

# Nucleon tomography with gluon saturation effects

Yoshikazu Hagiwara



# Abstract

In this thesis, we investigate the structure of the nucleons and nuclei in terms of partons (quarks and gluons) using the parton distribution functions (PDFs) and their multi-dimensional generalizations. In particular, we focus on the so-called Wigner distribution  $W(x, \mathbf{b}_\perp, \mathbf{k}_\perp)$  which depends not only on the longitudinal momentum fraction  $x$ , but also transverse momentum  $\mathbf{k}_\perp$  and impact parameter  $\mathbf{b}_\perp$ .

In high energy nucleon/nucleus scatterings, the number of partons becomes large and if the energy is enough high then the recombination effects of the gluon become appeared. In this situation, the number of partons is no longer increasing and saturated. In the realistic situation, these effects come into the scattering processes and we have to include this gluon saturation effects. The main topic of this thesis is the nucleon tomography with the gluon saturation effects.

Including the saturation effects, we need to solve the rapidity evolution equation called the Balitsky-Kovchegov equation or the finite color version B-JIMWLK equation with impact parameter dependence. The solution of these equations are related to scattering amplitude of high energy collisions. Recently it is discovered that the Wigner distributions are represented in the scattering amplitude at small Björken  $x$ . With using it, we investigate the properties of the Wigner distribution with the gluon saturation effects. We solve the impact parameter dependent BK equation. Also, we propose the way how to measure the Wigner distribution and applications to the elliptic flow in small systems.



# Acknowledgements

This thesis is my collection of research at graduate school of science of Kyoto University. The author greatly thank professor Yoshitaka Hatta from the Brookhaven National Laboratory for his teaching how to research and advice for my whole graduate course. The author gratitude professor Hideo Suganuma from Kyoto university for his careful attention to the research in the Nuclear theory group in Kyoto university including this thesis. The author gratitude professor Akira Ohnishi from Yukawa institute for theoretical physics for advising the thesis and many stuff around the thesis. The author thanks Dr. Hiroyuki Morita and Dr. Takahiro Ueda for advising to fix the code to calculate the numerical results in this thesis. The author thanks the member in the nuclear theory group in Kyoto university for providing many events to ease my stress from some troubles. The author thanks professor Roman Pasechnik from Lund University, professor Marek Tasevsky from Institute of Physics of Academy of Sciences of the Czech republic, professor Oleg Teryaev from Russia and Dubna International University, professor Bo-Wen Xiao from Central China Normal University and professor Feng Yuan from Lawrence Berkeley National Laboratory for collaborating researches. Finally, I thanks to my family for giving the chance to have research even we have in big trouble.



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# Chapter 1

## Introduction

What have we understood about the structure of the proton at high energy? From Rutherford to the present LHC, we always seek for the understanding of the world from scatterings. We are interested in the detailed structure of particles. We want to figure out the properties of particles from the fundamental theory. However it is not the exact theory, the parton model is intuitive phenomenological model to describe the scattering processes in the early days of the high energy collider physics[1]. After the discovery of the Standard model, it is the precise theory describing the high energy scattering in now days and has no contradiction to experimental data. Quantum Chromodynamics(QCD) in the SM which is  $SU(3)$  non-abelian Yang-Mills theory[2] can accurately describe the strong interaction which governs the dynamics of the quark and gluon. In this theses we investigate the structure of the nucleon in terms of the quarks and gluons.

QCD has two remarkable properties; asymptotic freedom at high energy[3, 4] and color confinement at low energy[5]. At high enough energy, the coupling of QCD is small and perturbation calculation is valid at this energy scale and it is called asymptotic freedom. From now on, there are no color multiplet states being observed in measurements, and it is the color confinement. From the color confinement phenomenon, we need to take into account the color singlet particles, called hadron, in the initial and final state of scatterings. The hadrons cannot be described only in a perturbation theory and need the non-perturbative effects. So we need to know the hadrons from experiments. To do so, we must separate the perturbative part and the non-perturbative part in the physical observables. These separations are rigorously proved in some scattering cross sections; deep inelastic scattering(DIS) and Drell-Yan process[6, 7, 8]. The non-perturbative parts are generally called parton(quark or gluon) distribution functions.

We cannot calculate the parton distribution functions from the perturbative theory in every energy scale, but the energy evolution of them can be calculated in the perturbative QCD framework. We need to take the initial conditions of the parton distribution functions of the energy evolution from the experiments and then evolve the distributions with energy evolution equations obtained by using the perturbative QCD. There are two well used evolution equations of energy; Dokshitzer-Gribov-Lipatov-Altarelli-Parisi(DGLAP)[9, 10, 11] and Balitsky-Fadin-Kuraev-Lipatov(BFKL)[12, 13, 14, 15] evolution equations. The DGLAP equation

is the evolution of the energy of the momentum transfer in  $s \sim Q$  region and the BFKL equation is the evolution of the Björken  $x \sim Q/s$  in  $s \gg Q$  region. Where  $s$  is the center-of-mass energy and  $Q$  is the momentum of the large scale in the process. The size of the probed quarks and gluons becomes smaller as the momentum becomes large. At small  $x$  and large momentum transfer, the number of the partons becomes large as  $x$  decreases. But the density of the partons is not increasing if  $x$  is fixed and it is always dilute in parton density through the DGLAP evolution. For the BFKL evolution, the magnitude of the momentum transfer is fixed and evolve the Björken  $x$  and the parton density in the nucleus increase.

The parton distribution functions have only one variable  $x$  (and renormalization scale  $Q$ ), so we can see only the longitudinal momentum distributions of the nucleon/nucleus. It is known that the detailed structures of the nucleus are indispensable to study the origin of the spin of the nucleon[16, 17], and it is necessary to predict the cross-section of the more exclusive scattering and semi-inclusive scattering than ordinal DIS process. Therefore observing the final state particles which have not only longitudinal momentum but also transverse momentum, it is useful to have the universal parton distribution functions which depend on the impact parameter and the transverse momentum.

There are two well-studied multi-dimensional parton distribution functions which depend not only on longitudinal momentum but also the transverse momentum or the impact parameter. One is called generalized parton distribution functions(GPDs) which depend on Fourier transformed impact parameter[18, 19, 20], the other is called transverse momentum dependent parton distribution functions(TMDs) which depend on the transverse momentum of the partons. They are extensively studied in the literature. To know the structure of the nucleon/nucleus, we must decide the distributions from some experiments to include the non-perturbative effects. It is known that the GPDs are extracted from deep virtual Compton scattering[21, 22, 23] and the TMDs are from semi-inclusive deep inelastic scattering.

The multidimensional distributions of partons in the nucleon/nucleus can reveal the dynamical structures of the nucleon/nucleus. For example, the angular momentum of the partons in the nucleus cannot be described without correlations of the impact parameter and transverse momentum. It is clear from an intuitive explanation that the angular momentum in the classical mechanics is written as  $\mathbf{r} \times \mathbf{k}$ , so we need the correlations of the impact parameter and the transverse momentum. Similar to this example, when you consider quantities related to correlations of the angle of produced particles, you need to consider both of the impact parameter and transverse momentum in a unified way. We can treat both dependencies using the most generalized parton distributions which are called generalized TMDs or its Fourier transformed Wigner distributions. GPDs and TMDs can be derived by integrating the Wigner distributions[24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34] and we can think the Wigner distribution as mother functions of the parton distribution functions. It is already investigated the relations between the gluon Wigner distributions and the orbital angular momentum of quarks and gluons in the nucleon[28, 35].

For other aspects in high energy, there are gluon saturation effects. At high enough energy, the number of gluons becomes large and the wave functions of gluons become overlapping and the recombination effects of the gluons become important

at small  $x$  region[37]. Then the number of gluons is saturated and it is the gluon saturation effects. We have no direct evidence of such effects, but there are many studies which explain the experimental data, and the effects of the gluon saturation cannot be ignored for good descriptions of the data. For the parton distribution functions at small  $x$ , we must include the gluon saturation effects for a realistic description of the nucleus/nucleon. The gluon saturation effects are expected to influence the structure of the nucleus at small  $x$ , and this is what we will see in this thesis.

In this thesis, we study Wigner distribution functions with gluon saturation effects. In chapter 2, we introduce a brief introduction to the parton distribution functions. Next chapter, we see what the gluon saturation is. Chapter 4 is one of the main parts of this thesis and we investigate the Wigner distribution function with the gluon saturation effects. In the following chapter, we see how to measure the Wigner distribution in the experiments. The last chapter, we see the applications of the Wigner distribution to phenomenology. Then we conclude this thesis.



## Chapter 2

# Parton distribution functions

In this chapter, we introduce the parton distribution functions of the nucleon/nucleus. PDFs are indispensable to study the nucleon/nucleus in terms of the QCD. One of the key ingredients of the perturbative QCD is the factorization theorem in high energy scattering. We show how the parton distribution functions come from the theorem. We take the proton moving the + light cone direction for simplicity.

### 2.1 Foundations of QCD

Quantum Chromodynamics is an SU(3) Yang-Mills theory[2] which represent the interaction of quarks and gluons. The Lagrangian of QCD is

$$\mathcal{L}_{QCD} = \sum_{\text{flavor}} \bar{\psi}_f (i\gamma_\mu D^\mu - m_f) \psi_f - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu} , \quad (2.1)$$

where  $\psi^f$  is a Dirac field with spin 1/2 and we abbreviate the color index in the quark field, and the superscript of the field strength of the gauge field  $A_\mu^a$  is the color index  $a = 1, \dots, 8$ . We defined the covariant derivative  $D_\mu$  as

$$D_\mu = \partial_\mu - ig t^a A_\mu^a , \quad (2.2)$$

where  $t^a$  is the generator of the SU(3) group. The field strength of the gauge field is defined as

$$G_{\mu\nu}^a \equiv \frac{i}{g} [D_\mu, D_\nu] = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^a A_\nu^b , \quad (2.3)$$

where  $f^{abc}$  is the structure constant of the SU(3) group.

