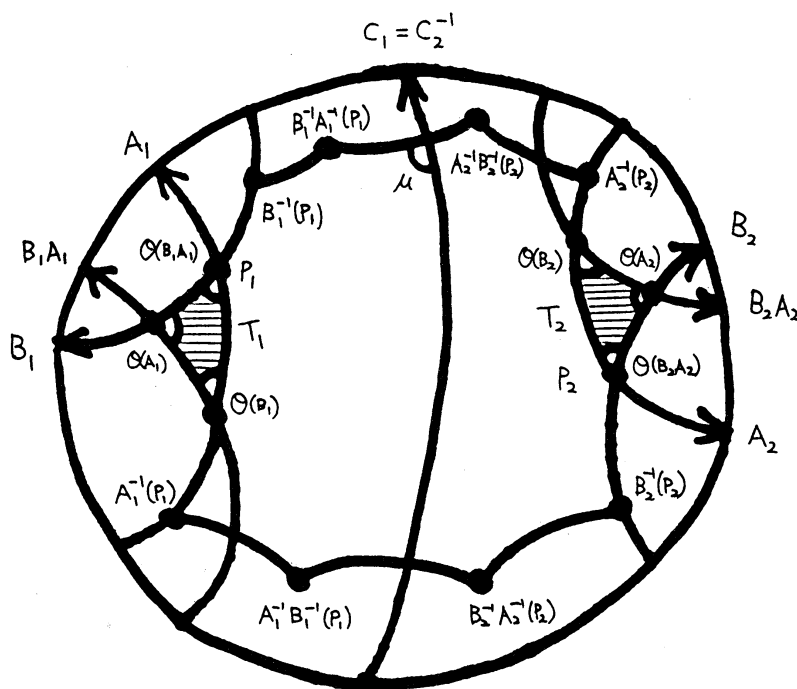


Figure 4.1.

**Lemma 4.1** *Seven angles  $\theta(A_j)$ ,  $\theta(B_j)$ ,  $\theta(B_jA_j)$  ( $j = 1, 2$ ) and  $\mu$  determine  $\Sigma_{(2,0,0)}$  global real analytically, up to conjugation by a Möbius transformation. This parameter space is defined by*

$$(4.1) \quad \theta(A_j), \theta(B_j), \theta(B_jA_j), \mu \in (0, \pi),$$

$$(4.2) \quad \theta(A_j) + \theta(B_j) + \theta(B_j A_j) < \pi,$$

$$(4.3) \quad F(\theta(A_1), \theta(B_1), \theta(B_1 A_1)) = F(\theta(A_2), \theta(B_2), \theta(B_2 A_2)) > 1,$$

where  $j = 1, 2$  and

$$F(x, y, z) = \frac{\cos^2 x + \cos^2 y + \cos^2 z + 2 \cos x \cos y \cos z - 1}{\sin x \sin y \sin z}.$$

**Remark 4.2** (4.1) and (4.2) imply that  $F(\theta(A_j), \theta(B_j), \theta(B_j A_j)) > 0$ .

Let  $R$  be a Riemann surface represented by  $\Sigma_{(2,0,0)}$ . Let  $(a_1, b_1, a_2, b_2)$  be a canonical homotopy basis of the fundamental group of  $R$  corresponding to  $\Sigma_{(2,0,0)}$ . We put same labels on a closed curve on  $R$  and the geodesic freely homotopic to it. Then  $\theta(A_j)$ ,  $\theta(B_j)$  and  $\theta(B_j A_j)$  are three interior angles of a triangle determined by  $a_j, b_j$  and  $a_j b_j$ . Let  $q_j$  be the intersection of  $a_j$  and  $b_j$ . The intersection angle of  $a_1 b_1 a_1^{-1} b_1^{-1}$  and the segment  $[q_1, q_2]$  is  $\mu$  (see Figure 4.2).

**Theorem 4.3**  $T(2, 0, 0)$  is parametrized global real analytically by the above seven angle parameters. This parameter space is defined by (4.1), (4.2) and (4.3).

