

An L^2 -index formula for monopoles with Dirac-type singularities

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Abstract

We prove the Fredholmness of the Dirac operators associated to monopoles with Dirac-type singularities on any complete oriented 3-dimensional Riemannian manifolds. We also calculate their L^2 -indices.

1 Introduction

Let (X, g) be a 3-dimensional complete oriented Riemannian manifold with bounded scalar curvature. We fix a spin structure on X . Let $Z \subset X$ be a finite subset. Let (V, h) be a Hermitian vector bundle on $X \setminus Z$ and A a connection on (V, h) . Let $\Phi \in \Gamma(X \setminus Z, \text{End}(V))$ be a skew-Hermitian endomorphism of (V, h) . The tuple (V, h, A, Φ) is called a monopole on $X \setminus Z$ if the tuple (V, h, A, Φ) satisfies the Bogomolny equation $F(A) = *\nabla_A(\Phi)$, where $F(A)$ is the curvature of A and $*$ is the Hodge operator. Moreover, a point $p \in Z$ is a Dirac-type singularity of (V, h, A, Φ) of weight $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^{\text{rank}(V)}$ if the monopole (V, h, A, Φ) satisfies a certain asymptotic behavior around $p \in Z$ (see Definition 2.1 (ii)). We set the Dirac operators $\not{D}_{(A,\Phi)}^\pm : \Gamma(X \setminus Z, V \otimes S_X) \rightarrow \Gamma(X \setminus Z, V \otimes S_X)$ of (V, h, A, Φ) to be $\not{D}_{(A,\Phi)}^\pm(s) := \not{D}_A(s) \pm \Phi \otimes \text{Id}_{S_X}$, where S_X is the spinor bundle on X and $\not{D}_A$ is the Dirac operator of (V, h, A) . We regard $\not{D}_{(A,\Phi)}^\pm$ as a closed operator $L^2(X \setminus Z, V \otimes S_X) \rightarrow L^2(X \setminus Z, V \otimes S_X)$ by considering derivation as a current. The main result is the following.

Theorem 1.1 (Theorem 4.5). Let (V, h, A, Φ) be a monopole of rank r on $X \setminus Z$ such that each $p \in Z$ is a Dirac-type singularity of (V, h, A, Φ) with weight $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^r$. We assume that (V, h, A, Φ) satisfies the following conditions (*the Råde condition*).

- We have $\Phi|_x, F(A)|_x = O(1)$ as $x \rightarrow \infty$.
- We have $\nabla_A(\Phi)|_x = o(1)$ as $x \rightarrow \infty$.
- There exists a compact region $Y \supset Z$ such that Y has a smooth boundary ∂Y , and the inequality $\inf_{x \in X \setminus Y} \{|\lambda| \mid \lambda \text{ is an eigenvalue of } \Phi|_x\} > 0$ is satisfied.

Then the Dirac operators $\not{D}_{(A,\Phi)}^\pm$ are Fredholm. Moreover, their indices $\text{Ind}(\not{D}_{(A,\Phi)}^\pm)$ are given as follows:

$$\text{Ind}(\not{D}_{(A,\Phi)}^\pm) = \mp \left\{ \sum_{p \in Z} \sum_{k_{p,i} > 0} k_{p,i} + \int_{\partial Y} \text{ch}(V^+) \right\} = \pm \left\{ \sum_{p \in Z} \sum_{k_{p,i} < 0} k_{p,i} + \int_{\partial Y} \text{ch}(V^-) \right\},$$

where $V^\pm$ is a subbundle of $V|_{\partial Y}$ spanned by the eigenvectors of $\mp\sqrt{-1}\Phi$ with positive eigenvalues.

Let $D : E \rightarrow F$ be an elliptic operator on an oriented manifold M . It is a fundamental problem to ask when D is Fredholm. Moreover, if D is Fredholm, it is desirable to obtain a formula to compute the index $\text{Ind}(D)$ of D . If M is compact, then D is always Fredholm. Moreover, according to the celebrated Atiyah-Singer index theorem the index $\text{Ind}(D)$ can be computed in a topological way. The result has been applied in a wide range of mathematics, including gauge theory, differential topology and complex geometry.

When M is non-compact, such a complete and definitive result has not yet been known, but there are many interesting studies. Let us recall some results in the case where $\dim(M)$ is odd. Note that if M is an odd dimensional compact manifold, then $\text{Ind}(D)$ is always 0. First we recall results for elliptic operators on non-compact complete manifolds. Callias [3] proved the index theorem of the Dirac operators of $SU(2)$ -bundles on $\mathbb{R}^{2n+1}$ that satisfies a certain asymptotic behavior at infinity. In [13], Råde studied the Dirac operators of vector bundles on any odd-dimensional complete spin manifolds with bounded scalar curvature. He formulated the asymptotic condition in Theorem 1.1, and obtained a generalization of Callias' index theorem. Next we consider results for elliptic operators defined on closed manifolds except for finitely many points. Pauly proved the index theorem of the deformation complexes of $SU(2)$ -monopoles with Dirac-type singularities on closed oriented 3-dimensional manifolds. His method was used by Kapustin and Witten in [6] to calculate the dimension of the moduli space of Hecke modifications on a Riemann surface. In [10], Moore, Royston and Van den Bleeken calculated the indices of the Dirac operators of the monopoles on associated vector bundles with principal G -bundles on $\mathbb{R}^3$ by generalizing Callias' argument but without using the method of Pauly, where G is a simple compact Lie group. Moreover, they conjectured that their index formula holds on any 3-dimensional complete Riemannian manifolds.

The proof of the main result is divided into two parts. First we extend Pauly's argument and calculate the indices of $\not{D}_{(A,\Phi)}^\pm$ when X is a closed manifold (Theorem 3.4 and Corollary 3.12). To explain in detail, we provide an overview of Pauly's original argument for the deformation complexes. For each $p \in Z$, Pauly made a local surgery on the deformation complex associated to (V, h, A, Φ) , and desingularize it around p . He calculated the difference of the indices before and after each surgery by using the relative index theorem, and obtain the index of the deformation complex inductively. However, the local surgery for the Dirac operator of (V, h, A, Φ) needs a condition $\sum_i k_{p,i} = 0$ for each $p \in Z$ essentially (see Lemma 4.4). Thus it is difficult to apply the argument to calculate the indices of the Dirac operators of (V, h, A, Φ) even if X is a closed manifold. Instead of local surgeries, we construct a global lift of (V, h, A, Φ) on a 4-dimensional closed manifold $\tilde{P}$ equipped with an S^1 -action. Then we obtain the index of the Dirac operator associated to (V, h, A, Φ) by calculating the S^1 -equivariant index of the Dirac operator of the lift of (V, h, A, Φ) . This global construction needs no assumption on the weights of the singularities. Next we combine our result and Råde's index theorem in [13], and obtain the index formula on any 3-dimensional complete Riemannian manifolds (Theorem 4.5).

Remark. Kottke [7] proved a generalization of Callias's index theorem on asymptotically conical manifolds with a weaker asymptotic condition than the Råde condition. Adopting his result instead of Råde's index theorem, we can weaken the asymptotic condition imposed in Theorem 1.1 in the case that X is an asymptotically conical manifold.

This result was obtained in the study of the inverse transform of the Nahm transform from L^2 -finite instantons on the product of $\mathbb{R}$ and a 3-dimensional torus T^3 to Dirac-type singular monopoles on the dual torus $\hat{T}^3$ of T^3 in [14].

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2 Preliminary

2.1 Monopoles with Dirac-type singularities

We recall the definition of monopoles with Dirac-type singularities following [5], [8] and [12].

Definition 2.1. Let (X, g) be an oriented 3-dimensional Riemannian manifold and $*_g$ be the Hodge operator on X . If there is no risk of confusion, then we abbreviate $*_g$ to just $*$.

- (i) Let (V, h) be a Hermitian vector bundle with a unitary connection A on X . Let Φ be a skew-Hermitian endomorphism of V . The tuple (V, h, A, Φ) is said to be a monopole on X if it satisfies the Bogomolny equation $F(A) = *\nabla_A(\Phi)$. The rank of (V, h, A, Φ) stands for the rank of V .
- (ii) Let $Z \subset X$ be a discrete subset. Let (V, h, A, Φ) be a monopole of rank $r \in \mathbb{N}$ on $X \setminus Z$. A point $p \in Z$ is called a Dirac-type singularity of the monopole (V, h, A, Φ) of weight $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^r$ if the following holds.
 - There exists a small neighborhood B of p such that $(V, h)|_{B \setminus \{p\}}$ is decomposed into a sum of Hermitian line bundles $\bigoplus_{i=1}^r F_{p,i}$ with $\deg(F_{p,i}) = \int_{\partial B} c_1(F_{p,i}) = k_{p,i}$.
 - In the above decomposition, we have the following estimates,

$$\begin{cases} \Phi = \frac{\sqrt{-1}}{2R_p} \sum_{i=1}^r k_{p,i} \cdot \text{Id}_{F_{p,i}} + O(1) \\ \nabla_A(R_p \Phi) = O(1), \end{cases}$$

where R_p is the distance from p .

For a monopole (V, h, A, Φ) on $X \setminus Z$, if each point $p \in Z$ is a Dirac-type singularity, then we call (V, h, A, Φ) a Dirac-type singular monopole on (X, Z) .

Remark 2.2. Let (V, h, A, Φ) be a Dirac-type singular monopole on (X, Z) of rank r and $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^r$ the weight of (V, h, A, Φ) at $p \in Z$.

- (i) The tuple $(\det(V), \det(h), \text{Tr}(A), \text{Tr}(\Phi))$ is also a Dirac-type singular monopole on (X, Z) , and the weight at $p \in Z$ is given as $\sum_i k_{p,i}$.

- (ii) We assume that X is compact. Then we obtain $\sum_{p \in Z} \sum_i k_{p,i} = \int_{\partial B(Z, \varepsilon)} c_1(V) = - \int_{X \setminus B(Z, \varepsilon)} d c_1(V) = 0$ by Stokes' theorem, where $\varepsilon > 0$ is a sufficiently small positive number and $B(Z, \varepsilon) := \coprod_{p \in Z} B(p, \varepsilon) = \coprod_{p \in Z} \{x \in X \mid d_g(x, p) < \varepsilon\}$. Here d_g stands for the distance function induced by g .

Remark. The notion of Dirac-type singularities of monopoles was defined first by Kronheimer on flat 3-dimensional Riemannian manifolds, and Pauly [12] generalized it to any 3-dimensional Riemannian manifolds. Charbonneau and Hurtubise [5] gave a new convenient definition.

For example, we recall the flat Dirac monopole of weight $k \in \mathbb{Z}$. Let $g_{i, \text{Euc}}$ denote the canonical metric on $\mathbb{R}^i$. For $i \in \mathbb{N}$, we denote by $r_i : \mathbb{R}^i \rightarrow \mathbb{R}$ the distance from $0 \in \mathbb{R}^i$. Let $p : \mathbb{R}^3 \setminus \{0\} \rightarrow S^2 (\simeq \mathbb{P}^1)$ be the projection. Let $\mathcal{O}(k)$ be a holomorphic line bundle on $\mathbb{P}^1$ of degree k . Let $h_{\mathcal{O}(k)}$ be a Hermitian metric of $\mathcal{O}(k)$ such that the Chern connection $A_{\mathcal{O}(k)}$ of $(\mathcal{O}(k), h_{\mathcal{O}(k)})$ has a constant scalar curvature. Then $(p^* \mathcal{O}(k), p^* h_{\mathcal{O}(k)}, p^* A_{\mathcal{O}(k)}, \sqrt{-1}k/2r_3)$ is a Dirac-type singular monopole on $(\mathbb{R}^3, \{0\})$. We call this monopole the flat Dirac monopole of weight k , and denote by (L_k, h_k, A_k, Φ_k) .

We will recall the equivalent condition proved by Pauly [12]. First we recall the notion of instantons.

Definition 2.3. Let (Y, g) be an oriented Riemannian 4-fold. For a Hermitian vector bundle (V, h) on Y and a connection A on (V, h) , the tuple (V, h, A) is an instanton if the ASD equation $F(A) = - * F(A)$ is satisfied.

Let $U \subset \mathbb{R}^3$ be a neighborhood of $0 \in \mathbb{R}^3$. Let g be a Riemannian metric on U . We assume that the standard coordinate of $\mathbb{R}^3$ is a normal coordinate of g at 0. Set the Hopf map $\pi : \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{R}^3 \setminus \{0\}$ to be $\pi(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1 z_2)$, where we set $z_i = x_i + \sqrt{-1}y_i$. We also set the $S^1 (= \mathbb{R}/2\pi\mathbb{Z})$ -action on $\mathbb{C}^2$ to be $\theta \cdot (z_1, z_2) := (e^{\sqrt{-1}\theta} z_1, e^{-\sqrt{-1}\theta} z_2)$. Then the restriction $\pi : \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{R}^3 \setminus \{0\}$ forms a principal S^1 -bundle. Then we have $\pi^* r_3 = r_4^2$.

Lemma 2.4 (Proposition 3 and 4 in [12]). There exist a harmonic function $f : U \setminus \{0\} \rightarrow \mathbb{R}$ with respect to the metric g and a 1-form ξ on $\pi^{-1}(U)$ such that the following hold.