#### 2.1.1 The light cone coordinate

In this thesis, we use the light cone perturbation. For any four vector  $p^\mu$ , we define the light cone coordinate as

$$p^+ = \frac{1}{\sqrt{2}}(p^0 + p^3), \quad p^- = \frac{1}{\sqrt{2}}(p^0 - p^3) . \quad (2.4)$$

Then the contraction of the two vectors  $p, q$  becomes

$$p^\mu q_\mu = p^+ q^- + p^- q^+ - \mathbf{p} \cdot \mathbf{q} . \quad (2.5)$$

### 2.1.2 Light cone wave function

It is useful to introduce the light cone wave function in the high enough energy region. We can think the wave function as a wave function in quantum mechanics[36]. We now assume that there is an incoming hadron state  $|\Psi\rangle$ . First, we define the Fock state of quarks and gluons

$$|n_g; n_q\rangle = |\{k_i^+, \mathbf{k}_i, \lambda_i, a_i\}; \{p_j^+, \mathbf{p}_j, \sigma_j, \alpha_j, f_j\}\rangle \quad . \quad (2.6)$$

For the gluon state  $i = 1, \dots, n_g$ ,  $\lambda_i$  is the polarization and  $a_i$  is the color number. For the quark state  $j = 1, \dots, n_q$ ,  $\sigma_j$  is the polarization,  $\alpha_j$  is the color number and  $f_j$  is the flavor. The state fulfill the complete relation

$$\sum_{n_g, n_q} \int \Omega_{n_g+n_q} |n_g; n_q\rangle \langle n_g; n_q| = 1_{id} \quad , \quad (2.7)$$

where the integration measure is defined as

$$\begin{aligned} \int \Omega_{n_g+n_q} = & \frac{2P^+(2\pi)^3}{S_n} \int \prod_{i=1}^{n_g} \sum_{\lambda_i, a_i} \frac{dk_i^+ d^2 \mathbf{k}_i}{2k_i^+ (2\pi)^3} \prod_{j=1}^{n_q} \sum_{\sigma_j, \alpha_j, f_j} \frac{dp_j^+ d^2 \mathbf{p}_j}{2p_j^+ (2\pi)^3} \\ & \times \delta \left( P^+ - \sum_{l_1}^{n_g} k_{l_1}^+ - \sum_{l_2}^{n_q} p_{l_2}^+ \right) \delta^2 \left( P^- - \sum_{l_1}^{n_g} \mathbf{k}_{l_1} - \sum_{l_2}^{n_q} \mathbf{p}_{l_2} \right) \quad , \end{aligned} \quad (2.8)$$

where  $S_n = n_g! N_q! \bar{N}_q!$  is a symmetry factor. Here  $N_q$  and  $\bar{N}_q$  are quark and anti-quark number respectively and  $n_q = N_q + \bar{N}_q$ . Finally, we define the light cone wave function

$$\Psi(n_g, n_q) := \langle n_g; n_q | \Psi \rangle \quad . \quad (2.9)$$

## 2.2 factorization theorem

Why do we need to know factorization theorem in QCD? A simple answer to this question is that we want to predict the observables which will be measured in the future and explain the measured observables from the fundamental theory. At low energy, the non-perturbative effects have influenced the physical observable in QCD, and we cannot predict the other observables without the experimental results. The factorization theorem states that we can separate the perturbatively decided parts and the non-perturbatively decided parts in the observable. If we have the non-perturbatively decided parts from experiments then if the other observable have the same non-perturbative parts, we can predict the observable.

In this section, we concentrate on the deep inelastic scattering(DIS) processes. It is already proved that the DIS cross sections have the factorized forms, and we see the simple form of the theorem in the parton model. For a more concrete example, we now consider the electron-proton scattering process. The momentum transfer

is  $k - k' = q$  and the square of the center-of-mass energy is  $s$ . The inclusive cross section of this process is

$$E \frac{d\sigma}{d^3\mathbf{k}} = \frac{2\alpha_{em}^2}{sQ^2} L_{\mu\nu} W^{\mu\nu} , \quad (2.10)$$

where  $\alpha_{em} = \frac{e^2}{4\pi}$  is the fine structure constant,  $Q^2 = -q^2$  and  $\mathbf{k}$  is the momentum of the final state electron.  $L_{\mu\nu}$  is the electron to electron and virtual photon scattering amplitude(leptonic tensor part) and  $W_{\mu\nu}$  is the inclusive virtual photon and proton scattering amplitude(hadronic tensor part). The leptonic tensor part can be calculated in QED, so there are no non-perturbative effects. The information of the hadron is contained in the hadronic tensor part and the non-perturbative effects are needed. The operator definition of the hadronic tensor part is

$$W^{\mu\nu} = \frac{1}{4\pi} \int d^4z e^{iqz} \langle P, S | j^\mu(z) j^\nu(0) | P, S \rangle , \quad (2.11)$$

where  $j^\mu(z) = \sum_f e_f \bar{\psi}_f \gamma^\mu \psi_f$  is electromagnetic current for quarks and  $\frac{1}{2} g^{\mu\nu} \partial_\mu A^i \partial_\nu A_j$  for gluon. We show the parton model calculations following[38]. In the parton model, we assume that the virtual photon scatters incoherently off the constituents of the nucleon. Then the hadronic tensor can be represented as the handbag diagram and the following formula for the quarks

$$\begin{aligned} W^{\mu\nu} &= \frac{1}{2\pi} \sum_f e_f^2 \sum_X \int \frac{d^3\mathbf{p}_X}{(2\pi)^3 p_X^0} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} \delta(l^2) \\ &\quad \times [\bar{u}(l) \gamma^\mu \phi(k, P)]^* [\bar{u}(l) \gamma^\nu \phi(k, P)] \\ &\quad \times (2\pi)^4 \delta^4(P - k - p_X) (2\pi)^4 \delta^4(k + q - l) , \end{aligned} \quad (2.12)$$

where  $f$  is the flavor of the quark and  $e_f$  is the electronic charge of the quark, and we introduce the nucleon-quark vertex

$$\phi(k, p) = \langle X | \psi(0) | N \rangle , \quad (2.13)$$

where  $|X\rangle$  is the final state and  $|N\rangle$  is the nucleon state. We define the quark-quark correlation matrix

$$\begin{aligned} \Phi_{ij}(k, P) &= \sum_X \int \frac{d^3\mathbf{p}_X}{(2\pi)^3 p_X^0} (2\pi)^4 \delta^4(P - k - p_X) \phi_i(k, P) \bar{\phi}_j(k, P) \\ &= \int d^4\xi e^{ik\xi} \langle N | \bar{\psi}_j(0) \psi_i(\xi) | N \rangle . \end{aligned} \quad (2.14)$$

Where we use the complete relation of the wave function inside the nucleon. At high enough energy, we can ignore the mass of the light quarks, and the hadronic tensor becomes

$$\begin{aligned} W^{\mu\nu} &= \frac{1}{2\pi} \sum_f e_f^2 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} \delta(l^2) (2\pi)^4 \delta^4(k + q - l) \text{tr}[\Phi \gamma^\mu \gamma^\rho k_\rho \gamma^\nu] \\ &= \frac{1}{2\pi} \sum_f e_f^2 \int \frac{d^4k}{(2\pi)^4} \delta((k + q)^2) \text{tr}[\Phi \gamma^\mu \gamma^\rho (k_\rho + q_\rho) \gamma^\nu] . \end{aligned} \quad (2.15)$$

We can also approximate the nucleon momentum as  $P^\mu = (P, 0, 0_\perp)$ , where we take the light cone + direction as the direction of the motion of the nucleon. The momentum of the virtual photon is  $q^\mu = (-xP, (P \cdot q), 0_\perp)$  and the quark momentum is  $k^\mu = (\alpha P, \frac{k^2 + \mathbf{k}_\perp^2}{2\alpha}, \mathbf{k}_\perp)$ . In the parton model, we assume that the quark is almost on-shell and the transverse momentum of the quark is small. Then we can approximate  $k^\mu = \alpha P^\mu$ . This is also true for the infinite momentum frame in QCD.

Using the on-shellness of the quark, the delta function in (2.15) can be approximate as

$$\delta((k+q)^2) \simeq \delta(-Q^2 + 2\alpha P \cdot q) = \frac{1}{2P \cdot q} \delta(\alpha - x) \quad . \quad (2.16)$$

Then we have

$$k^\mu = xP^\mu \quad . \quad (2.17)$$

The Björken  $x$  is represented as

$$x = \frac{k^+}{P^+} \quad . \quad (2.18)$$

We now back to the hadronic tensor part, the products of the gamma matrixes are decomposed as

$$\gamma^\mu \gamma^\rho \gamma^\nu = (g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho} - g^{\mu\nu} g^{\rho\sigma}) \gamma_\sigma - i\epsilon^{\mu\rho\nu\sigma} \gamma_\sigma \gamma^5 \quad . \quad (2.19)$$

In this thesis, we only consider the unpolarized deep inelastic scattering, so the symmetric part in the hadronic tensor contribute

$$\begin{aligned} W_S^{\mu\nu} &= \frac{1}{4\pi(P \cdot q)} \sum_f e_f^2 \int \frac{d^4 k}{(2\pi)^4} \delta(x - k^+/P^+) \\ &\quad \times [(k^\mu + q^\mu) \text{tr}(\Phi \gamma^\nu) + (k^\nu + q^\nu) \text{tr}(\Phi \gamma^\mu) \\ &\quad - g^{\mu\nu} (k_\rho + q_\rho) \text{tr}(\Phi \gamma^\rho)] \quad . \end{aligned} \quad (2.20)$$

In the parton model we have  $k^\mu + q^\mu = (P \cdot q) n^\mu$ , where  $n^\mu = (1, 0, 0_\perp)$ . Then we have

$$\begin{aligned} W_S^{\mu\nu} &= \frac{1}{4\pi} \sum_f e_f^2 \int \frac{d^4 k}{(2\pi)^4} \delta(x - k^+/P^+) \\ &\quad \times [n^\mu \text{tr}(\Phi \gamma^\nu) + n^\nu \text{tr}(\Phi \gamma^\mu) - g^{\mu\nu} n_\rho \text{tr}(\Phi \gamma^\rho)] \quad . \end{aligned} \quad (2.21)$$

Then we introduce the parton distribution function  $f(x)$

$$\begin{aligned} \int \frac{d^4 k}{(2\pi)^4} \delta(x - k^+/P^+) \text{tr}(\Phi \gamma^\nu) &= P^+ \int \frac{d\xi^-}{2\pi} e^{xP^+ \xi^-} \langle N | \bar{\psi}(0) \gamma^\nu \psi(0, \xi^-, 0_\perp) | N \rangle \\ &= 2f(x) P^\nu \quad . \end{aligned} \quad (2.22)$$

Then the parton distribution function is represented as

$$f(x) = \int \frac{d\xi^-}{4\pi} e^{xP^+ \xi^-} \langle N | \bar{\psi}(0) \gamma^+ \psi(0, \xi^-, 0_\perp) | N \rangle \quad , \quad (2.23)$$