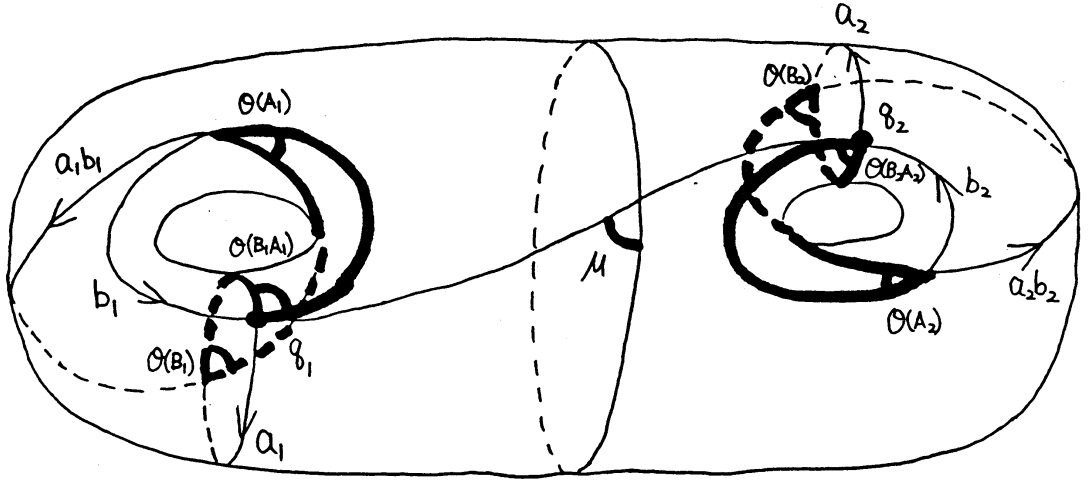


Figure 4.2.

## 5 The geometry of hyperbolic transformations

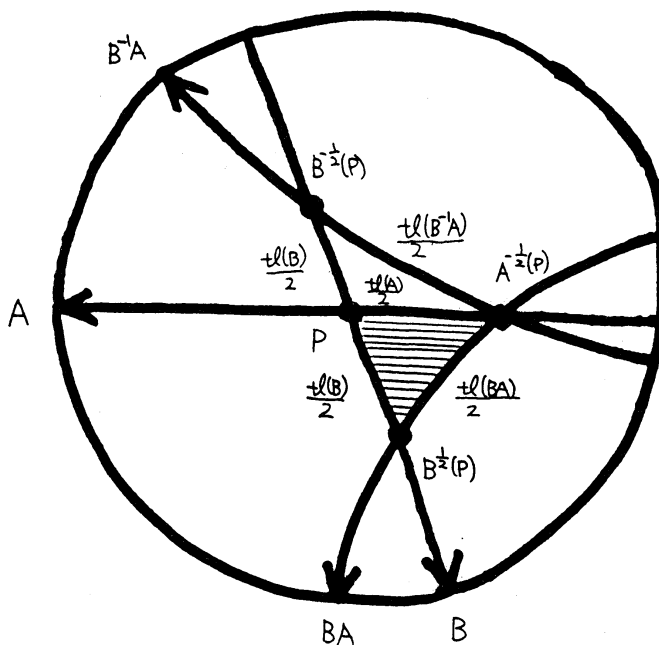
First we show the position of the axes of two hyperbolic transformations.

**Lemma 5.1** *Let  $A, B \in M(\mathbb{D})$  be hyperbolic. Then  $ax(A)$  and  $ax(B)$  intersect if and only if  $tr[\tilde{A}, \tilde{B}] < 2$ .*

**Lemma 5.2** *Let  $A, B \in M(\mathbb{D})$  be hyperbolic elements with intersecting axes. Then eight elements  $A^\varepsilon B^\eta, B^\varepsilon A^\eta$ ;  $\varepsilon, \eta \in \{\pm 1\}$  are hyperbolic. Let  $p$  be the intersection of  $ax(A)$  and  $ax(B)$ . Then*

$$\begin{aligned} ax(BA) &= L(A^{-1/2}(p), B^{1/2}(p)), \\ ax(B^{-1}A) &= L(A^{-1/2}(p), B^{-1/2}(p)), \\ d(A^{-1/2}(p), B^{1/2}(p)) &= \frac{tl(BA)}{2}, \\ d(A^{-1/2}(p), B^{-1/2}(p)) &= \frac{tl(B^{-1}A)}{2}. \end{aligned}$$

*Epecially,  $ax(A)$ ,  $ax(B)$  and  $ax(BA)$  determine a triangle with vertices  $p$ ,  $A^{-1/2}(p)$  and  $B^{1/2}(p)$  (see Figure 5.1).*



**Figure 5.1.** In the case that  $p(A)$ ,  $q(B)$ ,  $q(A)$  and  $p(B)$  are arranged clockwise in this order on the circle at infinity.