- The 1-form $\omega := (\pi^* f)\xi$ is a connection of $\pi : \mathbb{R}^4 \setminus \{0\} \rightarrow \mathbb{R}^3 \setminus \{0\}$, i.e. ω is S^1 -invariant, and we have $\omega(\partial_\theta) = 1$. Here ∂_θ is the generating vector field of the S^1 -action on $\mathbb{R}^4 \setminus \{0\}$.
- We have $d\omega = \pi^*(df)$.
- We have the following estimates:

$$\begin{cases} f = 1/2r_3 + o(1) \\ \xi = 2(-y_1 dx_1 + x_1 dy_1 + y_2 dx_2 - x_2 dy_2) + O(r_4^2). \end{cases}$$

- The symmetric tensor $g_4 = \pi^* f(\pi^* g + \xi^2)$ is a Riemannian metric of $L^2_{5, \text{loc}}$ -class on $\pi^{-1}(U)$, and we have an estimate $|g_4 - 2g_{4, \text{Euc}}|_{g_{4, \text{Euc}}} = O(r_4)$. Here a function on $\pi^{-1}(U)$ is of $L^2_{k, \text{loc}}$ -class if every derivative of f up to order k has a finite L^2 -norm on any compact subset of $\pi^{-1}(U)$.

Remark. The existence of f follows from $\Delta_g(r_3^{-1}) = \Delta_g(r_3^{-1}) - \Delta_{g_{3, \text{Euc}}}(r_3^{-1}) = O(r_3^{-1}) \in L^{3-\varepsilon}(U)$ for any $\varepsilon > 0$, where Δ_g is the Laplacian with respect to the metric g .

Proposition 2.5 (Proposition 5 in [12]). Let (V, h, A) be a Hermitian vector bundle on $U \setminus \{0\}$ of rank r , and $\Phi \in \Gamma(U \setminus \{0\}, \text{End}(V))$ be a skew-Hermitian endomorphism. The tuple (V, h, A, Φ) is a monopole on $U \setminus \{0\}$ if and only if the tuple $(\pi^*V, \pi^*h, \pi^*A - \xi \otimes \pi^*\Phi)$ is an instanton on $\pi^{-1}(U) \setminus \{0\}$ with respect to the metric $g_4 = \pi^*f(\pi^*g + \xi^2)$. Moreover, 0 is a Dirac-type singularity of (V, h, A, Φ) of weight $\vec{k} = (k_i) \in \mathbb{Z}^r$ if and only if the following hold.

- The instanton $(\pi^*V, \pi^*h, \pi^*A - \pi^*\Phi \otimes \xi)$ can be prolonged over $\pi^{-1}(U)$, and the prolonged connection is represented by an $L_{6, \text{loc}}^2$ -valued connection matrix. We will denote by (V_4, h_4, A_4) the prolonged instanton.
- The weight of the S^1 -action on the fiber $V_4|_0$ agrees with $\vec{k}$ up to a suitable permutation.

Remark 2.6.

- If $g = g_{3, \text{Euc}}$, we can choose $f = 1/2r_3$ and $\xi = 2(-y_1 dx_1 + x_1 dy_1 + y_2 dx_2 - x_2 dy_2)$. Then we have $g_4 = 2g_{4, \text{Euc}}$.
- By the Sobolev embedding theorem, the connection matrix of A_4 is of C^3 class.

Let $h_{\mathbb{C}}$ be the canonical Hermitian metric on $\mathbb{C}$. We set the Hermitian line bundle $(\tilde{L}, \tilde{h}) := (\pi^{-1}(U) \setminus \{0\}) \times_{U(1)} (\mathbb{C}, h_{\mathbb{C}})$ on $U \setminus \{0\}$ and take the connection $\tilde{A}$ induced by ω . Then $(\tilde{L}, \tilde{h}, \tilde{A}, \sqrt{-1}f)$ is a monopole on U with respect to g , and 0 is the Dirac-type singularity of weight 1. We call the monopole $(\tilde{L}_k, \tilde{h}_k, \tilde{A}_k, \sqrt{-1}kf) := (\tilde{L}^{\otimes k}, \tilde{h}^{\otimes k}, \tilde{A}^{\otimes k}, \sqrt{-1}kf)$ a Dirac monopole of weight k with respect to g . The following proposition is a partial generalization of [9, Proposition 5.2].

Proposition 2.7. Let (V, h, A, Φ) be a monopole on $U \setminus \{0\}$, and assume that the point 0 is a Dirac-type singularity of weight $\vec{k} = (k_i) \in \mathbb{Z}^r$. Then there exist a neighborhood $U' \subset U$ and a unitary isomorphism $\varphi : V|_{U' \setminus \{0\}} \simeq (\bigoplus_{i=1}^r \tilde{L}_{k_i})|_{U' \setminus \{0\}}$ such that the following estimates hold.

$$|A - \varphi^*(\bigoplus \tilde{A}_{k_i})| = O(1).$$

$$|\Phi - \varphi^*(\sum \sqrt{-1}k_i f \text{Id}_{L_{k_i}})| = O(1).$$

Proof. Let (V', h', A', Φ') be the monopole $\bigoplus_{i=1}^r (\tilde{L}_{k_i}, \tilde{h}_{k_i}, \tilde{A}_{k_i}, \sqrt{-1}k_i f)$. By Proposition 2.5, the instantons $(\pi^*V, \pi^*h, \pi^*A - \pi^*\Phi \otimes \xi)$ and $(\pi^*V', \pi^*h', \pi^*A' - \pi^*\Phi' \otimes \xi)$ can be prolonged over $\pi^{-1}(U)$, and denote by (V_4, h_4, A_4) and (V'_4, h'_4, A'_4) respectively. Then the weights of S^1 -actions on the fiber of V_4 and V'_4 at the origin coincide with each other, and the connections A_4 and A'_4 are S^1 -invariant. Hence there exist an S^1 -invariant neighborhood $U'_4 \subset \pi^{-1}(U)$ of 0 and an S^1 -equivariant unitary isomorphism $\varphi_4 : V_4|_{U'_4} \rightarrow V'_4|_{U'_4}$ such that $A_4 - \varphi_4^*(A'_4)$ vanishes at the origin. Since this condition involves only $\varphi_4|_0$ and $d\varphi_4|_0$, we may assume that φ_4 is sufficiently smooth. Hence we have $|A_4 - \varphi_4^*(A'_4)| = O(r_4)$. Since $f = 1/2r_3 + o(1)$ and ξ is orthogonal to $\pi^*(T^*\mathbb{R}^3)$ with the metric $g_4 = \pi^*f(\pi^*g + \xi^2)$, the unitary isomorphism $\varphi : V|_{U' \setminus \{0\}} \rightarrow V'|_{U' \setminus \{0\}}$ induced by φ_4 satisfies the desired estimates, where we put $U' := \pi(U'_4)$. $\square$

By the estimates in Lemma 2.4, we also obtain the following approximation.

Corollary 2.8. Let (V, h, A, Φ) be a monopole on $U \setminus \{0\}$, and assume that the point 0 is a Dirac-type singularity of weight $\vec{k} = (k_i) \in \mathbb{Z}^r$. Then there exist a neighborhood $U' \subset U$ and a unitary isomorphism $\varphi : V|_{U' \setminus \{0\}} \simeq (\bigoplus_{i=1}^r L_{k_i})|_{U' \setminus \{0\}}$ such that the following estimates hold.

$$|A - \varphi^*(\bigoplus A_{k_i})| = O(1).$$

$$|\Phi - \varphi^*(\frac{\sqrt{-1}}{2r_3} \sum k_i \text{Id}_{L_{k_i}})| = O(1).$$

2.2 Local properties of harmonic spinors of the flat Dirac monopoles

Let (X, g) be an n -dimensional oriented spin manifold with a fixed spin structure. We denote by S_X the spinor bundle on X , and by $\text{clif} : T^*X \rightarrow \text{End}(S_X)$ the Clifford product. If n is an odd number, then we assume $(\sqrt{-1})^{(n+1)/2} \text{clif}(\text{vol}_{(X,g)}) = -\text{Id}_{S_X}$, where we use the canonical linear isomorphism between the exterior algebra and the Clifford algebra. The spinor bundle S_X has the induced connection A_{S_X} by the Levi-Civita connection on X , and we set the Dirac operator $\not{D}_X : \Gamma(X, S_X) \rightarrow \Gamma(X, S_X)$ to be $\not{D}_X(f) := \text{clif} \circ \nabla_{A_{S_X}}(f)$. For a vector bundle (V, h) on X and a connection A on (V, h) , we also set the Dirac operator $\not{D}_A : \Gamma(X, S_X \otimes V) \rightarrow \Gamma(X, S_X \otimes V)$ to be $\not{D}_A(s) := \text{clif} \circ \nabla_{A_{S_X} \otimes A}(s)$. If n is even, then we have the decomposition $S_X = S_X^+ \oplus S_X^-$, and the Dirac operator $\not{D}_A$ is also decomposed into sum of the positive and negative Dirac operators $\not{D}_A^\pm : \Gamma(X, S_X^\pm \otimes V) \rightarrow \Gamma(X, S_X^\pm \otimes V)$. If $n = 3$, for a skew-Hermitian endomorphism $\Phi \in \Gamma(X, \text{End}(V))$ we set the Dirac operators $\not{D}_{(A, \Phi)}^\pm : \Gamma(X, V \otimes S_X) \rightarrow \Gamma(X, V \otimes S_X)$ to be $\not{D}_{(A, \Phi)}^\pm(f) := \not{D}_A(f) \pm (\Phi \otimes \text{Id}_{S_X})(f)$.

For a differential operator $P : \Gamma(X, V_1) \rightarrow \Gamma(X, V_2)$ between Hermitian vector bundles (V_1, h_1) and (V_2, h_2) on X , we regard P as the closed operator $P : L^2(X, V_1) \rightarrow L^2(X, V_2)$ with the domain $\text{Dom}(P) := \{s \in L^2(X, V_1) \mid P(s) \in L^2\}$, where $P(s)$ is the derivative as a current. We regard $\text{Dom}(P)$ as a Banach space equipped with the graph norm $\|s\|_P := \|s\|_{L^2} + \|P(s)\|_{L^2}$.

Remark. Any 3-dimensional oriented manifolds are parallelizable, and hence they have spin structures. In particular, any monopoles on a 3-dimensional oriented manifolds have the associated Dirac operators.

Let $S_{\mathbb{R}^3}$ be the spinor bundle on $\mathbb{R}^3$ with respect to the trivial spin structure, and d be the trivial connection on $S_{\mathbb{R}^3}$. By using the projection $p : \mathbb{R}^3 \setminus \{0\} \rightarrow S^2$, we describe the Dirac operators of the flat Dirac monopole $(L_k, h_k, A_k, \Phi_k) = (p^*\mathcal{O}(k), p^*h_{\mathcal{O}(k)}, p^*A_{\mathcal{O}(k)}, \sqrt{-1}k/2r_3)$ in terms of the Dirac operators $\not{D}_k^\pm$ of $(\mathcal{O}(k), A_{\mathcal{O}(k)})$ on $\mathbb{P}^1 = S^2$. Let $S_{S^2} = S_{S^2}^+ \oplus S_{S^2}^-$ be the spinor bundle on (S^2, g_{S^2}) , and $\not{D}_{S^2}^\pm : \Gamma(S^2, S_{S^2}^\pm) \rightarrow \Gamma(S^2, S_{S^2}^\pm)$ the Dirac operators on S^2 . By the isometry $\mathbb{R}^3 \setminus \{0\} \simeq (\mathbb{R}_+ \times S^2, (dr_3)^2 + r_3^2 g_{S^2})$ we obtain the unitary isomorphisms $S_{\mathbb{R}^3}|_{\mathbb{R}^3 \setminus \{0\}} \simeq p^*S_{S^2}$. According to Nakajima [11], under the identification $S_{\mathbb{R}^3}|_{\mathbb{R}^3 \setminus \{0\}} \simeq p^*S_{S^2}$ the Dirac operator $\not{D}_{\mathbb{R}^3}$ on $\mathbb{R}^3 \setminus \{0\}$ is written as follows:

$$\not{D}_{\mathbb{R}^3} = \frac{1}{r_3} \begin{pmatrix} \sqrt{-1}(r_3 \frac{\partial}{\partial r_3} + 1) & \not{D}_{S^2}^- \\ \not{D}_{S^2}^+ & -\sqrt{-1}(r_3 \frac{\partial}{\partial r_3} + 1) \end{pmatrix}.$$

Therefore we obtain the following equality.

$$\mathcal{D}_{(A_k, \Phi_k)}^\pm = \frac{1}{r_3} \begin{pmatrix} \sqrt{-1}(r_3 \frac{\partial}{\partial r_3} + \frac{2 \pm k}{2}) & \mathcal{D}_k^- \\ \mathcal{D}_k^+ & -\sqrt{-1}(r_3 \frac{\partial}{\partial r_3} + \frac{2 \mp k}{2}) \end{pmatrix}.$$

By the isomorphisms $S_{S^2}^+ \simeq \Omega^{0,0}(\mathcal{O}(-1))$, $S_{S^2}^- \simeq \Omega^{0,1}(\mathcal{O}(-1))$ and $\mathcal{D}_{S^2} = \mathcal{D}_{S^2}^+ + \mathcal{D}_{S^2}^- = \sqrt{2}(\bar{\partial}_{\mathcal{O}(-1)} + \bar{\partial}_{\mathcal{O}(-1)}^\star)$, we obtain $\text{Ker}(\mathcal{D}_k^+) \simeq H^0(\mathbb{P}^1, \mathcal{O}(k-1))$ and $\text{Ker}(\mathcal{D}_k^-) \simeq H^1(\mathbb{P}^1, \mathcal{O}(k-1))$. Here $\bar{\partial}_{\mathcal{O}(-1)}^\star$ is the formal adjoint of $\bar{\partial}_{\mathcal{O}(-1)}$. Let $f_\nu^\pm \in L^2(S^2, S_{S^2}^\pm \otimes \mathcal{O}(k))$ ($\nu \in \mathbb{N}$) be all the eigenvectors of the operators $\mathcal{D}_k^- \circ \mathcal{D}_k^+$ and $\mathcal{D}_k^+ \circ \mathcal{D}_k^-$ with non-zero eigenvalues respectively. We set $n_\nu > 0$ to be the eigenvalue of $f_\nu^\pm$, and assume $n_{\nu+1} \geq n_\nu$. Then, according to [1], we have $\{n_\nu\} = \{q^2 + |k|q ; q \in \mathbb{N}\}$. We set $q_\nu > 0$ to satisfy $n_\nu = q_\nu^2 + |k|q_\nu$. We may assume that $\{f_\nu^\pm\}$ forms an orthonormal system and satisfies the relations $\mathcal{D}_k^\pm(f_\nu^\pm) = \sqrt{n_\nu} f_\nu^\mp$ for any $\nu \in \mathbb{N}$. By the elliptic inequality and the Sobolev inequality on S^2 , there exist $C', C'' > 0$ such that $\|f_\nu^\pm\|_{L^6} < C'' \|f_\nu^\pm\|_{L_1^2} \leq C'(\|f_\nu^\pm\|_{L^2} + \|\mathcal{D}_k^\pm(f_\nu^\mp)\|_{L^2}) = C'(1 + \sqrt{n_\nu})$. Then by the interpolation inequality we obtain $\|f_\nu^\pm\|_{L^3} \leq (\|f_\nu^\pm\|_{L^2})^{1/2} \cdot (\|f_\nu^\pm\|_{L^6})^{1/2} = C\sqrt{1 + \sqrt{n_\nu}}$, where we put $C := \sqrt{C'}$. Hence we obtain the following lemma.