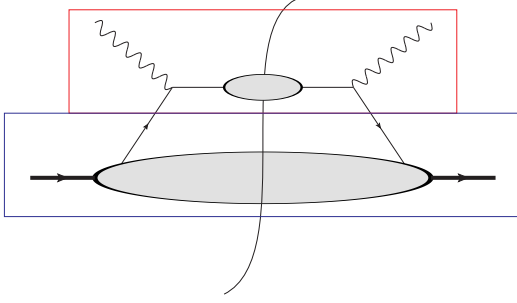


Figure 2.1: Factorization of DIS

where we assume that  $P^+$  is much larger than other components of the momentum of the nucleon, so only  $\gamma^+$  term survives. Inserting the parton distribution function to the hadronic tensor, we get

$$W_S^{\mu\nu} = \frac{1}{2\pi} \sum_f e_f^2 (n^\mu P^\nu + n^\nu P^\mu - g^{\mu\nu}) f_f(x) . \quad (2.24)$$

The factorization theorem also exists in the QCD, and the hadronic tensor of QCD can be written as

$$W_{\mu\nu} = \sum_j \int_{x^-}^{1^+} \frac{d\xi}{\xi} \text{Tr} C_j^{\mu\nu}(q, \xi P; \alpha_s, \eta) \rho_j(\xi; \eta) f_j(\xi; \eta) + p.s.c. \quad (2.25)$$

$C_j^{\mu\nu}$  are called hard-scattering coefficients and can be calculated in the perturbative QCD. The non-perturbative effects are included in the  $\rho_j f_j$  terms and we call them parton distribution functions. The *p.s.c* terms mean the power suppressed corrections and at enough high energy scattering it can be negligible. We show the pictorial representation of the factorization theorem at Fig. 2.1. The upper rectangle represent  $C^{\mu\nu}$  and the lower the parton distribution functions.

We show the operator definition of the parton distribution functions. We assume that the target nucleon/nucleus goes + direction in the light cone coordinate. The unpolarized parts of the parton distributions for quarks are written as

$$\rho_j f_j(\xi)|_{unp} = \int \frac{d\omega^-}{2\pi} e^{-i\xi P^+ \omega^-} \langle P | \bar{\psi}_j(0, \omega^-, 0_\perp) \frac{\gamma^+}{2} \psi_j(0) | P \rangle_c . \quad (2.26)$$

The subscript c in the matrix element represents that we only consider the connected part of the diagram. This formula is not invariant under gauge transformation. To extend this into the gauge invariant version, we have:

$$\rho_j f_j(\xi)|_{unp} = \int \frac{d\omega^-}{2\pi} e^{-i\xi P^+ \omega^-} \langle P | \bar{\psi}_j(0, \omega^-, 0_\perp) U(\omega^-, 0; 0_\perp) \frac{\gamma^+}{2} \psi_j(0) | P \rangle_c , \quad (2.27)$$

where the  $U(x, y; 0_\perp) = P\{e^{-ig \int_x^y dz^- A^+(0, z^-, 0_\perp)}\}$  is the Wilson line in the fundamental representation. The gluon PDF is:

$$\begin{aligned} \rho_g f_g(\xi)|_{unp} &= \frac{1}{\xi P^+} \int \frac{d\omega^-}{2\pi} e^{-i\xi P^+ \omega^-} \\ &\quad \times \langle P | \text{tr} G^{+j}(0, \omega^-, 0_\perp) \mathcal{U}[\omega, 0] G^{+j}(0) \mathcal{U}'^\dagger[0, \omega] | P \rangle_c , \end{aligned} \quad (2.28)$$

where  $G^{+j} = \partial A^j$  in the light cone gauge, and  $\mathcal{U}[a, b]$  and  $\mathcal{U}'[a, b]$  are the products of the Wilson lines which connect  $a$  to  $b$ . The following sections, we simply write  $\rho_j f_j$  as  $f_j$  and omit subscript  $c$  for convenient.

## 2.3 parton distribution functions

In this section, we show some properties of the parton distribution functions. At the previous section, we only consider the inclusive cross sections, but if we consider more exclusive cross section, we should generalize the parton distribution functions to have factorizations. The parton distribution function can be described by integration of the unintegrated parton distribution functions introduced above section:

$$f_j(x, \mathbf{k}, \Delta_\perp) = \int \frac{dz^- d^2 \mathbf{z}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{z} - ixP^+ z^-} \times \langle P + \Delta_\perp/2 | \bar{\psi}_j(0, z^-, \mathbf{z}) \mathcal{U} \frac{\gamma^+}{2} \psi_j(0) | P - \Delta_\perp/2 \rangle . \quad (2.29)$$

This function is called Generalized Transverse Momentum Dependent Parton Distribution Function (GTMD).

If  $\Delta_\perp = 0_\perp$  then we can prove that the parton distribution functions are non-negative functions. In the light cone gauge condition, the Wilson line in the definition of the GTMDs is the identify matrix. Then it is enough to consider the Dirac fields. We introduce the light front annihilation and creation operator

$$\psi(x) = \sum_\alpha \frac{1}{(2\pi)^3} \int_0^\infty \frac{dk^+}{k^+} d^2 \mathbf{k} \times [b_{\mathbf{k}, \alpha}(x^+) u_{\mathbf{k}, \alpha} e^{-ik^+ x^- + i\mathbf{k}\mathbf{x}} + d_{\mathbf{k}, \alpha}^\dagger(x^+) u_{\mathbf{k}, -\alpha} e^{ik^+ x^- - i\mathbf{k}\mathbf{x}}] . \quad (2.30)$$

Then the quark parton distribution functions are written as

$$f_j(x, \mathbf{k}, 0_\perp) = \sum_\alpha \frac{1}{2x(2\pi)^3} \frac{\langle P | b_{\mathbf{k}, \alpha, j} b_{\mathbf{k}, \alpha, j}^\dagger | P \rangle}{\langle P | P \rangle} . \quad (2.31)$$

This is nothing but a number density of quarks in the nucleus. So we can interpret the parton distribution function as the number density of partons in the nucleon/nucleus in QCD. This interpretation is only valid at the leading twist level and the light cone gauge. If we include the higher twist effects, then the parton distribution functions are not interpreted as the number density.

## 2.4 How to access the GPDs from experiments

The parton distributions have non-perturbative effects, thus we cannot access them only from the theoretical side and need to know from the experimental results. The factorization of cross sections help this situation. Parton distribution functions (PDFs) are universal to all processes, so if we access the PDFs in one process then we can predict the other all processes with inputting the measured PDFs.

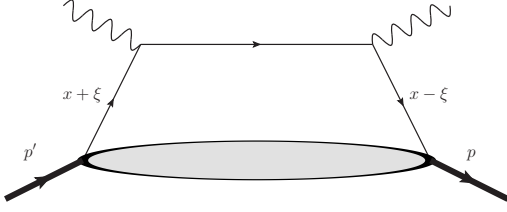


Figure 2.2: The diagram to access the GPDs

In this section, we simply consider the unpolarized nucleus. The GPDs appears at the deeply virtual Compton scattering or the deep virtual meson production process. The GPDs are obtained by these processes pictured at Fig.2.2. The GPDs have three parameter dependence:  $G(x, \xi, \Delta)$ . We see how these variables are decided in the processes.

We simply consider the deeply virtual Compton scattering(DVCS) process[39, 40, 41, 42]. The scattering amplitude for DVCS at leading order in  $\alpha_s$  is

$$\Im \mathcal{H}(\xi, t, Q^2) = \pi [H(\xi, \xi, t; Q^2) - H(-\xi, \xi, t; Q^2)] \quad (2.32)$$

$$\Re \mathcal{H}(\xi, t, Q^2) = \text{PV} \int_{-1}^1 dx H(x, x, t; Q^2) \left[ \frac{1}{\xi - x} - \frac{1}{\xi + x} \right] + D(t; Q^2) \quad , \quad (2.33)$$

where  $D(t)$  is related with Polyakov-Weiss  $D$ -term. For quarks,  $H$  in the above formula is related to matrix element:

$$\begin{aligned} & \int \frac{dz^-}{4\pi} e^{ixP^+z^-} \langle p', \Lambda' | \bar{\psi}_j(-\frac{1}{2}z) \gamma^+ \psi_j(\frac{1}{2}z) | p, \Lambda \rangle |_{z^+=0, z=0} \\ &= \frac{1}{2P^+} \bar{u}(p', \Lambda') \left[ H_j(x, \xi, t) \gamma_+ + E_j(x, \xi, t) \frac{i\sigma^{+\mu} \Delta_\mu}{2M_N} \right] u(p, \Lambda) \quad , \quad (2.34) \end{aligned}$$

where  $\Delta = p' - p$ ,  $t = \Delta^2$ ,  $\xi = \frac{p^+ - p'^+}{p^+ + p'^+}$ . The variable  $x$  is loop momentum and finally integrated out and we cannot see the GPDs in  $\xi = 0$ . From these formulas, we have  $H(\pm\xi, \xi, t : Q^2)$  and  $\int_{-1}^1 dx \frac{H(\mp x, \xi, t; Q^2)}{\xi \pm x}$  from experiments.



## Chapter 3

# Gluon saturation

What is the gluon saturation? At small Björken  $x$  region, the number of the gluon becomes larger than quarks. As the number of the gluons becomes large, then the nucleon/nucleus are filled with the gluons. In this situation, the recombination effects of gluons become important and the number of gluons is saturated. In the following sections, we see how the number of gluons becomes large and then saturated.

### 3.1 Increasing the number of gluons at high energy

We will see the gluon number density becomes large at small- $x$ . Intuitively, as the  $x$  becomes small, the center of mass energy becomes large, and more and more partons are produced. Gluons have the splitting vertex and the number of gluons quickly increases.