Let  $A, B \in M(\mathbb{D})$  be hyperbolic. If  $BA$  is not hyperbolic, then  $ax(A)$  and  $ax(B)$  do not intersect, by Lemma 5.2. If  $BA$  is hyperbolic, then  $ax(A)$ ,  $ax(B)$  and  $ax(BA)$  are characterized as follows:

**Lemma 5.3** *Let  $A, B, BA \in M(\mathbb{D})$  be hyperbolic.*

(i)  $ax(A)$ ,  $ax(B)$  and  $ax(BA)$  are disjoint, if and only if some two axes are disjoint.

(ii)  $ax(A)$ ,  $ax(B)$  and  $ax(BA)$  are not disjoint, if and only if these three axes do not intersect at one point and any two axes intersect each other. Thus there are three cases of the positions of axes illustrated in Figure 5.2. These cases are characterized by  $tr\tilde{A}$ ,  $tr\tilde{B}$  and  $tr\tilde{B}\tilde{A}$  as follows:

$$(a) \Leftrightarrow tr\tilde{A}tr\tilde{B}tr\tilde{B}\tilde{A} < 0, tr[\tilde{B}, \tilde{A}] > 18,$$

$$(b) \Leftrightarrow tr\tilde{A}tr\tilde{B}tr\tilde{B}\tilde{A} > 0, tr[\tilde{B}, \tilde{A}] > 2,$$

$$(c) \Leftrightarrow tr\tilde{A}tr\tilde{B}tr\tilde{B}\tilde{A} > 0, tr[\tilde{B}, \tilde{A}] < 2.$$

**Remark 5.4**  $tr[\tilde{B}, \tilde{A}] = tr^2\tilde{A} + tr^2\tilde{B} + tr^2\tilde{B}\tilde{A} - tr\tilde{A}tr\tilde{B}tr\tilde{B}\tilde{A} - 2$  and  $tr\tilde{A}tr\tilde{B}tr\tilde{B}\tilde{A}$  are invariant under the choice of matrix representations.

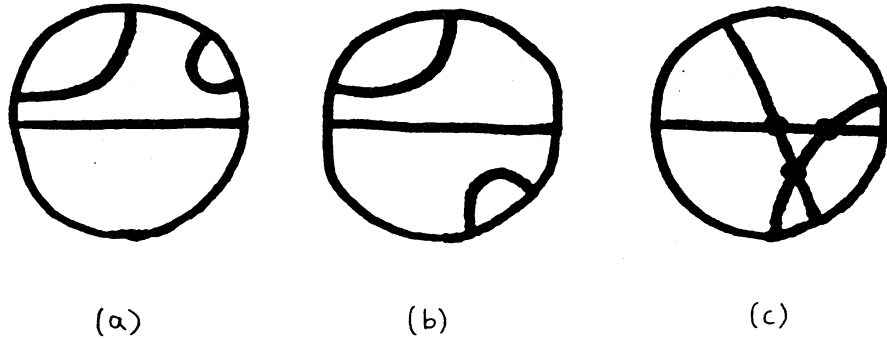


Figure 5.2.

**Remark 5.5** Similarly, for any non trivial elements  $A, B, BA \in M(\mathbb{D})$ , the position of their fixed points and the *direction* of their actions are characterized by such three traces.

Finally, we show the following result.



**Theorem 5.6** Let  $A, B \in M(D)$  be hyperbolic elements with intersecting axes. Let  $p$  be the intersection of these axes.

(i) The axes of  $A^*B^\eta$ ,  $B^*A^\eta$ ;  $\epsilon, \eta \in \{\pm 1\}$  determine the parallelogram with vertices  $A^{-1/2}(p)$ ,  $B^{1/2}(p)$ ,  $A^{1/2}(p)$  and  $B^{-1/2}(p)$ .