Lemma 2.9. We have the estimate $\|f_\nu^\pm\|_{L^3} = O(\sqrt{q_\nu})$.

Through the above arguments, we can solve the equation by separation of variables. Since $r_3^m \in L^2(B(r)^*)$ if and only if $m > -3/2$, we obtain the following proposition.

Proposition 2.10. Let s be a section of $L_k \otimes S_{\mathbb{R}^3}$ on a punctured ball $B(r)^* := \{x \in \mathbb{R}^3 \mid 0 < |x| < r\}$ for some $r > 0$.

- (i) If we have $s \in L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3}) \cap \text{Ker}(\mathcal{D}_{(A_k, \Phi_k)}^+)$, then there exists a sequence $\{c_\nu\} \subset \mathbb{C}$ such that we have

$$s = \sum_{\nu \in \mathbb{N}} c_\nu (a_\nu^+(r_3) f_\nu^+ + a_\nu^-(r_3) f_\nu^-).$$

Here the functions $a_\nu^\pm$ are given as follows:

$$\begin{aligned} a_\nu^+(r_3) &= r_3^{-1+q_\nu+|k|/2}, \\ a_\nu^-(r_3) &= \frac{q_\nu + \max(0, k)}{\sqrt{-1}\sqrt{q_\nu^2 + |k|q_\nu}} r_3^{-1+q_\nu+|k|/2}. \end{aligned}$$

- (ii) If we have $s \in L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3}) \cap \text{Ker}(\mathcal{D}_{(A_k, \Phi_k)}^-)$, then there exist a sequence $\{c_\nu\} \subset \mathbb{C}$, $\alpha^+ \in \text{Ker}(\mathcal{D}_k^+) = H^0(\mathbb{P}^1, \mathcal{O}(k-1))$ and $\alpha^- \in \text{Ker}(\mathcal{D}_k^-) = H^1(\mathbb{P}^1, \mathcal{O}(k-1))$ such that we have

$$s = \left(\sum_{+, -} \alpha^\pm \right) \rho(r_3) + \sum_{\nu \in \mathbb{N}} c_\nu (b_\nu^+(r_3) f_\nu^+ + b_\nu^-(r_3) f_\nu^-).$$

Here the functions $\rho^\pm$ and $b_\nu^\pm$ are given as follows:

$$\begin{aligned} \rho(r_3) &= r_3^{-1+|k|/2}, \\ b_\nu^+(r_3) &= r_3^{-1+q_\nu+|k|/2}, \\ b_\nu^-(r_3) &= \frac{q_\nu + \max(0, -k)}{\sqrt{-1}\sqrt{q_\nu^2 + |k|q_\nu}} r_3^{-1+q_\nu+|k|/2}. \end{aligned}$$

Remark 2.11.

- Since we have $\dim(\text{Ker}(\partial_k^\pm)) = \max(\pm k, 0)$, α^+ or α^- (or possibly both) is 0.
- We have $|s| = O(r_3^{-1/2})$.

By the above proposition we obtain the following corollary.

Corollary 2.12. For arbitrary positive numbers $r > r' > 0$, the restriction map $L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3}) \cap \text{Ker}(\partial_{(A_k, \Phi_k)}^\pm) \rightarrow L^2(B(r')^*, L_k \otimes S_{\mathbb{R}^3})$ is a compact map.

Proof. We prove for $\partial_{(A_k, \Phi_k)}^+$. We take an arbitrary $s \in L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3}) \cap \text{Ker}(\partial_{(A_k, \Phi_k)}^+)$. By Proposition 2.10, there exists a sequence $\{c_\nu\} \subset \mathbb{C}$ such that $s = \sum_{\nu \in \mathbb{N}} c_\nu (a_\nu^+(r_3)f_\nu^+ + a_\nu^-(r_3)f_\nu^-)$, where $a_\nu^\pm$ are given in Proposition 2.10. Then we have

$$\begin{aligned} \|s\|_{L^2(B(r)^*)}^2 &= \int_{B(r)^*} |s|^2 r_3^2 dr_3 \wedge d\text{vol}_{S^2} = \int_0^r dr_3 \int_{S^2} d\text{vol}_{S^2} |r_3 s|^2 \\ &= \sum_{\nu} |c_\nu|^2 (\|r_3 a_\nu^+\|_{L^2(0,r)}^2 + \|r_3 a_\nu^-\|_{L^2(0,r)}^2) < \infty. \end{aligned}$$

For $T > 0$ and $\nu \in \mathbb{N}$, we set $N^+(T, \nu) := \|r_3 a_\nu^+\|_{L^2(0,T)}^2 + \|r_3 a_\nu^-\|_{L^2(0,T)}^2$. By the definition of $a_\nu^\pm$, we have $N^+(T, \nu) = N^+(T', \nu) \cdot (T/T')^{2q_\nu + |k| + 1}$ for any $T, T' > 0$. Therefore we obtain

$$\|s\|_{L^2(B(r')^*)}^2 = \sum_{\nu} |c_\nu|^2 \cdot N^+(r, \nu) \cdot (r'/r)^{2q_\nu + |k| + 1}.$$

Since $(r'/r)^{2q_\nu + |k| + 1}$ converges to 0 as $\nu \rightarrow \infty$, the restriction map is compact by a standard argument about $l^2 = L^2(\mathbb{N})$. The proof for $\partial_{(A_k, \Phi_k)}^-$ is similar. $\square$

As a preparation of Proposition 2.14, we prove the following lemma.

Lemma 2.13. Let $t_0 > 0$ be a positive number and α a real number. Set the constant C_α to be

$$C_\alpha = \begin{cases} |2\alpha - 1|^{-1/2} & (\alpha \neq 1/2) \\ 1 & (\alpha = 1/2). \end{cases}$$

There exists a compact operator $K_\alpha : L^2(0, t_0) \rightarrow C^0([0, t_0])$ such that for any $f \in L^2(0, t_0)$, the function $g := K_\alpha(f)$ satisfies the estimate

$$\begin{aligned} |g(t)| &\leq C_\alpha \|f\|_{L^2} \cdot t^{1/2} (1 + \log(t_0/t)^{1/2}) \\ &\leq C_\alpha \|f\|_{L^2} \cdot \sqrt{t_0} (1 + 1/\sqrt{e}) \end{aligned}$$

and the differential equation $t\partial_t(g/t) + \alpha(g/t) = f$, where $C^0([0, t_0])$ is the Banach space consisting of bounded continuous functions on $[0, t_0]$. In particular, we have $\|g\|_{L^2} \leq C_\alpha t_0(1 + 1/\sqrt{e}) \cdot \|f\|_{L^2}$ and $\|t^{-1/6}g\|_{L^3} \leq t_0^{1/3} \cdot \|t^{-1/6}g\|_{C^0} \leq C_\alpha t_0^{2/3}(1 + \sqrt{3/2e})^{1/3} \cdot \|f\|_{L^2}$.

Proof. We set $g = K_\alpha(f)$ to be

$$g(t) := \begin{cases} t^{-\alpha+1} \int_0^t f(x) x^{\alpha-1} dx & (\alpha > 1/2) \\ -t^{-\alpha+1} \int_t^{t_0} f(x) x^{\alpha-1} dx & (\alpha \leq 1/2). \end{cases}$$

Then, by a direct calculation we have $t\partial_t(g/t) + \alpha(g/t) = f$. If $\alpha \neq 1/2$, then we obtain $|g(t)| \leq t^{-\alpha+1} \|f\|_{L^2} \sqrt{t^{2\alpha-1}/|2\alpha-1|} = |2\alpha-1|^{-1/2} \|f\|_{L^2} \cdot t^{1/2}$. If $\alpha = 1/2$, then we have $|g(t)| \leq \|f\|_{L^2} \cdot t^{1/2} \log(t_0/t)^{1/2}$. As a consequence of the above inequalities, we obtain the desired estimate. By these estimates and the differential equation, we obtain $|\partial_t g| \leq |f| + C_\alpha |1 - \alpha| (1 + \log(t_0/t)^{1/2}) \|f\|_{L^2} \cdot t^{-1/2} \in L^2([0, t_0])$. Hence the compactness of K_α follows from the Ascoli-Arzelà theorem. $\square$

Proposition 2.14. Let $r > 0$ be a positive number. There exists a compact map $G^\pm : L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3}) \rightarrow L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3})$ such that we have $R(G^\pm) \subset \text{Dom}(\mathcal{D}_{(A_k, \Phi_k)}^\pm)$ and $\mathcal{D}_{(A_k, \Phi_k)}^\pm \circ G^\pm = \text{Id}$, where $R(\cdot)$ means the range of the operator. Moreover, we have $R(G^\pm) \subset L^3(B(r)^*, L_k \otimes S_{\mathbb{R}^3})$, and $G^\pm : L^2 \rightarrow L^3$ is bounded.

Proof. The proof for $\mathcal{D}_{(A_k, \Phi_k)}^+$ remains valid for $\mathcal{D}_{(A_k, \Phi_k)}^-$. Hence we prove only for $\mathcal{D}_{(A_k, \Phi_k)}^+$. The subspace that is spanned by $\text{Ker}(\mathcal{D}_k^\pm)$ and $\{f_\nu^\pm\}$ is dense in $L^2(S^2, S_{S^2} \otimes \mathcal{O}(k))$. Hence for any $s \in L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3})$ there exist measurable maps $\alpha^\pm : (0, r) \rightarrow \text{Ker}(\mathcal{D}_k^\pm)$ and $s_\nu^\pm : (0, r) \rightarrow \mathbb{C}$ such that we have

$$s = \alpha^+ + \alpha^- + \sum_\nu (s_\nu^+ f_\nu^+ + s_\nu^- f_\nu^-)$$

and

$$\|s\|_{L^2}^2 = \|r_3 \alpha^+\|_{L^2}^2 + \|r_3 \alpha^-\|_{L^2}^2 + \sum_\nu (\|r_3 s_\nu^+\|_{L^2}^2 + \|r_3 s_\nu^-\|_{L^2}^2).$$

By some linear-algebraic operations and Lemma 2.13, we can take an element $t = \beta^+ + \beta^- + \sum_\nu (t_\nu^+ f_\nu^+ + t_\nu^- f_\nu^-) \in L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3})$ such that we have $\mathcal{D}_{(A_k, \Phi_k)}^+(t) = s$ and

$$\begin{aligned} \|t\|_{L^2} &\leq \|r_3 \beta^+\|_{L^2} + \|r_3 \beta^-\|_{L^2} + \sum_\nu (\|r_3 t_\nu^+\|_{L^2} + \|r_3 t_\nu^-\|_{L^2}) \\ &\leq r(1 + \sqrt{e}) \left\{ C_{1+k/2} \|r_3 \alpha^+\|_{L^2} + C_{1-k/2} \|r_3 \alpha^-\|_{L^2} \right. \\ &\quad \left. + \sum_\nu \left\{ \max(C_{1+(2q_\nu+k)/2}, C_{1-(2q_\nu+k)/2}) (\|r_3 s_\nu^+\|_{L^2} + \|r_3 s_\nu^-\|_{L^2}) \right\} \right\}, \end{aligned}$$

where C_α is the constant in Lemma 2.13. Then We set $G^+(s) := t$, and G^+ is linear because all constructions of G^+ are linear. Since $C_{1\pm(2q_\nu+k)/2} = o(1)$ ($\nu \rightarrow \infty$), the compactness of G^+ is deduced from the compactness of K_α in Lemma 2.13.