First, we define the unintegrated gluon distribution  $f(x, \mathbf{k})$  from the parton distribution function:

$$f(x, Q^2)_g = \int^{Q^2} d\mathbf{k}^2 f(x, \mathbf{k}^2) \quad . \quad (3.1)$$

At the lowest order, the splitting probability for a soft gluon emission from a high energy quark is

$$dP = C_R \frac{\alpha_s}{\pi^2} \frac{d^2 \mathbf{k}}{|\mathbf{k}|} \frac{dx}{x} \quad , \quad (3.2)$$

where  $C_R = N_c$ ,  $x \ll 1$  is a fraction of gluon's light cone momentum and  $\mathbf{k}$  is a relative transverse momentum. Then the one gluon emission probability is  $P_1 \sim \alpha_s \log(1/x)$ . In the small  $x$  region, the order of the probability becomes  $\mathcal{O}(1)$ , so we should sum the higher order terms. We can assume the strong longitudinal ordering of the subsequent emissions,  $x_1 \gg x_2 \gg \dots \gg x_n$ , to the successive emissions in the leading logarithmic approximation. Then the probability for  $n$  gluon emission becomes  $P_n \sim (\alpha_s \log(1/x))^n \sim \mathcal{O}(1)$  and cannot use perturbation as it is. We must resum these  $n$  gluon emission processes, and it is accomplished by the renormalization equation called the BFKL equation which can incorporate the leading logarithmic part  $(\alpha_s \log(1/x))^n$  in the perturbative calculations.

The unintegrated gluon distribution function  $f(x, \mathbf{k}^2)$  satisfy the BFKL equation which is the rapidity evolution equation:

$$\frac{\partial f(x, \mathbf{k}^2)}{\partial \log(1/x)} = \frac{N_c \alpha_s}{\pi^2} \int \frac{d^2 \mathbf{k}'}{(\mathbf{k} - \mathbf{k}')^2} \left[ f(x, \mathbf{k}'^2) - \frac{\mathbf{k}^2}{\mathbf{k}'^2 + (\mathbf{k} - \mathbf{k}')^2} f(x, \mathbf{k}^2) \right] . \quad (3.3)$$

The behavior of the solution of this equation is:

$$f(x, k^2) \sim \left( \frac{1}{x} \right)^\lambda , \quad (3.4)$$

where  $\lambda = \frac{N_c \alpha_s}{\pi} 4 \log(2) \sim 0.5$  at  $\alpha_s \sim 0.2$ . This behavior of the solution is easily understood by the simple example:  $\partial_Y S = CS \Rightarrow S = D e^{CY} = D(1/x)^C$  where  $C$  and  $D$  are constant. Thus we find that the rapidity increase then the number of the gluon is exponentially increasing. This behavior violates the Froissart-Martin bound[43, 44] i.e. unitarity of S matrix. The bound states that the total cross section cannot increase faster than the square of the logarithms of the center-of-mass energy.

We define an impact parameter dependent function:

$$N(x, \mathbf{r}) = \int d^2 \mathbf{k} (1 - e^{i\mathbf{k}\mathbf{r}}) \frac{1}{\mathbf{k}^2} f(x, \mathbf{k}^2) . \quad (3.5)$$

Then this function satisfies the BFKL equation in the coordinate space( $Y = \log(1/x)$ ):

$$\frac{\partial N(x, \mathbf{x}_{01})}{\partial Y} = \frac{N_c \alpha_s}{2\pi^2} \int d^2 \mathbf{x}_2 \frac{\mathbf{x}_{01}^2}{\mathbf{x}_{20}^2 \mathbf{x}_{21}^2} [N(x, \mathbf{x}_{02}) + N(x, \mathbf{x}_{21}) - N(x, \mathbf{x}_{01})] , \quad (3.6)$$

where  $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ .

### 3.2 Gluon saturation

As we see in the previous section, the number of the gluon is exponentially increasing as the rapidity increasing. However it has some physical problems, and we see that the gluon saturation effects can ease them. We have only considered the dilute regime of gluons, so the recombination effects of gluons have no big effects. When the Björken variable goes to a small region, the number of gluons are large and the wave function of gluons becomes overlapping. Then the recombination effects of gluons have an important role.

We consider the deep inelastic scattering process in the dipole picture. We assume that the nucleus is in the rest frame(the momentum of the nucleus is  $P = (m, 0, 0_\perp)$ ). Then the momentum of the virtual photon is  $q = (q^+, -Q^2/q^+, 0_\perp)$  in the light cone coordinate. From  $2P \cdot q = Q^2/x$ , we have

$$q^0 = \frac{Q^2}{2mx} \gg Q . \quad (3.7)$$

From the definition  $0 \leq Q^2 = -q^2 = (q^3)^2 - (q^0)^2$ , then we have  $q^3 \geq q^0 \gg Q$  and

$$q^+ \sim 2q^0 \quad (3.8)$$

$$q^- = \frac{q^+ q^-}{q^+} \sim \frac{Q^2}{2q^0} = -mx . \quad (3.9)$$

Then the coherent length of the virtual photon in the light cone + direction is

$$l_{coh} = x^+ \sim \frac{2}{|q^-|} = \frac{2}{mx} . \quad (3.10)$$

At small  $x$  region, the coherence length is much larger than the radius of the nucleus. The coherent length can be interpreted as the typical length of the interaction. Thus the length of the time which the virtual photon fluctuate into the dipole state is much longer than the decay of the dipole. Then in the DIS, the virtual photon becomes the dipole first and thereafter the dipole scatters with the nucleus.

Next, we see how the relative distance between the quark and the antiquark in the dipole change through the scattering with the nucleus. Let us take the relative transverse momentum coming from the scattering as  $\mathbf{k}$ , then the relative velocity between the quark and the antiquark is changed by  $\mathbf{k}/E \sim \mathbf{k}/q^0$ . Then the change of the relative distance is

$$\Delta x_{\perp} \sim R \frac{|\mathbf{k}|}{E} , \quad (3.11)$$

where  $R$  is the longitudinal size of the nucleus. In this situation, Björken variable  $x$  is

$$x = \frac{Q^2}{mq^+} \sim \frac{Q^2}{2mE} \quad (3.12)$$

And the uncertainty principle states  $Q \sim |\mathbf{k}_{\perp}| \sim 1/x_{\perp}$ . Thus

$$\frac{\Delta x_{\perp}}{x_{\perp}} \sim 2mxR = \frac{4R}{l_{coh}} \ll 1 . \quad (3.13)$$

In the end, the effects of the recoil that the dipole interacts with the nucleus can be ignored compared with the size of the dipole, and we ignore the recoil effects. This assumption is called the eikonal approximation.

Taking into account the above picture, the cross section for DIS is factorized in two parts. One is the splitting of the virtual photon into the dipole, and the second part is the scattering cross section of the dipole and the nucleus.

$$\sigma_{totL,T}^{\gamma^*A}(x, Q^2) = \int \frac{d\mathbf{x}}{4\pi} \int_0^1 \frac{dz}{z(1-z)} |\Psi_{T,L}^{\gamma^*}(\mathbf{x}_{\perp}, z)|^2 \sigma_{tot}^{q\bar{q}A}(\mathbf{x}, Y = \log(1/x)) \quad (3.14)$$

Where  $z$  is the momentum fraction of the quark in the dipole and  $\Psi_{T,L}$  is the light cone wave function

$$\begin{aligned} |\Psi_T^{\gamma^*}(\mathbf{x}, z)|^2 &= 2N_c \sum_f \frac{\alpha_{em} Z_f^2}{\pi} z(1-z) \\ &\quad \times \{a_f^2 K_1(a_f |\mathbf{x}|)^2 (z^2 + (1-z)^2) + m_f^2 K_0(a_f |\mathbf{x}|)^2\} \end{aligned} \quad (3.15)$$

$$|\Psi_L^{\gamma^*}(|\mathbf{x}|, z)|^2 = 2N_c \sum_f \frac{\alpha_{em} Z_f^2}{\pi} 4Q^2 z^3 (1-z)^3 K_0(a_f |\mathbf{x}|) , \quad (3.16)$$

where  $a_f^2 = Q^2 z(1-z) + m_f^2$ ,  $K_i$  are the modified Bessel functions and the definition of the cross sections are

$$\sigma_{totT}^{\gamma^*A}(x, Q^2) = \frac{4\pi^2\alpha_{em}}{q^0} W^{\mu\nu} \frac{1}{2} \sum_{\lambda=\pm} \epsilon_{T\mu}^\lambda \epsilon_{T\nu}^{\lambda*} \quad (3.17)$$

$$\sigma_{totT}^{\gamma^*A}(x, Q^2) = \frac{4\pi^2\alpha_{em}}{q^0} W^{\mu\nu} \epsilon_{L\mu} \epsilon_{L\nu}^* , \quad (3.18)$$

where  $\epsilon_{T\mu}^\lambda$  are the transverse polarization vectors of the virtual photon and  $\epsilon_{L\mu}$  are longitudinal polarization vectors.

Now we focus on the dipole-nucleus scattering cross section. We take the scattering amplitude of the dipole-nucleus scattering process as  $N(Y = \log(1/x), \mathbf{b}, \mathbf{r})$ . This amplitude is related to the total cross section of the dipole-nucleus scattering:

$$\sigma_{tot}^{ppA}(Y, \mathbf{r}) = 2 \int d^2\mathbf{b} N(Y, \mathbf{b} + \mathbf{r}/2, \mathbf{b} - \mathbf{r}/2) , \quad (3.19)$$

where  $\mathbf{b}$  is the impact parameter and  $\mathbf{r}$  is the relative distance between the quark and antiquark of the dipole.

The rapidity evolution of the scattering amplitude is governed by

$$\begin{aligned} \partial_Y N(Y, \mathbf{x}, \mathbf{y}) &= \frac{N_c \alpha_s}{2\pi^2} \int d^2\mathbf{z} \frac{(\mathbf{x} - \mathbf{y})^2}{(\mathbf{z} - \mathbf{x})^2 (\mathbf{z} - \mathbf{y})^2} \\ &\quad \times [N(Y, \mathbf{x}, \mathbf{z}) + N(Y, \mathbf{z}, \mathbf{y}) - N(Y, \mathbf{x}, \mathbf{y}) \\ &\quad - N(Y, \mathbf{x}, \mathbf{z}) N(Y, \mathbf{z}, \mathbf{y})] . \end{aligned} \quad (3.20)$$

This equation is called the Balitsky-Kovchegov(BK) equation[45, 46]. As we see from this equation, if we omit the non-linear term in the equation, it is nothing but the BFKL equation. If  $N$  is small compared to one then the equation becomes the BFKL equation, but if  $N$  approaches one then the non-linear term becomes affected and suppresses the rapid increase of the scattering amplitude. Thanks to this non-linear term, it is possible to suppress the rapid increase in the cross section.