Let  $C = [B, A]$  be hyperbolic. Let  $p(A)$ ,  $q(B)$ ,  $q(A)$  and  $p(B)$  be arranged clockwise in this order on the circle at infinity.

(ii)  $(A, B^{-1}A^{-1}B, C^{-1})$ ,  $(BA, B^{-1}A^{-1}, C^{-1})$  and  $(A^{-1}BA, B^{-1}, C^{-1})$  are canonical systems of generators of type  $(0, 0, 3)$ .

(iii) Let  $R \in M(D)$  be elliptic of order 2 with fixed point  $p$ . Then we have

$$\begin{aligned} C^{1/2} &= RBA, \\ A^{-1} &= RAR, \\ B^{-1} &= RBR, \\ A &= [R, A^{1/2}], \\ B &= [R, B^{1/2}], \\ \tilde{R} &= \frac{\pm 1}{\det(\tilde{B}\tilde{A} - \tilde{A}\tilde{B})^{1/2}}(\tilde{B}\tilde{A} - \tilde{A}\tilde{B}). \end{aligned}$$

(iv)  $C^{-1/2}A$ ,  $C^{-1/2}B^{-1}$  and  $C^{-1/2}BA$  are elliptic of order 2 satisfying

$$\begin{aligned} fp(C^{-1/2}A) &= (ABA)^{-1/2}(p) = (BA)^{-1/2}A^{-1/2}(p), \\ fp(C^{-1/2}B^{-1}) &= A^{-1/2}(p), \\ ax(ABA) &= L(fp(C^{-1/2}A), p). \end{aligned}$$

(v) Let  $A_{1/2}$  (resp.  $A_{-1/2}$ ) be elliptic of order 2 with fixed point  $A^{1/2}(p)$  (resp.  $A^{-1/2}(p)$ ), namely,  $A_{1/2} = A^{1/2}RA^{-1/2}$  and  $A_{-1/2} = A^{-1/2}RA^{1/2}$ . Similarly,  $B_{1/2}$  and  $B_{-1/2}$  are defined. Then we have

$$\begin{aligned} A &= RA_{-1/2} = A_{1/2}R, \\ B &= RB_{-1/2} = B_{1/2}R. \end{aligned}$$

Thus

$$\begin{aligned} BA &= B_{1/2}A_{-1/2}, \\ AB &= A_{1/2}B_{-1/2}, \\ C &= B_{-1/2}A_{1/2}B_{1/2}A_{-1/2}. \end{aligned}$$

Especially,  $C$  is determined by four elliptic transformations of order 2 whose fixed points are four vertices of the parallelogram (see Figure 5.3).

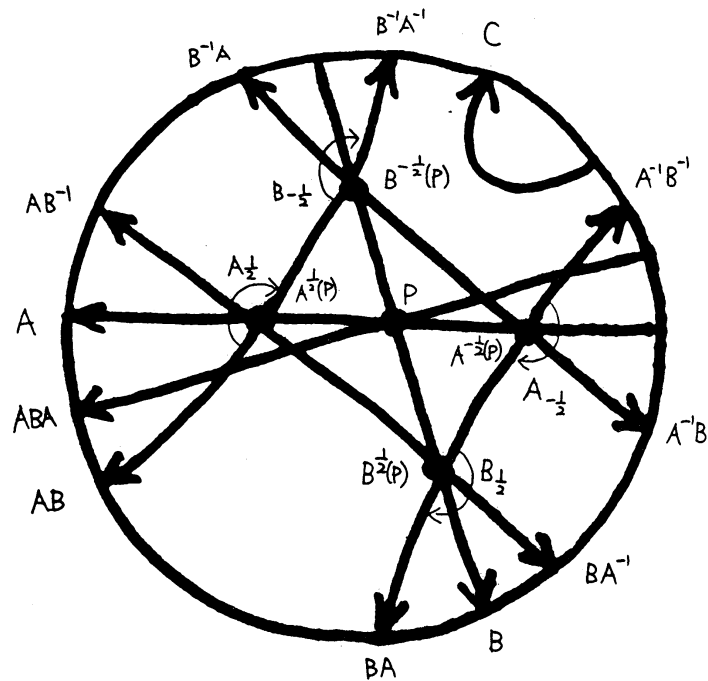


Figure 5.3.

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