By the triangle inequality, we also have

$$\|t\|_{L^3} \leq \|r_3^{2/3} \beta^+\|_{L^3} + \|r_3^{2/3} \beta^-\|_{L^3} + \sum_\nu \left(\|r_3^{2/3} t_\nu^+\|_{L^3} \cdot \|f_\nu^+\|_{L^3} + \|r_3^{2/3} t_\nu^-\|_{L^3} \cdot \|f_\nu^-\|_{L^3} \right).$$

By Lemma 2.13, we have $\|r_3^{2/3} \beta^+\|_{L^3} \leq C_{1+k/2} r^{2/3} (1 + \sqrt{3/2e})^{1/3} \cdot \|f\|_{L^2}$. Moreover, similar estimates hold for β^- and $s_\nu^\pm$ for any $\nu \in \mathbb{N}$. By the definition we have $2\sqrt{q_\nu} \cdot C_{1\pm(2q_\nu+k)/2} \rightarrow 1$ ($\nu \rightarrow \infty$). Hence $\|f_\nu^\pm\|_{L^3} \cdot C_{1\pm(2q_\nu+k)/2} = O(1)$ by Lemma 2.9. Therefore there exists $C > 0$ such that C is independent of s and we have $\|t\|_{L^3} < C \|s\|_{L^2}$. This is the desired estimate. $\square$

Corollary 2.15. For any positive numbers $r > r' > 0$, the restriction map $L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3}) \cap \text{Dom}(\mathcal{D}_{(A_k, \Phi_k)}^\pm) \rightarrow L^2(B(r')^*, L_k \otimes S_{\mathbb{R}^3})$ is a compact operator.

Proof. Let $\{f_n\}$ be a bounded sequence in $L^2(B(r)^*, L_k \otimes S_{\mathbb{R}^3}) \cap \text{Dom}(\not{D}_{(A_k, \Phi_k)}^\pm)$. By using $G^\pm$ in Proposition 2.14, we set $\tilde{f}_n := G^\pm(\not{D}_{(A_k, \Phi_k)}^\pm(f_n))$. Since $G^\pm$ is compact, there exists a subsequence $\{f_{n_k}\}$ such that $\{\tilde{f}_{n_k}\}$ is convergent. Hence we may assume that $\{\tilde{f}_n\}$ is convergent. Then we have $\not{D}_{(A_k, \Phi_k)}^\pm(f_n - \tilde{f}_n) = 0$. By Corollary 2.12, $\{(f_n - \tilde{f}_n)|_{B(r')}\}$ has a convergent subsequence. Therefore $\{f_n|_{B(r')}\}$ also has a convergent subsequence. $\square$

2.3 A local lift of the Dirac operators of the flat Dirac monopoles

Let $k \in \mathbb{Z}$ be an integer. Let (t, x, y) be the standard coordinate of $\mathbb{R}^3$. For the flat Dirac monopole $(V, h, A, \Phi) := (L_k, h_k, A_k, \sqrt{-1}k/2r_3)$ on $(\mathbb{R}^3, \{0\})$, we denote by (V_4, h_4, A_4) the prolongation of the instanton $(\pi^*V, \pi^*h, \pi^*A - \xi \otimes \pi^*\Phi)$ over $\mathbb{R}^4$, where $\xi = 2\{(x_1 dy_1 - y_1 dx_1) - (x_2 dy_2 - y_2 dx_2)\}$. We compare the Dirac operators $\not{D}_{(A, \Phi)}^\pm$ and $\not{D}_{A_4}^\pm$.

We denote by X and P the punctured spaces $\mathbb{R}^3 \setminus \{0\}$ and $\mathbb{R}^4 \setminus \{0\}$ respectively. Set the function $f : X = \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}_+$ to be $f(t, x, y) := 1/2r_3$. We also set $g_P := 2g_{4, \text{Euc}}$. Since $g_P = 2g_{4, \text{Euc}} = \pi^*f(\pi^*g + \xi^2)$, we have the orthogonal decomposition $TP \simeq \mathbb{R}\partial_\theta \oplus \pi^*TX$. Let $\mathcal{S}$ be the spin structure of $\mathbb{R}^3$ i.e. $\mathcal{S}$ is a principal $Spin(3)$ -bundle on $\mathbb{R}^3$ that satisfies $\mathcal{S} \times_{Spin(3)} (\mathbb{R}^3, g_{3, \text{Euc}}) \simeq T\mathbb{R}^3$. Let $\rho : Spin(3) \rightarrow Spin(4)$ be the lift of the homomorphism $SO(3) \rightarrow SO(4)$ which is induced by $\mathbb{R}^3 \ni p \rightarrow (0, p) \in \mathbb{R}^4$. We set $\mathcal{S}_4 := \pi^*(\mathcal{S}) \times_\rho Spin(4)$. Then we have $\mathcal{S}_4 \times_{Spin(4)} (\mathbb{R}^4 \setminus \{0\}) \simeq \underline{\mathbb{R}} \oplus \pi^*TX$, where $\underline{\mathbb{R}}$ is a trivial bundle on P . Hence $\mathcal{S}_4$ is a spin structure on P . Under the isomorphisms $Spin(3) \simeq SU(2)$ and $Spin(4) \simeq SU(2)_+ \times SU(2)_-$, the homomorphism ρ is written as $\rho(g) = (g, g)$. Therefore we have the following proposition.

Proposition 2.16. The following claims are satisfied.

- We have the unitary isomorphisms $\pi^*S_X \simeq S_P^\pm$.
- Under the above isomorphisms, the Clifford product on P can be represented as follows:

$$\begin{aligned} \text{clif}_P(\xi) &= (\pi^*f)^{-1/2} \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}. \\ \text{clif}_P(\pi^*\alpha) &= (\pi^*f)^{-1/2} \begin{pmatrix} 0 & \text{clif}_X(\alpha) \\ \text{clif}_X(\alpha) & 0 \end{pmatrix} \quad (\alpha \in \Gamma(X, \Omega^1(X))). \end{aligned}$$

Since the isomorphisms $\pi^*S_X \simeq S_P^\pm$ are unitary, we have $\|\pi^*s\|_{L^2(P)}^2 = \int_P |\pi^*s|^2 (-\pi^*f^2 \cdot \xi \wedge \pi^*d\text{vol}_X) = 2\pi \|f^{1/2}s\|_{L^2(X)}^2$ for any $s \in \Gamma(X, V \otimes S_X)$. Hence the operator $\pi_\pm^\dagger(s) := \pi^*((2\pi f)^{-1/2}s)$ are isometric isomorphisms between $L^2(X, V \otimes S_X)$ and $L^2(P, V_4 \otimes S_P^\pm)^{S^1}$ respectively.

On one hand, we take a global flat unitary frame $e^3 = (e_1^3, e_2^3)$ of S_X on which $\text{clif}_X(dt)$, $\text{clif}_X(dx)$ and $\text{clif}_X(dy)$ act as the Pauli matrices σ_1 , σ_2 and σ_3 , respectively. Here the Pauli matrices are defined as follows:

$$\sigma_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

On the other hand, we have the isomorphisms $S_P^+ \simeq \Omega_{\mathbb{C}^2}^{0,0} \oplus \Omega_{\mathbb{C}^2}^{0,2}$ and $S_P^- \simeq \Omega_{\mathbb{C}^2}^{0,1}$. Moreover, under the isomorphisms we also have $\not{D}_{A_4} = \sqrt{2}(\bar{\partial}_{A_4} + \bar{\partial}_{A_4}^\star)$ and $\text{clif}_P(\alpha) = \sqrt{2}(\alpha^{(0,1)} \wedge$

$-\lrcorner(\alpha^{(1,0)})^\flat$ for a 1-form α on P , where $\lrcorner$ means the interior product and $(\alpha)^\flat$ is the image of α under the isomorphism $\Omega_{\mathbb{C}^2}^{1,0} \simeq T^{(0,1)}\mathbb{C}^2$ induced by the metric g_P . Here we set S^1 -invariant global unitary frames $e^\pm = (e_1^\pm, e_2^\pm)$ of $S_P^\pm$ to be the following.

$$\begin{aligned} e_1^+ &:= 1. \\ e_2^+ &:= -(\pi^*(-\xi)^{0,1}/|\pi^*(-\xi)^{0,1}|) \wedge (\pi^*(d\bar{z})/|\pi^*(d\bar{z})|). \\ e_1^- &:= \pi^*(-\xi)^{0,1}/|\pi^*(-\xi)^{0,1}|. \\ e_2^- &:= \pi^*(d\bar{z})/|\pi^*(d\bar{z})|. \end{aligned}$$

Then, with respect to the frames $e^\pm$ and e^3 , the representations of Clifford products of X and P coincide as in the sense of Proposition 2.16. Therefore we may assume $\pi^*e^3 = e^\pm$. Hence by a direct calculation we obtain the following proposition.

Proposition 2.17. For the flat Dirac monopole (V, h, A, Φ) , the equalities

$$\pi_-^\dagger \circ \left(\not\partial_{(A, \Phi)}^+ f^{-1/2} \right) (s) = \not\partial_{A_4}^+ \circ \pi_+^\dagger (s)$$

and

$$\pi_+^\dagger \circ \left(f^{-1/2} \not\partial_{(A, \Phi)}^- \right) (s) = \not\partial_{A_4}^- \circ \pi_-^\dagger (s)$$

are satisfied for any $s \in \Gamma(X, V \otimes S_X)$.

3 An index formula of Dirac operators on compact 3-dimensional manifold

Let (X, g) be a closed oriented spin 3-dimensional manifold and Z a finite subset. Let $\mathcal{S}$ be a spin structure on (X, g) i.e. $\mathcal{S}$ is a principal $Spin(3)$ -bundle on X such that $\mathcal{S} \times_{Spin(3)} (\mathbb{R}^3, g_{3, \text{Euc}}) \simeq (TX, g)$. Let (V, h, A, Φ) be a Dirac-type singular monopole on (X, Z) of rank r , and we denote by $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^r$ the weight of (V, h, A, Φ) at each $p \in Z$.

3.1 Fredholmness of Dirac operators

Let (x_p^1, x_p^2, x_p^3) be a normal coordinate at p on $B(p, \varepsilon)$, and set the flat metric g' on $B(Z, \varepsilon)$ to be $g'|_{B(p, \varepsilon)} := \sum_i (dx_p^i)^2$. We take a smooth bump function $\rho : X \rightarrow [0, 1]$ satisfying $\rho(B(Z, \varepsilon/2)) = 1$ and $\rho(X \setminus B(Z, 3\varepsilon/4)) = 0$, and set a metric $\tilde{g} := (1 - \rho)g + \rho \cdot g'$. Then we have $g|_{X \setminus B(Z, \varepsilon)} = \tilde{g}|_{X \setminus B(Z, \varepsilon)}$ and $|g - \tilde{g}|_g = O(R_p^2)$ on $B(p, \varepsilon)$ for any $p \in Z$, where R_p is the distance from p . Hence there exists an isometric isomorphism $\mu : (TX, g) \simeq (TX, \tilde{g})$ such that $\mu|_{X \setminus B(Z, \varepsilon)} = \text{Id}_{TX}$ and $|\mu - \text{Id}_{TX}|_g = O(R_p^2)$ on $B(p, \varepsilon)$ for any $p \in Z$. Therefore we obtain the following lemma.

Lemma 3.1. For a 1-form α , we have an equality $\text{clif}_{(X, g)}(\alpha)|_{X \setminus B(Z, \varepsilon)} = \text{clif}_{(X, \tilde{g})}(\alpha)|_{X \setminus B(Z, \varepsilon)}$ and an estimate $|\text{clif}_{(X, g)}(\alpha) - \text{clif}_{(X, \tilde{g})}(\alpha)| = |\alpha| \cdot O(R_p^2)$ on $B(p, \varepsilon)$ for any $p \in Z$, where $\text{clif}_{(X, g)}$ and $\text{clif}_{(X, \tilde{g})}$ denote the Clifford product with respect to g and $\tilde{g}$ respectively.

We also take a direct sum of the flat Dirac monopoles (V', h', A', Φ') on $(B(Z, \varepsilon) \setminus Z, g')$ to be $(V', h', A', \Phi')|_{B(p, \varepsilon)} = \bigoplus_{i=1}^r (L_{k_{p,i}}, h_{k_{p,i}}, A_{k_{p,i}}, \Phi_{k_{p,i}})$ for any $p \in Z$. By Corollary 2.8, there exists a unitary isomorphism $\varphi : V|_{B(Z, \varepsilon) \setminus Z} \simeq V'$ such that the estimates in Corollary 2.8 are satisfied. We set a connection $\tilde{A} := (1 - \rho)A + \rho \cdot \varphi^* A'$ and an endomorphism

$\tilde{\Phi} := (1 - \rho)\Phi + \rho \cdot \varphi^* \Phi'$. Then for each $p \in Z$ the restriction $(V, h, \tilde{A}, \tilde{\Phi})|_{B(p, \varepsilon/2) \setminus \{p\}}$ is a direct sum of the flat Dirac monopoles, and $|A - \tilde{A}|$ and $|\Phi - \tilde{\Phi}|$ are bounded on $X \setminus Z$.

We denote by $\tilde{\partial}_{(A, \Phi)}^\pm$ and $\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm$ the Dirac operators of the tuples (V, h, A, Φ) and $(V, h, \tilde{A}, \tilde{\Phi})$ with respect to the metrics $\tilde{g}$ respectively. In Proposition 3.3, we show the Fredholmness of $\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm$. Consequently, we will prove the Fredholmness of $\tilde{\partial}_{(A, \Phi)}^\pm$ in Theorem 3.4.

Proposition 3.2. The injection maps $\text{Dom}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm) \rightarrow L^2(X \setminus Z, V \otimes S_X)$ are compact.

Proof. The norm $\|s\|_1 := \|s|_{X \setminus B(Z, \varepsilon/8)}\|_{L^2} + \|s|_{B(Z, \varepsilon/4) \setminus Z}\|_{L^2}$ on $L^2(X \setminus Z, V \otimes S_X)$ is equivalent to the ordinary L^2 -norm on X . By the Rellich-Kondrachov theorem, the restriction maps $\text{Dom}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm) \ni s \rightarrow s|_{X \setminus B(Z, \varepsilon/8)} \in L^2(X \setminus B(Z, \varepsilon/8), S_X \otimes V)$ are compact. By Corollary 2.15, the restriction maps $\text{Dom}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm) \ni s \rightarrow s|_{B(Z, \varepsilon/4) \setminus Z} \in L^2(B(Z, \varepsilon/4) \setminus Z, S_X \otimes V)$ are also compact. Hence the injection maps $\text{Dom}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm) \rightarrow L^2(X \setminus Z, V \otimes S_X)$ are compact. $\square$

Proposition 3.3. The Dirac operators $\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm : L^2(X \setminus Z, V \otimes S_X) \rightarrow L^2(X \setminus Z, V \otimes S_X)$ are closed Fredholm operators and adjoint to each other.