We show that if the initial conditions for the BK equation satisfy  $0 \leq N(Y = 0) \leq 1$  then the solution also satisfy  $0 \leq N(Y) \leq 1$ . We prove it by contradiction following the proof for the BMS equation in the jet physics[47]. We first assume that there is a rapidity  $Y_1$  which satisfies  $S(Y, \mathbf{x}, \mathbf{y}) = 1 - N(Y_1, \mathbf{x}, \mathbf{y}) > 1$ . Then there is  $Y$  which satisfies  $\max_{\mathbf{x}, \mathbf{y}} \{S(Y, \mathbf{x}, \mathbf{y})\} = 1$  and  $\partial_Y S(Y, \mathbf{x}, \mathbf{y}) > 0$  in the same  $\mathbf{x}, \mathbf{y}$  which satisfies the previous equation. Then for any  $\mathbf{z}$ ,  $S(Y, \mathbf{x}, \mathbf{z})S(Y, \mathbf{z}, \mathbf{y}) - S(Y, \mathbf{x}, \mathbf{y}) < 0$  and from the BK equation for  $S$  we find that  $\partial_Y S(Y, \mathbf{x}, \mathbf{y}) < 0$ . Then there is a contradiction and  $S \leq 1$  is proved. This is also true for  $S \leq 0$ .

This lemma shows that the S-matrix cannot exceed the black disc limit, and ease the FM bound problem, but it is not enough to satisfy the FM bound and we need some cut off to include non-perturbative effects.

### 3.3 Deriving the BK equation in the dipole model

In this section, we show a short review of the derivation of the BK equation. First, we introduce the Mueller's dipole model[48]. Using this model, it is easy to investigate

the high energy scattering at small- $x$ . As is already stated in the previous section, DIS cross section can be made by the photon splitting term and the dipole-nucleus scattering cross section.

We denote the dipole light cone wave function by  $\Psi_{\sigma\sigma'}(\mathbf{k}, z)$ , where  $\mathbf{k}$  is the relative momentum of the  $q\bar{q}$  pair,  $z = \frac{k^+}{p^+}$  is the fraction of the light cone momentum of the dipole carried by the quark and  $\sigma, \sigma'$  are the polarization of the quark and antiquark. The transverse relative distance is invariant through the scattering, so we use the transverse coordinate and longitudinal momentum dependent dipole wave function

$$\Psi_{\sigma\sigma'}^{(0)}(\mathbf{x}, z) = \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{b}} \Psi_{\sigma\sigma'}^{(0)}(\mathbf{k}, z) . \quad (3.21)$$

As the energy increases, the dipole produces gluons. We first consider one gluon production and see the probability of this production. In the dipole picture, we can see this as a process which is one dipole produce two succeeding dipoles

$$\psi^{(1)}(\mathbf{b}_1, x_1, \mathbf{b}_2, x_2) = \frac{ig}{\pi} t^A \psi^{(0)}(\mathbf{b}_1, x_1) \left( \frac{\mathbf{b}_{21}}{b_{21}^2} - \frac{\mathbf{b}_{20}}{b_{20}^2} \right) \cdot \epsilon_2 , \quad (3.22)$$

where we omit the polarization for simplicity and define a  $n$ -dipole wave function  $\Psi^{(n)}(\mathbf{b}_1, x_1, \dots, \mathbf{b}_n, x_n)$ . Then the finding probability density of the two dipole is

$$|\psi^{(1)}(\mathbf{b}_1, x_1, \mathbf{b}_2, x_2)|^2 = 4C_F \frac{\alpha_s}{\pi} |\psi^{(0)}(\mathbf{b}_1, x_1)|^2 \frac{b_{10}^2}{b_{20}^2 b_{21}^2} . \quad (3.23)$$

In this formula, we sum up the polarization and color. The probability of finding the daughter dipole is

$$\int_{x_0}^{\min\{x_1, 1-x_1\}} \frac{dx_2}{x_2} \int \frac{d\mathbf{b}_2}{4\pi} |\psi^{(1)}|^2 . \quad (3.24)$$

Instead of considering the scattering amplitude  $N$ , we calculate the S-matrix for simplicity

$$S(Y, \mathbf{x}, \mathbf{y}) := 1 - N(Y, \mathbf{x}, \mathbf{y}) . \quad (3.25)$$

At large  $N_c$ , we can consider the scattering as the scattering of the cascade of daughter dipoles and the nucleus. Then we can express the S-matrix as

$$\begin{aligned} S(Y, \mathbf{b} + \mathbf{r}/2, \mathbf{b} - \mathbf{r}/2) &= \sum_{k=1}^{\infty} \frac{1}{k!} \int d^2\mathbf{r}_1 d^2\mathbf{b}_1 \dots d^2\mathbf{r}_k d^2\mathbf{b}_k \\ &\quad \times |\Psi^{[n]}(\mathbf{b}_1, \mathbf{r}_1, \dots, \mathbf{b}_n, \mathbf{r}_k, Y)|^2 s_0(\mathbf{r}_1, \mathbf{b}_1) \dots s_0(\mathbf{r}_k, \mathbf{b}_k) . \end{aligned} \quad (3.26)$$

Where  $|\Psi^{[n]}|^2$  is the probability of finding the  $n$ -dipoles with rapidity integrating in  $[0, Y]$  interval and  $s_0$  is the S-matrix for the scattering of one dipole and nucleus. Now we have the rapidity evolution of this S-matrix and it is represented by the

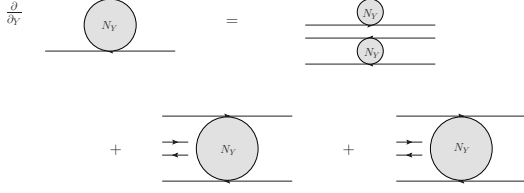


Figure 3.1: Picture of the rapidity evolution of the S-matrix

figure 3.1.

$$\partial_Y S(Y, \mathbf{x}_1, \mathbf{x}_0) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 \mathbf{x}_2 \frac{\mathbf{x}_{01}^2}{\mathbf{x}_{20}^2 \mathbf{x}_{21}^2} (S(Y, \mathbf{x}_1, \mathbf{x}_2) S(Y, \mathbf{x}_2, \mathbf{x}_0) - S(Y, \mathbf{x}_1, \mathbf{x}_0)) \quad . \quad (3.27)$$

Then we have an equation for the scattering amplitude

$$\begin{aligned} \partial_Y N(Y, \mathbf{x}_1, \mathbf{x}_0) = & \frac{\alpha_s N_c}{2\pi^2} \int d^2 \mathbf{x}_2 \frac{\mathbf{x}_{01}^2}{\mathbf{x}_{20}^2 \mathbf{x}_{21}^2} \\ & \times (N(Y, \mathbf{x}_1, \mathbf{x}_2) + N(Y, \mathbf{x}_2, \mathbf{x}_0) - N(Y, \mathbf{x}_1, \mathbf{x}_0) \\ & - N(Y, \mathbf{x}_1, \mathbf{x}_2) N(Y, \mathbf{x}_2, \mathbf{x}_0)) \quad . \end{aligned} \quad (3.28)$$

If we ignore the last non-linear term, this equation is nothing but the BFKL equation. When the scattering amplitude is small compared to one, we consider the BK equation as the BFKL equation.

For simplicity, we assume that the solution depends only on the relative distance of the parent dipole

$$N(Y, \mathbf{x}, \mathbf{y}) = N(Y, \mathbf{x} - \mathbf{y}) \quad . \quad (3.29)$$

In this situation, we can rewrite the BK equation as

$$\begin{aligned} \partial_Y N(Y, \mathbf{x}_{10}) = & \frac{\alpha_s N_c}{2\pi^2} \int d^2 \mathbf{x}_2 \frac{\mathbf{x}_{01}^2}{\mathbf{x}_{20}^2 \mathbf{x}_{21}^2} \\ & \times (N(Y, \mathbf{x}_{12}) + N(Y, \mathbf{x}_{20}) - N(Y, \mathbf{x}_{10}) \\ & - N(Y, \mathbf{x}_{12}) N(Y, \mathbf{x}_{20})) \quad . \end{aligned} \quad (3.30)$$

### 3.3.1 Saturation momentum

In this section, we introduce the characteristic momentum in the gluon saturation called saturation momentum. This represents the typical momentum of the partons in the nucleus at small  $x$ . The saturation momentum  $Q_s(Y)$  is defined as

$$N(Y, \mathbf{x} = 1/Q_s(Y), \mathbf{y} = 1/Q_s(Y)) = \text{const} \quad , \quad (3.31)$$

where *const* is order of one. At the leading order, we can ignore the few magnifications of the *constant*. In the following sentences in this section, we ignore the impact parameter dependence of the amplitude  $N(Y, \mathbf{r}) = N(Y, \mathbf{b} + \mathbf{r}/2, \mathbf{b} - \mathbf{r}/2)$ .

When the scattering amplitude is small compared to one, then we can ignore the non-linear term in the BK equation, and we have the exact solution of the equation which is the same as the BFKL equation. At this region, the solution of the equation is written as[49]

$$N(Y, \mathbf{x}) = \int_{-\infty}^{\infty} d\nu C_\nu \exp[\bar{\alpha}_s \chi(0, \nu) Y + (1 + 2i\nu) \log(|\mathbf{x}| Q_{s0})] \quad , \quad (3.32)$$

where the  $C_\nu$  is decided by the initial condition for the equation and  $Q_{s0}$  is the typical momentum in the nucleus at  $Y = 0$ . Using the steepest decent method, the saddle point  $\mu_{sp}$  is calculated by

$$\bar{\alpha}_s \chi'(0, \nu_{sp}) Y + 2i \log(|\mathbf{x}| Q_{s0}) = 0 \quad . \quad (3.33)$$

Then the amplitude becomes

$$N(Y, \mathbf{x}) \propto (|\mathbf{x}| Q_{s0})^{1+2i\nu_{sp}} e^{\bar{\alpha}_s \chi(0, \nu_{sp}) Y} \quad . \quad (3.34)$$

The condition of the amplitude  $N(Y, 1/Q_s(Y)) = \text{const}$  leads

$$\bar{\alpha}_s \chi'(0, \nu_0) Y + (1 + 2i\nu_0) \log(Q_{s0}/Q_s(Y)) = 0 \quad . \quad (3.35)$$

Also we have the saddle point condition

$$\bar{\alpha}_s \chi'(0, \nu_0) Y + 2i \log(Q_{s0}/Q_s(Y)) = 0 \quad . \quad (3.36)$$

Where  $\nu_0 := \nu_{sp}(1/Q_s(Y), Y)$ . Then the saturation momentum is

$$Q_s(Y) = Q_{s0} \exp \left[ \bar{\alpha}_s \frac{\chi(0, \nu_0)}{1 + 2i\nu_0} Y \right] \quad . \quad (3.37)$$

And the  $\nu_0$  satisfy

$$\frac{\chi'(0, \nu_0)}{\chi(0, \nu_0)} = \frac{2i}{1 + 2i\nu_0} \quad , \quad (3.38)$$

and numerically  $\nu_0 \sim -0.1275i$  Then we have

$$N(Y, \mathbf{x}) \propto (|\mathbf{x}| Q_{s0})^{1+2i\nu_{sp}} \left( \frac{Q_s(Y)}{Q_{s0}} \right)^{(1+2i\nu_0) \frac{\chi(0, \nu_{sp})}{\chi(0, \nu_0)}} \quad . \quad (3.39)$$

In our saddle point approximation, we have  $\nu_{sp} \sim \nu_0$  and get

$$N(Y, \mathbf{x}) \propto (|\mathbf{x}| Q_s(Y))^{1+2i\nu_0} \sim (|\mathbf{x}| Q_s(Y))^{1.255} \quad . \quad (3.40)$$

From this formula, we can see that the scattering amplitude is the function of the only one variable  $|\mathbf{x}| Q_s(Y)$ , and this behavior is called the geometric scaling[37, 50, 51]. This scaling is one of the strong evidence of the gluon saturation effects. In HERA data of electron-proton structure function in  $x < 0.01$  region, the scaling is observed [52]. It is confirmed that the scaling is exist in the charged hadron spectrum of proton-proton and heavy ion collisions[53, 54].

### 3.4 McLerran-Venugopalan model

In this section, we introduce the McLerran-Venugopalan(MV) model[55, 56, 57]. First, we separate the degrees of freedom of the hard(large  $x$ ) and soft(small  $s$ ) gluons in the nucleon. At high energy, the fast partons in the nucleon do not evolve during the scattering. Then we can regard them as time-independent objects(like glass) moving the speed of light. In this region, gluons are dominant, so we now treat the fast gluon as the valence degrees of freedom characterized by source charge  $\rho(x)$ . The soft gluons are treated as the dynamical gauge field obtained by solving the classical Yang-Mills equation of motion

$$[D_\mu, F^{\mu\nu}] = J^\nu \quad , \quad (3.41)$$

where  $J_a^\nu = \delta^{\nu+}\delta(x^-)\rho(x^-, \mathbf{x})$ , it is because the nucleon is localized  $x^- \sim 0$  at high energy, then gluons in the nucleon are light cone time independent. So the color charge of the gluons does not depend on  $x^+$ . In the lorentz gauge  $\partial_\nu A^\nu = 0$ , the equation becomes

$$-\nabla_\perp^2 b_a(x) = \rho_a(x) \quad , \quad (3.42)$$

where  $A_a^\mu = \delta^{\mu+}b_a$ . The expectation values of observables constructed by the gauge fields are obtained by the average of fluctuated color sources of the nucleon. It may be valid to assume that the nucleons have no correlation among them in the large nucleus, and from the central limit theorem the color sources are distributed as the Gaussian distribution. This distribution is called the McLerran-Venugopalan model, and we have:

$$\langle \rho_a(x^-, \mathbf{x}) \rho_b(y^-, \mathbf{y}) \rangle = g^2 \delta_{ab} \delta(x^- - y^-) \delta^{(2)}(\mathbf{x} - \mathbf{y}) \mu^2(x^-, \mathbf{x}) \quad . \quad (3.43)$$

Then we get:

$$\langle A_a^+(x^-, \mathbf{x}) A_b^+(y^-, \mathbf{y}) \rangle = g^2 \delta_{ab} \delta(x^- - y^-) \gamma(x^-, \mathbf{x}, \mathbf{y}) \quad , \quad (3.44)$$

where

$$\gamma(x^-, \mathbf{x}, \mathbf{y}) = \int d^2z \, G(\mathbf{x} - \mathbf{z}) G(\mathbf{y} - \mathbf{z}) \mu^2(x^-, \mathbf{z}) \quad , \quad (3.45)$$

and

$$\begin{aligned} G(\mathbf{x}) &= \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\mathbf{x}}}{\mathbf{k}^2 + m^2} \\ &= \frac{1}{2\pi} K_0(m|\mathbf{x}|) \quad , \end{aligned} \quad (3.46)$$

where  $m$  is the infrared cutoff. The weight functional is written as

$$W_Y[\rho] = \exp \left[ - \int dx^- d^2\mathbf{x} \frac{\rho_i(x^-, \mathbf{x}) \rho_i(x^-, \mathbf{x})}{2g^2 \mu^2(x^-, \mathbf{x})} \right] \quad . \quad (3.47)$$

The charge density  $\mu$  is related to the saturation momentum

$$Q_s^2 = \frac{g_s^2 C_F}{2\pi} \int dx^- d^2 \mathbf{x} \mu^2(x) . \quad (3.48)$$

Now we assume that the color charge distribution is related to nuclear thickness function  $T_A(b)$  for target nucleus

$$\mu_A(b) = \mu_0 A T_A(b) , \quad (3.49)$$

where the thickness function is defined by

$$T_A(b) = \int dz \rho_A(\sqrt{\mathbf{b}^2 + z^2}) , \rho_A(x) = \frac{N_A}{\exp\left(\frac{x-R_A}{\delta} + 1\right)} , \quad (3.50)$$

where  $\rho_A$  is the Woods-Saxon type potential and represent the configuration of the nucleon in the target nucleus,  $R_A = (1.12A^{1/3})\text{fm}$  is the nuclear radius and  $\delta = 0.54\text{fm}$  is the radial distance where the density rapidly decrease. There are  $A$  nucleons inside the nucleus and the distribution of each nucleon is the thickness function, then the scattering amplitude is

$$\begin{aligned} S(\mathbf{b}, r) &= \int \prod_{i=1}^A d^2 \mathbf{b}_i T_A(\mathbf{b}_i) e^{-\sum_{i=1}^A N_2(\mathbf{b}-\mathbf{b}_i)} \\ &= \left( \int d^2 \mathbf{b}' T_A(\mathbf{b}') e^{-N_2(\mathbf{b}-\mathbf{b}', r)} \right)^A , \end{aligned} \quad (3.51)$$

where  $N_2((\mathbf{x} + \mathbf{y})/2, \mathbf{x} - \mathbf{y}) = \frac{g^2 C_F}{2} (\gamma(\mathbf{x}, \mathbf{x}) + \gamma(\mathbf{y}, \mathbf{y}) - 2\gamma(\mathbf{x}, \mathbf{y}))$  is the two gluon exchange process between quark and a nucleon in the nucleus.

### 3.4.1 The Color Glass Condensate formalism

In any energy region, the color charge source does not distribute as a Gaussian but more general probability distribution  $W_Y[A]$ . In this formalism, observables are calculated as

$$\langle O[A] \rangle_Y = \frac{\int \mathcal{D}[A] O[A] W_Y[A]}{\int \mathcal{D}[A] W_Y[A]} , \quad (3.52)$$

where  $W_Y$  is a weight function with dependence on rapidity  $Y$ . As the energy increase, fast partons are more and more increasing, and the evolution of the probability distributions is governed by the B-JIMWLK equation[45, 58, 59, 60, 61, 62]:

$$\partial_Y W_Y = H_Y W_Y . \quad (3.53)$$

In the eikonal approximation, this scattering amplitude can be represented using the Wilson line which describes the multiple scattering of the quark and nucleus:

$$N(Y, \mathbf{x}, \mathbf{y}) = 1 - \frac{1}{N_c} \langle \text{tr}[U(\mathbf{x}) U^\dagger(\mathbf{y})] \rangle_Y , \quad (3.54)$$

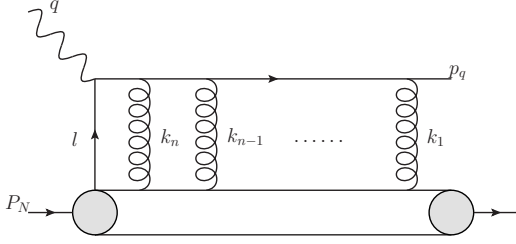


Figure 3.2: Multi-gluon exchange process

where  $U(\mathbf{x})$  is the Wilson line and defined as

$$U(z) = \text{P exp} \left\{ \frac{ig_s}{2} \int dz^- A^+(z^+ = 0, z^-, z) \right\} . \quad (3.55)$$

where  $P$  denotes the path ordering of  $A$  in  $z^-$  space. The color current has only a light cone  $+$  direction which is the direction of the nucleus moving. Thus in the covariant gauge, we can make the gauge field have only the light cone  $+$  direction.

### 3.5 Wilson lines

The Wilson lines are introduced to evaluate the confinement of the quarks in the proton[5]. The Wilson lines are defined as the path-ordered exponential of the integral of the gluon field along the two points.

$$\mathcal{U}[x; y] = P \exp \left( ig \int_{[x; y]} dz^\mu A_\mu^a(z) t^a \right) , \quad (3.56)$$

where  $[x; y]$  is the path along  $x$  to  $y$ .