Proof. We show that $\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm$ are adjoint to each other. For a densely defined closed operator F , we denote by F^* the adjoint of F . Take $\alpha \in \text{Dom}((\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm)^*)$. Then we have $\langle (\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm)^*(\alpha), \varphi \rangle_{L^2} = \langle \alpha, \tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm(\varphi) \rangle_{L^2}$ for any $\varphi \in C_0^\infty(X \setminus Z, V \otimes S_X)$, where $C_0^\infty(X \setminus Z, V \otimes S_X)$ denotes the set of compact-supported smooth sections. Therefore $\alpha \in \text{Dom}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\mp)$ and $(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm)^*(\alpha) = \tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\mp(\alpha)$. We show the converse. Take $a \in \text{Dom}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\mp)$ and $b \in \text{Dom}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm)$. Because of the elliptic regularity, Proposition 2.10 and Proposition 2.14, we obtain $|a|, |b| \in L^3(X \setminus Z)$. Let $\kappa : \mathbb{R} \rightarrow [0, 1]$ be a smooth function that satisfies the conditions $\kappa((-\infty, -1]) = \{0\}$, $\kappa([-1/2, \infty)) = \{1\}$. Set $\psi_n(x) = \kappa(n^{-1} \cdot \log(d_{\tilde{g}}(x, Z)))$ for $n \in \mathbb{N}$, where we set $d_{\tilde{g}}(x, Z) := \min\{d_{\tilde{g}}(x, p) \mid p \in Z\}$. Since $\psi_n a$ has a compact support on $X \setminus Z$, we have $\langle \psi_n a, \tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm(b) \rangle_{L^2} = \langle \tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\mp(\psi_n a), b \rangle_{L^2} = \langle \psi_n \tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\mp(a), b \rangle_{L^2} + \langle \text{clif}_X(d\psi_n)a, b \rangle_{L^2}$. Since we have $|d\kappa(n^{-1} \log(x))/dx| \leq (x|\log(x)|)^{-1} \cdot \|\kappa'\|_{L^\infty}$ for $0 < x < 1$, $|d\psi_n|$ is dominated by an L^3 -function that is independent of n . Hence we obtain $\langle a, \tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm(b) \rangle_{L^2} = \langle \tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\mp(a), b \rangle_{L^2}$ by the dominated convergence theorem. Therefore $a \in \text{Dom}((\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm)^*)$ and $(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm)^*(a) = \tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\mp(a)$.

We show that the kernel of $\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm$ is finite-dimensional. By Proposition 3.2, the identity map of $\text{Ker}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm)$ is a compact operator. Hence we obtain $\dim(\text{Ker}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm)) < \infty$. Since the Dirac operators $\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm$ are adjoint to each other, the claim $\dim(R(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm)^\perp) < \infty$ can be deduced from $\dim(\text{Ker}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm)) < \infty$, where $\perp$ means the orthogonal complement in L^2 .

To prove that $R(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm)$ is closed, it suffices to show that there exists a constant $C > 0$ such that the condition $\|s\|_{L^2} < C\|\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm(s)\|_{L^2}$ holds for any $s \in \text{Dom}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm) \cap (\text{Ker}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm))^\perp$. Suppose that there is no such a constant $C > 0$, then we can take a sequence $\{s_n\} \subset \text{Dom}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm) \cap (\text{Ker}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm))^\perp$ such that the conditions $\|s_n\| = 1$ and $\|\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm(s_n)\|_{L^2} <$

$1/n$ are satisfied. By Proposition 3.2, we may assume that $\{s_n\}$ converges to some $s_\infty \in \left(\text{Ker}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm)\right)^\perp$. Since $\|s_n\|_{L^2} = 1$ for any $n \in \mathbb{N}$, we have $s_\infty \neq 0$. However, by definition $\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm(s_n)$ converges to 0 as $n \rightarrow \infty$. Thus we obtain $\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm(s_\infty) = 0$, and hence $s_\infty \in \text{Ker}(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm)$, which is impossible. Therefore the condition holds for some $C > 0$ and $R(\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm)$ is closed. $\square$

Theorem 3.4. The Dirac operators $\partial_{(A, \Phi)}^\pm$ are closed Fredholm operators and adjoint to each other, and they have the same indices of $\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm$.

Proof. Since $|A - \tilde{A}|$ and $|\Phi - \tilde{\Phi}|$ are bounded on $X \setminus Z$, by Proposition 3.2 the operators $\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm$ are closed Fredholm operators and adjoint to each other, and they have the same indices of $\tilde{\partial}_{(\tilde{A}, \tilde{\Phi})}^\pm$.

We will prove that $\tilde{\partial}_{(A, \Phi)}^\pm$ and $\partial_{(A, \Phi)}^\pm$ have the same domains and indices. First we show $\text{Dom}(\partial_{(A, \Phi)}^\pm) \subset \text{Dom}(\tilde{\partial}_{(A, \Phi)}^\pm)$. By Lemma 3.1, there exists $C_0 > 0$ such that the estimate

$$|\tilde{\partial}_{(A, \Phi)}^\pm(s) - \partial_{(A, \Phi)}^\pm(s)| < C_0 R_p^2 \cdot |\nabla_{A_X \otimes A}(s)| \quad (1)$$

holds on a neighborhood of $p \in Z$ for any $s \in \Gamma(X \setminus Z, S_X \otimes V)$. Let $\kappa : [0, \infty] \rightarrow [0, 1]$ be a smooth bump function satisfying

$$\kappa(x) = \begin{cases} 0 & (1 \leq x) \\ 1 & (3/8 \leq x \leq 3/4) \\ 0 & (x \leq 1/3). \end{cases}$$

For $\delta > 0$, we set a function $\varphi_\delta(x) = \kappa(\delta^{-1}d_g(x, Z))$ on X . By the Weitzenböck formula $\nabla_{A_X \otimes A}^{\star g} \nabla_{A_X \otimes A} = \partial_{(A, \Phi)}^- \partial_{(A, \Phi)}^+ - \Phi^2 + Sc(g)$, we have $\|\nabla_{A_{S_X} \otimes A}(\varphi_\delta s)\|_{L^2}^2 = \|\partial_{(A, \Phi)}^+(\varphi_\delta s)\|_{L^2}^2 + \|\Phi(\varphi_\delta s)\|_{L^2}^2 + \int_X Sc(g)|\varphi_\delta s|^2 d\text{vol}_M$ for any $s \in \Gamma(X \setminus Z, V \otimes S_X)$, where $\nabla_{A_X \otimes A}^{\star g}$ is the formal adjoint of $\nabla_{A_X \otimes A}$ with respect to g and $Sc(g)$ is the scalar curvature of g . Therefore, there exists $C_1 > 0$ such that for any sufficiently small $\delta > 0$ we have $\|\nabla_{A_{S_X} \otimes A}(s)\|_{L^2(U_p(3\delta/8, 3\delta/4))} \leq C_1(\|\partial_{(A, \Phi)}^\pm(s)\|_{L^2(U_p(\delta/3, \delta))} + \delta^{-1}\|s\|_{L^2(U_p(\delta/3, \delta))})$ holds for any $s \in \Gamma(X \setminus Z, V \otimes S_X)$, where we put $U_p(\delta_1, \delta_2) := \{x \in X \mid \delta_1 < d_g(p, x) < \delta_2\}$. We set $\delta_i := 4\varepsilon/(3 \cdot 2^i)$ for $i \in \mathbb{Z}_{\geq 0}$. Then we have

$$\begin{aligned} & \|\tilde{\partial}_{(A, \Phi)}^\pm(s) - \partial_{(A, \Phi)}^\pm(s)\|_{L^2(B(p, \varepsilon))} \\ & \leq C_0 \|R_p^2 \cdot \nabla_{A_X \otimes A}(s)\|_{L^2(B(p, \varepsilon))} \\ & \leq C_0 \sum_{i=0}^{\infty} \|R_p^2 \cdot \nabla_{A_X \otimes A}(s)\|_{L^2(U_p(3\delta_i/8, 3\delta_i/4))} \\ & \leq C_0 C_1 \sum_{i=0}^{\infty} \left\{ \delta_i^2 \|\partial_{(A, \Phi)}^\pm(s)\|_{L^2(U_p(\delta_i/3, \delta_i))} + \delta_i \|s\|_{L^2(U_p(\delta_i/3, \delta_i))} \right\} \\ & \leq C_0 C_1 \left(\delta_0^2 \|\partial_{(A, \Phi)}^\pm(s)\|_{L^2(B(p, \delta_0))} + \delta_0 \|s\|_{L^2(B(p, \delta_0))} \right) \end{aligned}$$

Hence there exists $C_2 = C_2(\varepsilon) > 0$ such that $\|\tilde{\partial}_{(A, \Phi)}^\pm(s) - \partial_{(A, \Phi)}^\pm(s)\|_{L^2} \leq C_2(\|s\|_{L^2} + \|\partial_{(A, \Phi)}^\pm(s)\|_{L^2})$, and we have $C_2 = O(\varepsilon)$ ($\varepsilon \rightarrow 0$). Hence we obtain $\text{Dom}(\partial_{(A, \Phi)}^\pm) \subset \text{Dom}(\tilde{\partial}_{(A, \Phi)}^\pm)$.

We show the converse. Let $\tilde{A}_X$ denote the connection on S_X induced by the Levi-Civita connection of $(X, \tilde{g})$. By the definition of Dirac-type singularity, we have $|\nabla_A(\Phi)| = |F(A)| = O(R_p^{-2})$ around $p \in Z$. Therefore from the Weitzenböck formula $\nabla_{\tilde{A}_X \otimes A}^{\star \tilde{g}} \nabla_{\tilde{A}_X \otimes A} = \tilde{\partial}_{(A, \Phi)}^- \tilde{\partial}_{(A, \Phi)}^+ - \Phi^2 + \text{clif}(\nabla_A(\Phi) - *_g F(A)) + Sc(\tilde{g})$ and a similar argument as above, it follows that there exists $C_3 = C_3(\varepsilon) > 0$ such that $\|\tilde{\partial}_{(A, \Phi)}^\pm(s) - \partial_{(A, \Phi)}^\pm(s)\|_{L^2} \leq C_3(\|s\|_{L^2} + \|\tilde{\partial}_{(A, \Phi)}^\pm(s)\|_{L^2})$, and $C_3 = O(\varepsilon)$ ($\varepsilon \rightarrow 0$). Therefore $\text{Dom}(\tilde{\partial}_{(A, \Phi)}^\pm) = \text{Dom}(\partial_{(A, \Phi)}^\pm)$. Moreover, their graph norms are also equivalent. It is a well-known fact that sufficiently small deformations of Fredholm operators remain Fredholm. Hence the operators $\tilde{\partial}_{(A, \Phi)}^\pm$ are closed Fredholm operators, and they have the same indices as ones of $\partial_{(\tilde{A}, \tilde{\Phi})}^\pm$. $\square$

3.2 An index calculation on a compact 3-dimensional manifolds

3.2.1 A lift of singular monopoles to closed 4-dimensional manifolds

For an arbitrary 3-dimensional manifold N , a point $x \in N$ and a principal S^1 -bundle P on $N \setminus \{x\}$, we set $\deg_x(P) := \int_{\partial B} c_1(P)$, where B is a small neighborhood of x .

We take a finite subset $Z' \subset X$ satisfying the conditions $|Z'| = |Z|$ and $Z \cap Z' = \emptyset$, and set $\tilde{Z} = Z \cup Z'$. Here we consider the following lemma.

Lemma 3.5. There exists a principal S^1 -bundle $\pi : P \rightarrow X \setminus \tilde{Z}$ such that we have $\deg_p(P) = 1$ for $p \in Z$ and $\deg_{p'}(P) = -1$ for $p' \in Z'$.

Proof. By taking the first Chern class, the set of isomorphism classes of S^1 -principal bundles on a manifold M is bijective to $H^2(M, \mathbb{Z})$. Moreover, this bijection is compatible with the pull-back by a continuous map. Hence we consider the cohomology group on $X \setminus \tilde{Z}$.

We fix a sufficiently small $\varepsilon > 0$. On $B_\varepsilon(\tilde{Z}) \setminus \tilde{Z}$, we can take a principal S^1 -bundle P_0 such that $\deg_p(P_0) = 1$ for $p \in Z$ and $\deg_{p'}(P_0) = -1$ for $p' \in Z'$. Then the cohomology class $\alpha = (a_p)_{p \in \tilde{Z}} \in H^2(B_\varepsilon(\tilde{Z}) \setminus \tilde{Z}, \mathbb{Z}) \simeq \mathbb{Z}^{\tilde{Z}}$ corresponding to P_0 by the above bijection is given as follows:

$$a_p = \begin{cases} 1 & (p \in Z) \\ -1 & (p \in Z'), \end{cases}$$

where $\mathbb{Z}^{\tilde{Z}}$ is the set of maps from $\tilde{Z}$ to $\mathbb{Z}$. We consider an open covering $X = B_\varepsilon(\tilde{Z}) \cup (X \setminus \tilde{Z})$, and obtain an exact sequence $H^2(X \setminus \tilde{Z}, \mathbb{Z}) \rightarrow H^2(B_\varepsilon(\tilde{Z}) \setminus \tilde{Z}, \mathbb{Z}) \simeq \mathbb{Z}^{\tilde{Z}} \rightarrow H^3(X, \mathbb{Z}) \simeq \mathbb{Z}$ by the Mayer-Vietoris argument and $H^2(B_\varepsilon(\tilde{Z}), \mathbb{Z}) = 0$. Then, the boundary map $\delta : H^2(B_\varepsilon(\tilde{Z}) \setminus \tilde{Z}, \mathbb{Z}) \rightarrow H^3(X, \mathbb{Z})$ is written as $\delta((b_p)_{p \in \tilde{Z}}) = \sum b_p$ under the isomorphisms $H^2(B_\varepsilon(\tilde{Z}) \setminus \tilde{Z}, \mathbb{Z}) \simeq \mathbb{Z}^{\tilde{Z}}$ and $H^3(X, \mathbb{Z}) \simeq \mathbb{Z}$. Since we have $|Z| = |Z'|$, the equality $\delta(\alpha) = 0$ holds. Therefore there exists a principal S^1 -bundle P such that $P|_{B_\varepsilon(\tilde{Z}) \setminus \tilde{Z}} \simeq P_0$, which proves the lemma. $\square$

By the above lemma, we take a principal S^1 -bundle P on $X \setminus \tilde{Z}$. We take a metric $\hat{g}$ on X such that $\tilde{g}$ is flat on $B(\tilde{Z}, \varepsilon/2)$ in a similar manner to $\tilde{g}$. Let $f : X \setminus \tilde{Z} \rightarrow \mathbb{R}_+$ be a smooth function. Let $\omega \in \Omega^1(P, \mathbb{R})$ be a connection of P . We assume that for any $p \in Z$ (resp. Z') the tuple $((P, \omega) \times_{S^1} (\mathbb{C}, h_{\mathbb{C}}), \sqrt{-1}f)|_{B(p, \varepsilon/2)}$ (resp. $((P, \omega) \times_{S^1} (\mathbb{C}, h_{\mathbb{C}}), -\sqrt{-1}f)|_{B(p, \varepsilon/2)})$ is the flat Dirac monopole of weight 1 (resp. -1) with respect to $\hat{g}$. Set a one-form $\xi := \omega/\pi^*f$ and a metric $g_P := \pi^*f(\pi^*\hat{g} + \xi^2)$ on P . We choose the global 4-form $\pi^*f^2(-\xi \wedge \pi^*\text{vol}_{(X, \hat{g})})$ as the orientation of P . As in subsection 2.3, we take the homomorphism $\rho : \text{Spin}(3) \rightarrow \text{Spin}(4)$ and a spin structure $\mathcal{S}_4 := (\pi^*\mathcal{S}) \times_\rho \text{Spin}(4)$ of P , where $\mathcal{S}$ is the spin structure on X .

Hence, as Proposition 2.16 we obtain the unitary isomorphisms $S_P^\pm \simeq \pi^*(S_X|_{X \setminus \tilde{Z}})$ and the following equalities:

$$\begin{aligned} \text{clif}_P(\xi) &= (\pi^* f)^{-1/2} \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix} \\ \text{clif}_P(\pi^* \alpha) &= (\pi^* f)^{-1/2} \begin{pmatrix} 0 & \text{clif}_X(\alpha) \\ \text{clif}_X(\alpha) & 0 \end{pmatrix} \quad (\alpha \in \Omega^1(X \setminus \tilde{Z})). \end{aligned}$$

For $p \in Z$, the restriction $\pi : \pi^*(B(p, \varepsilon/2) \setminus \{p\}) \rightarrow B(p, \varepsilon/2) \setminus \{p\}$ can be identified with the Hopf fibration $(\mathbb{R}^4 \setminus \{0\}) \rightarrow (\mathbb{R}^3 \setminus \{0\})$. For $p' \in Z'$, we can also identify $\pi : \pi^*(B(p', \varepsilon/2) \setminus \{p'\}) \rightarrow B(p', \varepsilon/2) \setminus \{p'\}$ with the inverse-oriented Hopf fibration $(-\mathbb{R}^4 \setminus \{0\}) \rightarrow (\mathbb{R}^3 \setminus \{0\})$, where $-\mathbb{R}^4$ is the differentiable manifold $\mathbb{R}^4$ with the inverse orientation of the standard one of $\mathbb{R}^4$. Hence by taking the one-point compactification on the closure of each $\pi^*(B(p, \varepsilon/2) \setminus \{p\})$, we obtain a closed 4-dimensional manifold $\tilde{P}$ equipped with an S^1 -action. Then g_P can be prolonged to a metric on $\tilde{P}$ as in Lemma 2.4. We extend the projection $\pi : P \rightarrow X \setminus \tilde{Z}$ to the smooth map $\tilde{P} \rightarrow X$, and we denote this map by the same letter π by abuse of notation. Set $Z_4 := \pi^{-1}(Z)$, $Z'_4 := \pi^{-1}(Z')$ and $\tilde{Z}_4 := \pi^{-1}(\tilde{Z})$. Then $\pi|_{\tilde{Z}_4} : \tilde{Z}_4 \rightarrow \tilde{Z}$ is a bijection. We have $\tilde{P} = P \sqcup \tilde{Z}_4$ and $\text{codim}(\tilde{P}, \tilde{Z}_4) = 4$. Hence we obtain natural isomorphisms $\pi_1(P) \simeq \pi_1(\tilde{P})$ and $H^2(P, \mathbb{Z}/2\mathbb{Z}) \simeq H^2(\tilde{P}, \mathbb{Z}/2\mathbb{Z})$. Therefore the orientation and the spin structure of P induce the unique ones of $\tilde{P}$. Thus we obtain the following lemma.

Lemma 3.6. We have the unitary isomorphisms $S_{\tilde{P}}^\pm|_P \simeq S_P^\pm$. Under these isomorphisms, we have $\text{clif}_{\tilde{P}}(v)|_P = \text{clif}_P(v)$ for $v \in \Omega^1(\tilde{P})$.

For the Dirac-type singular monopole (V, h, A, Φ) on (X, Z) , we take a connection $\hat{A}$ and a skew-Hermitian endomorphism $\hat{\Phi}$ that satisfy the following conditions.

- For any $p \in Z$, $(V, h, \hat{A}, \hat{\Phi})|_{B(p, \varepsilon/2) \setminus \{p\}}$ is a direct sum of the flat Dirac monopoles with respect to the metric $\hat{g}$.
- For any $p' \in Z'$, $(V, h, \hat{A})|_{B(p', \varepsilon/2)}$ is a flat unitary bundle and $\hat{\Phi}|_{B(p', \varepsilon/2)} = 0$.
- The differences $|A - \hat{A}|$, $|\Phi - \hat{\Phi}|$ are bounded on $X \setminus \tilde{Z}$.

We denote by $\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^\pm$ the Dirac operators of $(V, h, \hat{A}, \hat{\Phi})$ with respect to the metric $\hat{g}$. By the same argument as Proposition 3.3 and Theorem 3.4, $\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^\pm$ are Fredholm and adjoint to each other, and the indices of $\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^\pm$ are the same as the ones of $\mathcal{D}_{(A, \Phi)}^\pm$.

We set $(V_4, h_4, A_4) := (\pi^* V, \pi^* h, \pi^* \hat{A} - \xi \otimes \pi^* \hat{\Phi})$ on $P \sqcup Z'_4$. By Proposition 2.5, (V_4, h_4, A_4) can be prolonged over $\tilde{P}$, and we denote it by the same symbols. Let $\mathcal{D}_{A_4}^\pm : \Gamma(\tilde{P}, S_P^\pm \otimes V_4) \rightarrow \Gamma(\tilde{P}, S_P^\mp \otimes V_4)$ be the Dirac operators of (V_4, h_4, A_4) . By using the isomorphisms $\pi^*(V \otimes S_X) \simeq (V_4 \otimes S_P^\pm)|_P$, we set operators $\pi_\pm^\dagger : L^2(X \setminus \tilde{Z}, V \otimes S_X) \rightarrow L^2(\tilde{P}, V_4 \otimes S_P^\pm)$ to be $\pi_\pm^\dagger(s) := \pi^*((2\pi f)^{-1/2} s)$. Then $\pi_\pm^\dagger$ preserves L^2 -norms as in subsection 2.3. Since P is a principal S^1 -bundle on X , $\pi_\pm^\dagger$ is an isometric isomorphism from $L^2(X \setminus \tilde{Z}, V \otimes S_X)$ to $L^2(\tilde{P}, V_4 \otimes S_P^\pm)^{S^1}$, where $L^2(\tilde{P}, V_4 \otimes S_P^\pm)^{S^1}$ is the closed subspace of $L^2(\tilde{P}, V_4 \otimes S_P^\pm)$ consisting of S^1 -invariant sections. For $i = 1, 2$, take smooth functions $\lambda_i^\pm : X \setminus \tilde{Z} \rightarrow \mathbb{R}_+$ satisfying the following conditions.

- The equality $\lambda_1^\pm \lambda_2^\pm = f^{-1/2}$ holds.

- The equality $\lambda_1^\pm = \lambda_2^\mp$ holds.
- For any $p \in Z$, $\lambda_1^+|_{B(p,\varepsilon)\setminus\{p\}} = 1$.
- For any $p' \in Z'$, $\lambda_2^+|_{B(p',\varepsilon)\setminus\{p'\}} = 1$.

By Lemma 3.6 and Proposition 2.17, there exist compact-supported smooth endomorphisms $\epsilon^\pm \in \Gamma(X \setminus \tilde{Z}, \text{End}(S_X \otimes V))$ such that we have $\pi_\mp^\dagger(\epsilon^\pm(s)) = \not\partial_{A_4}^\pm \circ \pi_\pm^\dagger(s) - \pi_\mp^\dagger \circ (\lambda_1^\pm \hat{\not\partial}_{(\hat{A}, \hat{\Phi})}^\pm \lambda_2^\pm)(s)$ for any $s \in \Gamma(X \setminus Z, S_X \otimes V)$. Let $D^\pm$ be the differential operator $\lambda_1^\pm \hat{\not\partial}_{(\hat{A}, \hat{\Phi})}^\pm \lambda_2^\pm + \epsilon^\pm$ on $X \setminus \tilde{Z}$. We denote by $\text{Ind}(\not\partial_{A_4}^\pm)^{S^1}$ the S^1 -equivariant index of the closed operator $\not\partial_{A_4}^\pm : L^2(\tilde{P}, V_4 \otimes S_{\tilde{P}}^\pm)^{S^1} \rightarrow L^2(\tilde{P}, V_4 \otimes S_{\tilde{P}}^\mp)^{S^1}$.

Proposition 3.7. Under the isometric isomorphism $\pi^\dagger$, the operators $D^\pm$ and $\not\partial_{A_4}^\pm$ determine the same closed operators respectively. In particular, the operators $D^\pm$ are closed Fredholm operator adjoint to each other, and satisfy $\text{Ind}(D^\pm) = \text{Ind}(\not\partial_{A_4}^\pm)^{S^1}$.

Proof. We take an arbitrary $a \in \text{Dom}(\not\partial_{A_4}^\pm)^{S^1}$, and set $b := \not\partial_{A_4}^\pm(a)$. We will show $(\pi_\pm^\dagger)^{-1}(a) \in \text{Dom}(D^\pm)$ and $D^\pm((\pi_\pm^\dagger)^{-1}(a)) = (\pi_\pm^\dagger)^{-1}(b)$. Let φ be a compact-supported smooth section of $V \otimes S_X$ on $X \setminus \tilde{Z}$. Then $\pi_\mp^\dagger(\varphi)$ also has a compact support. Hence we have $\langle a, (\not\partial_{A_4}^\pm)^\star(\pi_\mp^\dagger(\varphi)) \rangle_{L^2} = \langle b, \pi_\mp^\dagger(\varphi) \rangle_{L^2}$. Since $(\pi_\pm^\dagger)$ is isometric, we obtain $\langle (\pi_\pm^\dagger)^{-1}(a), (D^\pm)^\star(\varphi) \rangle = \langle (\pi_\pm^\dagger)^{-1}(b), \varphi \rangle$. Therefore we have $(\pi^\dagger)^{-1}(a) \in \text{Dom}(D^\pm)$ and $D^\pm((\pi_\pm^\dagger)^{-1}(a)) = (\pi_\pm^\dagger)^{-1}(b)$. We prove the converse. We take an arbitrary $c \in \text{Dom}(D^\pm)$, and set $d := D^\pm(c)$. Let χ be a compact-supported smooth section of $V_4 \otimes S_{\tilde{P}}^\mp$ on $\tilde{P} \setminus \tilde{Z}_4$. We take the orthogonal decomposition $\chi = \chi^{S^1} + \chi^\perp \in L^2(\tilde{P}, V_4 \otimes S_{\tilde{P}}^\mp)^{S^1} \oplus (L^2(\tilde{P}, V_4 \otimes S_{\tilde{P}}^\mp)^{S^1})^\perp$. Then χ^{S^1} and $\chi^\perp$ are also compact-supported smooth sections on $\tilde{P} \setminus \tilde{Z}_4$, and we have $(\not\partial_{A_4}^\pm)^\star(\chi^{S^1}) \in L^2(\tilde{P}, V_4 \otimes S_{\tilde{P}}^\pm)^{S^1}$ and $(\not\partial_{A_4}^\pm)^\star(\chi^\perp) \in (L^2(\tilde{P}, V_4 \otimes S_{\tilde{P}}^\pm)^{S^1})^\perp$. Hence we obtain $\langle \pi_\pm^\dagger(c), (\not\partial_{A_4}^\pm)^\star(\chi) \rangle_{L^2} = \langle \pi_\pm^\dagger(c), (\not\partial_{A_4}^\pm)^\star(\chi^{S^1}) \rangle_{L^2} = \langle c, (\pi_\pm^\dagger)^{-1}((\not\partial_{A_4}^\pm)^\star(\chi^{S^1})) \rangle_{L^2} = \langle c, (D^\pm)^\star((\pi_\pm^\dagger)^{-1}(\chi^{S^1})) \rangle_{L^2} = \langle d, (\pi_\pm^\dagger)^{-1}(\chi^{S^1}) \rangle_{L^2} = \langle \pi_\mp^\dagger(d), \chi^{S^1} \rangle_{L^2} = \langle \pi_\mp^\dagger(d), \chi \rangle_{L^2}$. Therefore $\not\partial_{A_4}^\pm(\pi_\pm^\dagger(c)) = \pi_\mp^\dagger(d)$ holds on $P = \tilde{P} \setminus \tilde{Z}_4$. Here we consider the following lemma.