We show the way how the Wilson line comes at high energy scatterings[63]. We consider the deep inelastic scattering process with multi-gluon exchange pictured at Fig.3.2. First, we consider one gluon exchange process. The amplitude of this process is

$$\langle p_q, P_f | j_{1\nu}(0) | P_N \rangle = g \bar{u}(p_q) \int \frac{d^4 k_1}{(2\pi)^4} \langle N_f | A_\mu \gamma^\mu S(p_q - k_1) \gamma_\nu \psi(0) | P_N \rangle , \quad (3.57)$$

where the quark propagator is

$$S(k) = \frac{k_\mu \gamma^\mu}{k^2 + i\epsilon} . \quad (3.58)$$

From the momentum conservation, we have

$$p_q = q + l + k_1 . \quad (3.59)$$

At high enough energy, we can use the Björken limit  $q_- \rightarrow \infty$ . The outgoing quark is on-shell  $p_q^2 = 0$ , then

$$l_+ + k_{1+} = x_B \quad (3.60)$$

holds. From this, we approximate the quark propagator as

$$S(p_q - k_1) \sim \frac{q_- \gamma_+}{2q_-(l_+ - x_B) + i\epsilon} \sim \frac{-\gamma_+}{2k_{1+} - i\epsilon} . \quad (3.61)$$

In this situation, the gluon field becomes

$$A_\mu \gamma^\mu \sim A_+ \gamma_- . \quad (3.62)$$

Using the relation  $\bar{u} \gamma_- \gamma_+ = 2\bar{u}$ , the integration in the amplitude becomes

$$\int \frac{d^4 k_1}{(2\pi)^4} \frac{1}{k_{1+} - i\epsilon} A_+(k_1) = i \int_{-\infty}^{\infty} dz_- \Theta(z_-) A_+(z_-, 0, 0_\perp) . \quad (3.63)$$

Then the one gluon exchange amplitude is

$$\langle p_q, P_f | j_{1\nu}(0) | P_N \rangle = \bar{u}(p_q) \gamma_\nu \langle N_f | (-ig) \int_{-\infty}^{\infty} dz_- \Theta(z_-) A_+(z_-, 0, 0_\perp) \psi(0) | P_N \rangle . \quad (3.64)$$

We now generalize this argument to  $n$ -gluon exchange processes and the amplitude of this process is

$$\begin{aligned} \langle p_q, P_f | j_{(n)\nu}(0) | P_N \rangle &= g^n \bar{u}(p_q) \int \prod_{i=1}^n \frac{d^4 k_i}{(2\pi)^4} \\ &\quad \times \langle N_f | A_\mu(k_1) \gamma^\mu S(p_q - k_1) A_\mu(k_2) \gamma^\mu S(p_q - k_1 - k_2) \dots \\ &\quad \times A_\mu(k_n) \gamma^\mu S(p_q - k_1 - \dots - k_n) \gamma_\nu \psi(0) | P_N \rangle . \end{aligned} \quad (3.65)$$

Applying the same formula for one gluon exchange process, we get the  $n$  gluon exchange amplitude

$$\begin{aligned} \langle p_q, P_f | j_{(n)\nu}(0) | P_N \rangle &= (-g)^n \bar{u}(p_q) \gamma_\nu \int \prod_{i=1}^n \frac{d^4 k_i}{(2\pi)^4} \\ &\quad \times \frac{1}{k_{1+} - i\epsilon} \frac{1}{k_{1+} + k_{2+} - i\epsilon} \dots \frac{1}{k_{1+} + \dots + k_{n+} - i\epsilon} \\ &\quad \times \langle N_f | A_\mu(k_1) A(k_2) \dots A(k_n) \psi(0) | P_N \rangle . \end{aligned} \quad (3.66)$$

The integrations can be simplified

$$\begin{aligned} &\int \prod_{i=1}^n \frac{d^4 k_i}{(2\pi)^4} \frac{1}{k_{1+} - i\epsilon} \frac{1}{k_{1+} + k_{2+} - i\epsilon} \dots \frac{1}{k_{1+} + \dots + k_{n+} - i\epsilon} \\ &\quad \times A_\mu(k_1) A(k_2) \dots A(k_n) \\ &= i^n \int_{-\infty}^{\infty} \prod_{i=1}^n dz_{i-} \Theta(z_{i-} - z_{(i+1)-}) A_+(z_{1-}) \dots \times A_+(z_{n-}) , \end{aligned} \quad (3.67)$$

where we define  $z_{(n+1)-} = 0$ . Finally, the  $n$  gluon exchange scattering amplitude is

$$\begin{aligned} \langle p_q, P_f | j_{(n)\nu}(0) | P_N \rangle &= (-ig)^n \bar{u}(p_q) \gamma_\nu \langle N_f | \int_0^\infty dz_{1-} \int_0^{z_{2-}} \dots \int_0^{z_{(n-1)-}} dz_{n-} \\ &\quad \times A_+(z_{1-}) A_+(z_{2-}) \dots A_+(z_{n-}) \psi(0) | P_N \rangle . \end{aligned} \quad (3.68)$$

Therefore we have the resummed amplitude with respect to the multi gluon exchange

$$\begin{aligned}
\langle p_q, P_f | j_\nu(0) | P_N \rangle &= \sum_{n=0}^{\infty} \langle p_q, P_f | j_{(n)\nu}(0) | P_N \rangle \\
&= \bar{u}(p_q) \gamma_\nu \langle N_f | P \exp \left( -ig \int_0^\infty dz_- A_+(z_-) \right) \psi(0) | P_N \rangle .
\end{aligned} \tag{3.69}$$

Then we see how a Wilson line comes from the multiple scattering.

## 3.6 Diffraction in DIS

### 3.6.1 Basics of the high energy diffractive processes

The diffraction is known in the optics. Light goes through an object and the strength of the light is recorded at some distance from the object. We see that the strength of the light has a diffractive pattern in the records. The same things happen in the high energy elastic scattering. The notion of the diffraction is not only limited to the elastic scattering but also generalizable to break up scatterings in the high energy collisions. The definition of the diffractive scattering is that there is a rapidity gap between projectile and target after the collisions. We may think that the diffractive processes are rare events but 15% of the collisions in the LHC are diffractive processes.

The definition of the diffractive processes is

*A reaction in which no quantum numbers are exchanged between the colliding particles.*

We derive the formula of the scattering amplitude of the diffractive scattering with using the color glass condensate formalism[64]. We define the wave function of the whole state  $|\Psi(t)\rangle$  which includes the target and the projectile. For simplicity, we consider the dipole-hadron scattering. Then the wave function before the scattering( $t = 0^-$ ) can be written as

$$|\Psi(t = 0^-)\rangle = |O(Y - Y_0)\rangle \otimes |h(Y_0)\rangle , \tag{3.70}$$

where  $|O\rangle$  is dipole state and  $|h\rangle$  is hadron state. In this expression, we assume that particles in the dipole state are not largely separated in rapidity. The dipole state is composed of the  $q\bar{q}$  pair and gluons and we define the  $q\bar{q}$  pair and  $n - 1$  gluons state as  $|n\rangle$ . Then we have

$$|O(Y - Y_0)\rangle = \sum_n C_n(Y - Y_0) |n\rangle . \tag{3.71}$$

The hadron state is represented as the average of the gauge field in the hadron

$$|h(Y_0)\rangle = \int D[A] \Phi[A](Y_0) |A\rangle , \tag{3.72}$$

where the square of the weight function  $W_Y := |\Phi[A](Y)|^2$  obey the B-JIMWLK equation

$$\partial_Y W_Y = H_{B-JIMWLK}(Y) W_Y \quad . \quad (3.73)$$

After the scattering, we need the S-matrices  $s(A)$  which come from the scattering of the dipole or gluon to the target hadron. Then the state of the system after the scattering is

$$\begin{aligned} |\Psi(t=0^+)\rangle &= \hat{S} |\Psi(t=0^-)\rangle = \int D[A] \Phi[A](Y_0) \\ &\times \sum_n C_n(Y-Y_0) \prod_{i=1}^n s_i[A] |n\rangle \otimes |A\rangle \quad , \end{aligned} \quad (3.74)$$

here,  $\hat{S}$  is the S-matrix. We now define the scattering amplitude for the projectile.

$$S_Y^{Op}[A] := \langle O(Y) | \hat{S} | p \rangle \quad , \quad (3.75)$$

where  $p$  is a wave function of the projectile. The expectation value for any observable  $\hat{O}$  is expressed as

$$\begin{aligned} \langle \hat{O} \rangle &= \int \mathcal{D}[A] \mathcal{D}[A'] \sum_{p'p''} (\delta^{Op'} - S_{Y-Y_0}^{Op'\dagger}[A']) (\delta^{p''O} - S_{Y-Y_0}^{p''O}[A]) \\ &\times \Phi^*[A'](Y_0) \Phi[A](Y_0) \mathcal{O}^{p'p''}[A, A'] \quad . \end{aligned} \quad (3.76)$$

Elastic scattering is a process that the projectile and the target are staying intact after scattering, and the cross section of it can be obtained by setting  $\hat{O} = |\Psi(t=0^-)\rangle \langle \Psi(t=0^-)|$

$$\sigma_{el} = \left| \int \mathcal{D}[A] (1 - S_{Y-Y_0}^{OO}[A]) |\Phi[A](Y_0)|^2 \right|^2 \quad . \quad (3.77)$$

For a more concrete example, we consider a di-jet production in the deep inelastic scattering process. As we described before, we need to know the dipole-nucleon scattering amplitude. It is represented as the dipole-nucleon scattering amplitude.

As an example, we consider the diffractive di-jet production in DIS. We assume that the rapidity separation between the two jets are small and can be treated as a kind of an elastic scattering process. As is usual to the DIS process, the virtual photon split into the quark-antiquark pair, and the dipole scatters with the nucleon. Before averaging the color sources from the target nucleon, the scattering amplitude is

$$\begin{aligned} 2\pi 2q^+ \delta(k_0^+ + k_1^+ - q^+) \mathcal{M}_{\gamma \rightarrow q\bar{q}} &= \int \frac{d^2 \mathbf{x}_0}{2\pi} \int \frac{d^2 \mathbf{x}_1}{2\pi} e^{-i\mathbf{p}_0 \cdot \mathbf{x}_0 - i\mathbf{p}_1 \cdot \mathbf{x}_1} \\ &\times \left( \left[ U(\mathbf{x}_0) U^\dagger(\mathbf{x}_1) \right]_{\beta_0 \beta_1} - \delta_{\beta_0 \beta_1} \right) \\ &\times \Psi_{q\bar{q}}(z_0, z_1, \mathbf{x}_0, \mathbf{x}_1) \quad . \end{aligned} \quad (3.78)$$

From our assumption, the cross section for this process becomes

$$(2\pi)^6 2p_0^+ 2p_1^+ \frac{d\sigma}{dp_0^+ dp_1^+ d^2\mathbf{p}_0 d^2\mathbf{p}_1} = \sum_{\beta_i h_i} (2p^+) 2\pi \delta(k_0^+ + k_1^+ - q^+) |\langle \mathcal{M}_{\gamma \rightarrow q\bar{q}} \rangle|^2, \quad (3.79)$$

where we take the color source average  $\langle O \rangle = \int \mathcal{D}[A] W_Y[A] O[A]$ .

$$\begin{aligned} (2p^+) 2\pi \delta(k_0^+ + k_1^+ - q^+) \langle \mathcal{M}_{\gamma \rightarrow q\bar{q}} \rangle &= \int \frac{d^2\mathbf{x}_0}{2\pi} \int \frac{d^2\mathbf{x}_1}{2\pi} e^{-i\mathbf{p}_0 \cdot \mathbf{x}_0 - i\mathbf{p}_1 \cdot \mathbf{x}_1} \\ &\quad \times \delta_{\beta_0 \beta_1} (-N(Y, \mathbf{x}_0, \mathbf{x}_1)) \\ &\quad \times \Psi_{q\bar{q}}(z_0, z_1, \mathbf{x}_0, \mathbf{x}_1) . \end{aligned} \quad (3.80)$$

The light cone wave function of a transverse polarized photon is

$$\begin{aligned} \Psi_T &= 2\pi e e_f \sqrt{z_0 z_1} \delta_{\beta_0, \beta_1} \delta_{h_0, -h_1} (z_1 - z_0 - (2h_0)\lambda) \int \frac{d^2\mathbf{k}}{2\pi} \frac{\epsilon_\lambda \cdot \mathbf{k}}{\mathbf{k}^2 + Q^2 z_0 z_1} \\ &\quad \times (-\tilde{N}(-\mathbf{k} + \mathbf{p}_0, \mathbf{k} + \mathbf{p}_0)) \frac{1}{(2\pi)^2} , \end{aligned} \quad (3.81)$$

where  $\tilde{N}$  is the Fourier transformed scattering amplitude.