Lemma 3.8. Take arbitrary $u \in L^2(\tilde{P}, V_4 \otimes S_{\tilde{P}}^\pm)$ and $v \in L^2(\tilde{P}, V_4 \otimes S_{\tilde{P}}^\mp)$. If u and v satisfy $\not\partial_{A_4}^\pm(u) = v$ on P , then we have $\not\partial_{A_4}^\pm(u) = v$ on whole $\tilde{P}$.

If we admit this lemma, then we obtain $\not\partial_{A_4}^\pm(\pi_\pm^\dagger(c)) = \pi_\mp^\dagger(d)$ on $\tilde{P}$. Hence the proof is complete. $\square$

proof of Lemma 3.8. Take $\varphi \in \Gamma(\tilde{P}, V_4 \otimes S_{\tilde{P}}^\mp)$. Let $\kappa : \mathbb{R} \rightarrow [0, 1]$ be a smooth function that satisfies $\kappa((-\infty, -1)) = \{0\}$ and $\kappa((-1/2, \infty)) = \{1\}$. Set $\psi_n : \tilde{P} \rightarrow [0, 1]$ to be $\psi_n(x) := \kappa(n^{-1} \log(d_{g_P}(x, \tilde{Z}_4)))$ for $n \in \mathbb{N}$. Then $\psi_n \cdot \varphi$ has a compact support on $\tilde{P} \setminus \tilde{Z}_4$. Hence we obtain $\langle u, (\not\partial_{A_4}^\pm)^\star(\psi_n \cdot \varphi) \rangle_{L^2} = \langle u, \psi_n \cdot (\not\partial_{A_4}^\pm)^\star(\varphi) \rangle_{L^2} + \langle u, \text{clif}_{\tilde{P}}(d\psi_n)\varphi \rangle_{L^2} = \langle v, \psi_n \cdot \varphi \rangle_{L^2}$. Since we have an estimate $|d\kappa(n^{-1} \log(x))/dx| \leq (x|\log(x)|)^{-1} \|\kappa'\|_{L^\infty}$ for $0 < x < 1$, $|d\psi_n|$ is dominated by an L^2 -function that is independent of n . Therefore we obtain $\langle u, (\not\partial_{A_4}^\pm)^\star(\varphi) \rangle_{L^2} = \langle v, \varphi \rangle$ by the dominated convergence theorem. $\square$

We will associate the S^1 -invariant indices of $\not\partial_{A_4}^\pm$ and the indices of $\hat{\not\partial}_{(\hat{A}, \hat{\Phi})}^\pm$.

Proposition 3.9. We have $\text{Ind}(\not\partial_{A_4}^\pm)^{S^1} = \text{Ind}(\hat{\not\partial}_{(\hat{A}, \hat{\Phi})}^\pm)$.

Proof. If we prove $\text{Ind}(\hat{\mathcal{D}}_{A_4}^+)^{S^1} = \text{Ind}(\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^+)$, then we obtain $\text{Ind}(\hat{\mathcal{D}}_{A_4}^-)^{S^1} = -\text{Ind}(\hat{\mathcal{D}}_{A_4}^+)^{S^1} = -\text{Ind}(\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^+) = \text{Ind}(\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^-)$ because $\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^\pm$ are adjoint to each other. Hence we only need to prove $\text{Ind}(\hat{\mathcal{D}}_{A_4}^+)^{S^1} = \text{Ind}(\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^+)$. By Proposition 3.7, it suffices to show $\text{Ind}(\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^+) = \text{Ind}(D^+)$. Since the support of ϵ^+ is compact in $X \setminus Z$, $\lambda_1^+ \hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^+ \lambda_2^+$ is a closed Fredholm operator and it has the same index as D^+ . By the same asymptotic analysis in Proposition 2.10, for any solutions $s \in \Gamma(X \setminus \tilde{Z}, S_X \otimes V)$ of the equation $\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^+(s) = 0$, we have $s \in L^2$ if and only if $(\lambda_2^+)^{-1} s \in L^2$. Hence we have the natural equality $\text{Ker}(\lambda_1^+ \hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^+ \lambda_2^+) \cap L^2 = (\lambda_2^+)^{-1} \cdot (\text{Ker}(\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^+) \cap L^2)$, where $(\lambda_2^+)^{-1} \cdot (\text{Ker}(\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^+) \cap L^2)$ means the set $\{(\lambda_2^+)^{-1} \cdot s \mid s \in \text{Ker}(\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^+) \cap L^2\}$. By a similar way, we also have $\text{Cok}(\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^+) \cap L^2 = \text{Ker}(\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^-) \cap L^2$ and $\text{Cok}(\lambda_1^+ \hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^+ \lambda_2^+) \cap L^2 = \text{Ker}(\lambda_1^- \hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^- \lambda_2^-) \cap L^2 = (\lambda_2^-)^{-1} \cdot (\text{Ker}(\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^-) \cap L^2)$. Therefore we obtain $\text{Ind}(\hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^+) = \text{Ind}(\lambda_1^+ \hat{\mathcal{D}}_{(\hat{A}, \hat{\Phi})}^+ \lambda_2^+) = \text{Ind}(D^+)$, which completes the proof. $\square$

By following [2], we calculate the S^1 -equivariant index $\text{Ind}(\hat{\mathcal{D}}_{A_4}^\pm)^{S^1}$.

Lemma 3.10. For $p \in Z_4$ (resp. Z'_4), the weights of the fiber $S_{\tilde{P}}^+|_p$ and $S_{\tilde{P}}^-|_p$ are $(0, 0)$ and $(-1, 1)$ (resp. $(-1, 1)$ and $(0, 0)$) respectively.

Proof. For $p \in Z_4$, the projection $\pi|_{B(p, \varepsilon)} : B(p, \varepsilon) \rightarrow \pi(B(p, \varepsilon))$ can be identified with the Hopf fibration $\mathbb{R}^4 = \mathbb{C}^2 \rightarrow \mathbb{R}^3$ in Section 1. By the natural isomorphisms $S_{\mathbb{C}^2}^+ \simeq \Omega_{\mathbb{C}^2}^{0,0} \oplus \Omega_{\mathbb{C}^2}^{0,2}$ and $S_{\mathbb{C}^2}^- \simeq \Omega_{\mathbb{C}^2}^{0,1}$, the weights of $S_{\tilde{P}}^+|_p$ and $S_{\tilde{P}}^-|_p$ are $(0, 0)$ and $(-1, 1)$ respectively. As a similar way, for $p' \in Z'_4$, the projection $\pi|_{B(p', \varepsilon)} : B(p', \varepsilon) \rightarrow \pi(B(p', \varepsilon))$ can be identified with the inverse-oriented Hopf fibration $-\mathbb{R}^4 \rightarrow \mathbb{R}^3$. Therefore the weights of $S_{\tilde{P}}^+|_{p'}$ and $S_{\tilde{P}}^-|_{p'}$ are $(-1, 1)$ and $(0, 0)$ respectively. $\square$

Proposition 3.11. The S^1 -invariant index $\text{Ind}(\hat{\mathcal{D}}_{A_4}^\pm)^{S^1}$ is given as

$$\text{Ind}(\hat{\mathcal{D}}_{A_4}^\pm)^{S^1} = \mp \sum_{p \in Z} \sum_{k_{p,i} > 0} k_{p,i},$$

where $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^r$ is the weight of the monopole (V, h, A, Φ) at $p \in Z$.

Proof. According to [2], The S^1 -invariant index $\text{Ind}(\hat{\mathcal{D}}_{A_4}^\pm)^{S^1}$ is given as

$$\text{Ind}(\hat{\mathcal{D}}_{A_4}^\pm)^{S^1} = (2\pi)^{-1} \int_{S^1} \sum_{p \in \tilde{Z}_4} \frac{\text{tr}_\theta((S_{\tilde{P}}^\pm \otimes V_4)|_p) - \text{tr}_\theta((S_{\tilde{P}}^\mp \otimes V_4)|_p)}{\text{tr}_\theta(\bigwedge^{-1}(T_p \tilde{P} \otimes \mathbb{C}))} d\theta, \quad (2)$$

where tr_θ is trace of the action by $\theta \in S^1$ and $\bigwedge^{-1}(T_p \tilde{P} \otimes \mathbb{C})$ means the virtual vector space $\bigoplus_{i=0}^\infty (-1)^i \bigwedge^i (T_p \tilde{P} \otimes \mathbb{C})$. Then by Lemma 3.10 we have

$$\begin{aligned} \text{tr}_\theta \left((S_{\tilde{P}}^\pm \otimes V_4)|_p \right) - \text{tr}_\theta \left((S_{\tilde{P}}^\mp \otimes V_4)|_p \right) &= \pm 2(1 - \cos \theta) \sum_i \exp(\sqrt{-1} k_{p,i} \theta) \quad (p \in Z) \\ \text{tr}_\theta \left((S_{\tilde{P}}^\pm \otimes V_4)|_{p'} \right) - \text{tr}_\theta \left((S_{\tilde{P}}^\mp \otimes V_4)|_{p'} \right) &= \mp 2r(1 - \cos \theta) \quad (p' \in Z'). \end{aligned}$$

Since the weight of the complexified tangent space $T_p \tilde{P} \otimes \mathbb{C}$ is given by $(1, 1, -1, -1)$, we obtain

$$\text{tr}_\theta \left(\bigwedge^{-1} (T_p \tilde{P} \otimes \mathbb{C}) \right) = 4(1 - \cos \theta)^2 \quad (\tilde{p} \in \tilde{Z}).$$

Applying the above calculations and the equality $\sum_p \sum_i k_{p,i} = 0$ in Remark 2.2 to (2), we obtain the conclusion. $\square$

Hence we obtain the following corollary.

Corollary 3.12. The indices of the Dirac operators $\not{D}_{(A,\Phi)}^\pm$ are given as follows:

$$\text{Ind}(\not{D}_{(A,\Phi)}^\pm) = \mp \sum_{p \in Z} \sum_{k_{p,i} > 0} k_{p,i} = \pm \sum_{p \in Z} \sum_{k_{p,i} < 0} k_{p,i},$$

where $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^r$ is the weight of the monopole (V, h, A, Φ) at $p \in Z$.

4 An index formula of Dirac operators on 3-dimensional complete manifolds

Let (X, g) be an oriented 3-dimensional complete Riemannian manifold such that the scalar curvature $Sc(g)$ is bounded. We fix a spin structure on X . Let $i : Y \hookrightarrow X$ be a relative compact region with a smooth boundary ∂Y . We take the orientation of ∂Y to satisfy that $\nu \wedge \text{vol}_{\partial Y}$ is positive for the inward normal unit 1-form $\nu \in i^* \Omega^1(X)$.

4.1 The non-singular case

Following [13], we recall the non-singular case. Let (V, h, A) be a Hermitian bundle with a connection on X and Φ be a skew-Hermitian endomorphism on V . We assume the following conditions.

- We have $\Phi|_x, F(A)|_x = O(1)$ as $x \rightarrow \infty$.
- We have $\nabla_A(\Phi)|_x = o(1)$ as $x \rightarrow \infty$.
- The inequality $\inf_{x \in X \setminus Y} \{|\lambda| \mid \lambda \text{ is an eigenvalue of } \Phi|_x\} > 0$ is satisfied.

We call this conditions *the Råde condition*. In [13], Råde proved the following theorem.

Theorem 4.1. The Dirac operators $\not{D}_{(A,\Phi)}^\pm = \not{D}_A \pm \Phi : L^2(X, V \otimes S_X) \rightarrow L^2(X, V \otimes S_X)$ are closed Fredholm, and their indices are given as follows:

$$\text{Ind}(\not{D}_{(A,\Phi)}^\pm) = \mp \int_{\partial Y} \text{ch}(V^+) = \pm \int_{\partial Y} \text{ch}(V^-),$$

where $V^\pm$ is a subbundle of $V|_{\partial Y}$ spanned by the eigenvectors of $\mp \sqrt{-1}\Phi$ with positive eigenvalues.

4.2 The indices of twisted flat Dirac monopoles

Let (L_k, h_k, A_k, Φ_k) be the flat Dirac monopole of weight $k \in \mathbb{Z}$. For $a \in \mathbb{R} \setminus \{0\}$, we set $\Phi_{a,k} := \sqrt{-1}(a + (k/2r_3))$. Then $(L_k, h_k, A_k, \Phi_{a,k})$ is also a Dirac-type monopole on $(\mathbb{R}^3, \{0\})$.

Proposition 4.2. The operators $\not{D}_{(A_k, \Phi_{a,k})}^\pm$ are Fredholm. Moreover, we have $\text{Ind}(\not{D}_{(A_k, \Phi_{a,k})}^\pm) = 0$ if $ak > 0$.

Proof. The Fredholmness of $\mathcal{D}_{(A_k, \Phi_{a,k})}^\pm$ follows from Corollary 2.15 and some standard arguments. The proof for the case $a > 0$ and $k > 0$ works for the case $a < 0$ and $k < 0$ mutatis mutandis. Hence we may assume $a > 0$ and $k > 0$. We take $f_\nu^\pm \in L^2(S^2, S_{S^2}^\pm \otimes \mathcal{O}(k))$ ($\nu \in \mathbb{N}$) and $n_\nu > 0$ as in subsection 2.2. We set vector subspaces $W_0 := H^0(\mathbb{P}^1, \mathcal{O}(k-1)) \times \{0\}$ and $W_\nu := (f_\nu^+, 0)\mathbb{C} \oplus (0, f_\nu^-)\mathbb{C}$ of $L^2(S^2, \mathcal{O}(k) \otimes S_{S^2}) = L^2(S^2, \mathcal{O}(k) \otimes S_{S^2}^+) \times L^2(S^2, \mathcal{O}(k) \otimes S_{S^2}^-)$. Then we have a decomposition $L^2(S^2, \mathcal{O}(k) \otimes S_{S^2}) = \widehat{\bigoplus_{\nu \geq 0} W_\nu}$, where $\widehat{\bigoplus}$ means L^2 -completion of the direct sum. Hence we obtain the decomposition $L^2(\mathbb{R}^3 \setminus \{0\}, L_k \otimes S_{\mathbb{R}^3}) = \widehat{\bigoplus_{\nu \geq 0} W_\nu} \otimes L^2(\mathbb{R}_+, r^2 dr)$, where $L^2(\mathbb{R}_+, r^2 dr)$ is the weighted L^2 -space on $\mathbb{R}_+$ with the norm $\|f\|^2 = \int_{\mathbb{R}_+} r^2 |f(r)|^2 dr$. We denote by E_ν the space $W_\nu \otimes L^2(\mathbb{R}_+, r^2 dr)$. The Dirac operators $\mathcal{D}_{(A_k, \Phi_{a,k})}^\pm$ preserves this decomposition, and hence we obtain $\text{Ind}(\mathcal{D}_{(A_k, \Phi_{a,k})}^\pm) = \sum_\nu \text{Ind}(\mathcal{D}_{(A_k, \Phi_{a,k})}^\pm|_{E_\nu} : E_\nu \rightarrow E_\nu)$. Here we prepare the following lemma.

Lemma 4.3. We take Hermitian matrices

$$A_\nu = \begin{pmatrix} -(2+k)/2 & \sqrt{-1}n_\nu \\ -\sqrt{-1}n_\nu & -(2-k)/2 \end{pmatrix}, \quad B = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}.$$

We set the closed operator $P_\nu : \mathbb{C}^2 \otimes L^2(\mathbb{R}_+, r^2 dr) \rightarrow \mathbb{C}^2 \otimes L^2(\mathbb{R}_+, r^2 dr)$ to be $P_\nu(v) := \partial_r v - (A_\nu v/r + Bv)$. Then P_ν is closed Fredholm and $\text{Ind}(P_\nu) = 0$.

By this lemma we have $\text{Ind}(\mathcal{D}_{(A_k, \Phi_{a,k})}^\pm|_{E_\nu}) = 0$ unless $\nu = 0$. Hence $\text{Ind}(\mathcal{D}_{(A_k, \Phi_{a,k})}^\pm) = \text{Ind}(\mathcal{D}_{(A_k, \Phi_{a,k})}^\pm|_{E_0})$. Moreover, we obtain $\text{Ind}(\mathcal{D}_{(A_k, \Phi_{a,k})}^\pm|_{E_0}) = 0$ by a straightforward calculation. $\square$

(proof of Lemma 4.3). For any $f \in C_0^\infty(\mathbb{R})$ and $c \in \mathbb{R}$, we have $\|(\partial_t + c)f\|_{L^2(\mathbb{R})}^2 = \|\partial_t f\|_{L^2(\mathbb{R})}^2 + \|cf\|_{L^2(\mathbb{R})}^2$. Hence there exists $C, K > 0$ such that $\|f\|_{L^2(\mathbb{R}, r^2 dr)} < C(\|P_\nu(f)\|_{L^2(\mathbb{R}, r^2 dr)} + \|f\|_{L^2([0, K], r^2 dr)})$ for any $f \in \text{Dom}(P_\nu)$. Hence the Fredholmness can be proved by a similar way in Proposition 3.3. We take a function $C_\nu : \mathbb{R}_+ \rightarrow \text{Mat}(2, \mathbb{C})$ as

$$C_\nu(r) := \begin{cases} A_\nu/r & (r \leq 1) \\ B & (r > 1) \end{cases}$$

and set a differential operator $\tilde{P}_\nu$ to be $\tilde{P}_\nu(v) := \partial_r v - C_\nu(r)v$. We also set differential operators $P_\nu^s := (1-s)P_\nu + s\tilde{P}_\nu$ for $s \in [0, 1]$. Then P_ν^s is also closed Fredholm. Moreover, we have $\text{Dom}(P_\nu^s) = \text{Dom}(P_\nu^{s'})$ for any $s, s' \in [0, 1]$. Hence $\{P_\nu^s\}$ forms a continuous family of Fredholm operators from $\text{Dom}(P_\nu)$ to $L^2(\mathbb{R}_+, r^2 dr)$. In particular, $P_\nu^0 = P_\nu$ and $P_\nu^1 = \tilde{P}_\nu$ have the same indices. We can write any elements of the kernels of $\tilde{P}_\nu$ and the adjoint operator $\tilde{P}_\nu^\star$ explicitly, and there are no non-zero L^2 -solutions of $\tilde{P}_\nu(v) = 0$ and $\tilde{P}_\nu^\star(v) = 0$. Hence we obtain $\text{Ind}(P_\nu) = \text{Ind}(\tilde{P}_\nu) = 0$, which is the desired equality. $\square$

4.3 The general case

Let $Z \subset Y$ be a finite subset. Let (V, h, A, Φ) be a Dirac-type singular monopole on (X, Z) of rank r which satisfies the Råde condition. We denote by $\vec{k}_p = (k_{p,i}) \in \mathbb{Z}^r$ the weight of (V, h, A, Φ) at $p \in Z$. The following lemma is used in Theorem 4.5.

Lemma 4.4. Smooth Hermitian vector bundles on S^2 is classified by the rank and the first Chern class.

Proof. Easily seen from $\pi_1(U(n)) \simeq \mathbb{Z}$. $\square$

Theorem 4.5. The Dirac operators $\not{D}_{(A,\Phi)}^\pm$ are Fredholm, and the indices of them are given as follows:

$$\text{Ind}(\not{D}_{(A,\Phi)}^\pm) = \mp \left\{ \sum_{p \in Z} \sum_{k_{p,i} > 0} k_{p,i} + \int_{\partial Y} \text{ch}(V^+) \right\} = \pm \left\{ \sum_{p \in Z} \sum_{k_{p,i} < 0} k_{p,i} + \int_{\partial Y} \text{ch}(V^-) \right\}.$$

Proof. We may assume that X is connected. Fredholmness of $\not{D}_{(A,\Phi)}^\pm$ is an easy consequences of Corollary 2.15 and Theorem 4.1. We calculate the indices of $\not{D}_{(A,\Phi)}^\pm$ by using the excision formula in [4, Appendix B]. We set $k := \sum_{p \in Z} \sum_i k_{p,i}$.

First we consider the case $k = 0$. Let (V_0, h_0, A_0) be a trivial Hermitian bundle of rank r with the trivial connection on S^3 and Φ_0 the zero endomorphism on V_0 . Let U_N be the northern closed ball of S^3 . We take a compact neighborhood U of Z that is diffeomorphic to a closed ball. Then ∂U and ∂U_N are diffeomorphic to S^2 . Here we fix the orientations of ∂U and ∂U_N as stated in the first paragraph in this section. Fix sufficiently small tubular neighborhoods T and T_N of ∂U and ∂U_N respectively, and take an orientation-preserving diffeomorphism $\varphi_0 : T \rightarrow T_N$ such that $\varphi_0|_{\partial U}$ is also an orientation-preserving diffeomorphism between ∂U and ∂U_N . We may assume that $T \cap Z = \emptyset$. We set manifolds $\tilde{X}_0 := ((X \setminus U) \cup T) \sqcup (U_N \cup T_N) / \sim$ and $\tilde{S}_0^3 := (U \cup T) \sqcup ((S^3 \setminus U_N) \cup T_N) / \sim$, where both of the equivalence relations $\sim$ are induced by φ_0 . Then we can construct a complete Riemannian metric on $\tilde{X}_0$ from the metrics on X and S^3 . Since we have $\int_{\partial U} c_1(V) = (\int_{\partial U} - \int_{\partial B(Z, \varepsilon)}) c_1(V) = -\int_{U \setminus B(Z, \varepsilon)} dc_1(V) = 0$ by Stokes' theorem, we obtain $c_1(V)|_{\partial U} = 0$. Hence by Lemma 4.4, $(V, h)|_{\partial U}$ and $(V_0, h_0)|_{\partial U_N}$ are isomorphic under the identification by $\varphi_0|_{\partial U}$. We extend the isomorphism over T and T_N under φ_0 , and denote it by τ_0 . By making compact-supported smooth perturbations on (A, Φ) and (A_0, Φ_0) if necessary, we can glue $(V, h, A, \Phi)|_{(X \setminus U) \cup T}$ and $(V_0, h_0, A_0, \Phi_0)|_{U_N \cup T_N}$ along τ_0 , and obtain $(\tilde{V}^0, \tilde{h}^0, \tilde{A}^0, \tilde{\Phi}^0)$ on $\tilde{X}_0$. By the same method we also obtain $(\tilde{V}_0, \tilde{h}_0, \tilde{A}_0, \tilde{\Phi}_0)$ on $\tilde{S}_0^3 \setminus Z$. Then by the excision formula we have $\text{ind}(\not{D}_{(A,\Phi)}^\pm) + \text{ind}(\not{D}_{(A_0, \Phi_0)}^\pm) = \text{ind}(\not{D}_{(\tilde{A}^0, \tilde{\Phi}^0)}^\pm) + \text{ind}(\not{D}_{(\tilde{A}_0, \tilde{\Phi}_0)}^\pm)$. Hence by Corollary 3.12 and Theorem 4.1 we obtain

$$\text{ind}(\not{D}_{(A,\Phi)}^\pm) = \mp \left\{ \sum_{p \in Z} \sum_{k_{p,i} > 0} k_{p,i} + \int_{\partial Y} \text{ch}(V^+) \right\}.$$

Next we consider the case $k \neq 0$. The proof for the case $k > 0$ remain valid for $k < 0$ mutatis mutandis. Hence we may assume $k > 0$. We set the tuple $(V_1, h_1, A_1, \Phi_1) := (L_{-k}, h_{-k}, A_{-k}, \Phi_{-k, -1}) \oplus (\mathbb{C}^{r-1}, h_{\mathbb{C}^{r-1}}, \underline{d}, -\sqrt{-1}\text{Id}_{\mathbb{C}^{r-1}})$ on $\mathbb{R}^3 \setminus \{0\}$, where $\underline{d}$ is the trivial connection. By Theorem 4.1 and Proposition 4.2 we have $\text{Ind}(\not{D}_{(A_1, \Phi_1)}^\pm) = 0$. Let T_1 be a sufficiently small tubular neighborhood of $\partial(\mathbb{R}^3 \setminus B(0, 1))$ and $\varphi_1 : T \rightarrow T_1$ an orientation-preserving diffeomorphism such that $\varphi_1|_{\partial U}$ is also an orientation-preserving diffeomorphism between ∂U and $\partial(\mathbb{R}^3 \setminus B(0, 1))$. We set manifolds $\tilde{X}_1 := ((X \setminus U) \cup T) \sqcup ((\mathbb{R}^3 \setminus B(0, 1)) \cup T_1) / \sim$ and $\tilde{S}_1^3 := (U \cup T) \sqcup (B(0, 1) \cup T_1) / \sim$ as in the case $k = 0$. Then we can construct a complete metric on $\tilde{X}_1$ as $\tilde{X}_0$. By Lemma 4.4, we take an isomorphism τ_1 between $(V, h)|_T$ and $(V_1, h_1)|_{T_1}$ under φ_1 . Using τ_1 we obtain a tuple $(\tilde{V}^1, \tilde{h}^1, \tilde{A}^1, \tilde{\Phi}^1)$ on $\tilde{X}_1$ and a tuple $(\tilde{V}_1, \tilde{h}_1, \tilde{A}_1, \tilde{\Phi}_1)$ on $\tilde{S}_1^3 \setminus (Z \cup \{0\})$ as in the case $k = 0$. The tuple $(\tilde{V}^1, \tilde{h}^1, \tilde{A}^1, \tilde{\Phi}^1)$ satisfies the Råde condition. Therefore the operators $\not{D}_{(\tilde{A}^1, \tilde{\Phi}^1)}^\pm$ are Fredholm. By the excision formula we

have $\text{Ind}(\not{D}_{(A,\Phi)}^\pm) + \text{Ind}(\not{D}_{(A_1,\Phi_1)}^\pm) = \text{Ind}(\not{D}_{(\tilde{A}^1,\tilde{\Phi}^1)}^\pm) + \text{Ind}(\not{D}_{(\tilde{A}_1,\tilde{\Phi}_1)}^\pm)$. By Corollary 3.12, Theorem 4.1 and $\text{Ind}(\not{D}_{(A_1,\Phi_1)}^\pm) = 0$, we obtain $\text{Ind}(\not{D}_{(A,\Phi)}^\pm) = \mp \left\{ \sum_{k_{p,i}>0} k_{p,i} + \int_{\partial Y} ch(V^+) \right\}$, which completes the proof. $\square$

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