### 3.7 Solving the impact parameter dependent BK equation

As we will see in the following chapter, we need to solve the impact parameter dependent BK equation to get the phase space distribution functions. It is because that the phase space distribution is localized in the impact parameter space, so we cannot use the translational invariance of the nucleus which is always used in many calculations to match the experimental results.

We can solve the impact parameter dependent BK equation directly, but it is not so precise[65]. So we would like to use some symmetry to reduce the parameters of the solutions. It has been found by Gubser[66] that this equation has SO(3) symmetry. This symmetry is already noticed in the jet physics[67]. Although the realistic initial conditions for the BK equation is not invariant under this symmetry, it is observed in numerical calculations that the symmetry is dynamically restored by evolving the rapidity. So we assume that the solution of the equation has the SO(3) symmetry. Let us show that the BK equation has the symmetry. First, we consider the 2-dimensional transverse plane as a complex plane  $\mathbf{x} = (x_1, x_2) \rightarrow x = x_1 + ix_2$ . The BK equation can be rewritten as

$$\begin{aligned} \partial_Y N(Y, x, y) &= \frac{\alpha_s N_c}{\pi} \int \frac{d^2 z}{2\pi} \frac{|x - y|^2}{|x - z|^2 |z - y|^2} \\ &\quad \times (N(Y, x, z) + N(Y, z, y) - N(Y, x, y) \\ &\quad - N(Y, x, z) N(Y, z, y)) . \end{aligned} \quad (3.82)$$

This equation is invariant under the symmetry

$$z \rightarrow \frac{\alpha z + \beta}{\gamma z + \delta} , \quad (3.83)$$

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where  $\alpha, \beta, \gamma$  and  $\delta$  are complex numbers and satisfy  $\alpha\delta - \beta\gamma = 1$ . To check this invariance, we mention that the one form  $\frac{x-y}{(x-z)(z-y)}dz$  preserve this symmetry, then it is easy to see the BK equation is invariant under this symmetry. The  $SO(3)$  symmetry is generated by inversion  $z \rightarrow -1/z$ . So it is easy to confirm the invariance. Under this assumption, the solution of this equation depends on the only one variable  $d^2$

$$d^2(\mathbf{x}, \mathbf{y}) = \frac{R^2(\mathbf{x} - \mathbf{y})^2}{(R^2 + \mathbf{x}^2)(R^2 + \mathbf{y}^2)} = \frac{R^2 r^2}{(R^2 + b^2 + \frac{r^2}{4})^2 - \frac{b^2 r^2}{2} - \frac{b^2 r^2}{2} \cos 2\phi_{br}} \quad , \quad (3.84)$$

where  $R$  is an arbitrary parameter with the dimension of length,  $\mathbf{b} = \frac{\mathbf{x} + \mathbf{y}}{2}$  and  $\mathbf{r} = \mathbf{x} - \mathbf{y}$ . This is the stereographic projection of the square of the distance function of the Poincaré sphere to the complex plane.

We show that the distance function is limited to 0 to 1. Using the Cauchy-Schwartz inequality

$$(\mathbf{x} \cdot \mathbf{y})^2 \leq \mathbf{x}^2 \mathbf{y}^2 \quad . \quad (3.85)$$

Then we find

$$\begin{aligned} R^4 + \mathbf{x}^2 \mathbf{y}^2 + 2R^2 \mathbf{x} \cdot \mathbf{y} &\geq R^4 + (\mathbf{x} \cdot \mathbf{y})^2 + 2R^2 \mathbf{x} \cdot \mathbf{y} \\ &= (R^2 + (\mathbf{x} \cdot \mathbf{y}))^2 \geq 0 \quad . \end{aligned} \quad (3.86)$$

From this inequality, we get

$$\begin{aligned} R^4 + \mathbf{x}^2 \mathbf{y}^2 + 2R^2 \mathbf{x} \cdot \mathbf{y} \geq 0 &\Leftrightarrow R^2(\mathbf{x} - \mathbf{y})^2 \leq (R^2 + \mathbf{x}^2)(R^2 + \mathbf{y}^2) \\ &\Leftrightarrow d^2(\mathbf{x}, \mathbf{y}) \leq 1 \quad . \end{aligned} \quad (3.87)$$

We finally get the inequality for the distance function  $0 \leq d^2 \leq 1$ .

This distance function is invariant under the symmetry  $r \rightarrow r_m^2/r, r_m = 2\sqrt{b^2 + R^2}$ . This behavior is already observed in the numerical calculations of the impact parameter dependent BK equation[68, 69].

Let us set  $\mathbf{y} = -\mathbf{x}$ , then we have a bijective map  $0 \leq |\mathbf{x}| \leq R$  to  $0 \leq d^2 \leq 1$ . From this map, we consider the solution of the BK equation depends on an one variable  $|\mathbf{x}|$ , so it is enough to consider  $S_Y(\mathbf{x}, -\mathbf{x}) = 1 - N(Y, \mathbf{x}, -\mathbf{x})$  in the  $0 \leq |\mathbf{x}| \leq R$  region. Now we split the integral region into two pieces in the BK equation.

$$\begin{aligned} \partial_Y S_Y(\mathbf{x}, -\mathbf{x}) &= \frac{\alpha_s N_c}{\pi} \left( \int_{|\mathbf{z}| \leq R} + \int_{|\mathbf{z}| > R} \right) \frac{d^2 \mathbf{z}}{2\pi} \frac{4x^2}{(\mathbf{x} - \mathbf{z})^2 (\mathbf{z} + \mathbf{x})^2} \\ &\quad \times (S_Y(\mathbf{x}, \mathbf{z}) S_Y(\mathbf{z}, -\mathbf{x}) - S_Y(\mathbf{x}, -\mathbf{x})) \quad , \end{aligned} \quad (3.88)$$

First we consider the region  $|\mathbf{x}| \leq R$ . For any vectors  $\mathbf{x}, \mathbf{z}$ , we define the associated vector  $\mathbf{x}_I$  by

$$d^2(\mathbf{x}, \mathbf{z}) = d^2(\mathbf{x}_I, \mathbf{x}_I) \quad . \quad (3.89)$$

Then we find

$$\mathbf{x}_I^2 = R^2 \left\{ -1 + \frac{2}{d^2(\mathbf{x}, \mathbf{z})} - \frac{2}{d^2(\mathbf{x}, \mathbf{z}) \sqrt{1 - d^2(\mathbf{x}, \mathbf{z})}} \right\} \quad . \quad (3.90)$$

Using the  $SO(3)$  symmetry, the region  $|\mathbf{x}| \lg R$  can be mapped to the region  $|\mathbf{x}| \leq R$ . If we consider the 2 dimensional plane as a complex plane  $\mathbf{x} = (x_1, x_2) \rightarrow \omega = x_1 + ix_2$ , then the BK equation is invariant under  $\omega \rightarrow -\frac{R^2}{\omega}$  as stated above. Now we choose  $\mathbf{x} = (x, 0)$ , we can rewrite the  $|\mathbf{z}| > R$  integration as

$$\frac{\alpha_s N_c}{\pi} \int_{|\mathbf{z}| \leq R} \frac{d^2 \mathbf{z}'}{2\pi} \frac{4x'^2}{(\mathbf{x}' - \mathbf{z}')^2 (\mathbf{z}' + \mathbf{x}')^2} (S_Y(\mathbf{x}', \mathbf{z}') S_Y(\mathbf{z}', -\mathbf{x}') - S_Y(\mathbf{x}', -\mathbf{x}')) , \quad (3.91)$$

where  $\mathbf{x}' = (-R^2/x, 0)$ . We parametrize the solution  $h_Y(x) := S_Y(\mathbf{x}, -\mathbf{x})$ , then the BK equation becomes

$$\begin{aligned} \partial_Y h_Y(x) = & \frac{\alpha_s N_c}{\pi} \int_{|\mathbf{z}| \leq R} \frac{d^2 \mathbf{z}}{2\pi} \left\{ \right. \\ & \frac{4x^4}{(\mathbf{x} - \mathbf{z})^2 (\mathbf{z} + \mathbf{x})^2} (h_Y(x_I(\mathbf{x}, \mathbf{z})) h_Y(x_I(\mathbf{z}, -\mathbf{x})) - h_Y(x)) \\ & \left. + \frac{4x'^4}{(\mathbf{x}' - \mathbf{z})^2 (\mathbf{z} + \mathbf{x}')^2} (h_Y(x_I(\mathbf{x}', \mathbf{z})) h_Y(x_I(\mathbf{z}, -\mathbf{x}')) - h_Y(x)) \right\} , \end{aligned} \quad (3.92)$$

where  $x_I(\mathbf{x}', -\mathbf{x}') = R^2/x'^2 = x^2$ . The scattering amplitude is given by

$$N(Y, \mathbf{x}, \mathbf{y}) = 1 - h_Y(x_I(\mathbf{x}, \mathbf{y})) . \quad (3.93)$$

We use this solution to investigate the Wigner distribution.

We also study another parameterization of the solution. We set  $\mathbf{y} = 0$ , then we have a bijective map  $0 \leq |\mathbf{x}| \leq \infty$  to  $0 \leq d^2 = \mathbf{x}^2/(\mathbf{x}^2 + R^2) \leq 1$ . After this, the BK equation becomes

$$\partial_Y S_Y(\mathbf{x}, 0) = \frac{\alpha_s N_c}{\pi} \int \frac{d^2 \mathbf{z}}{2\pi} \frac{\mathbf{x}^2}{(\mathbf{x} - \mathbf{z})^2 \mathbf{z}^2} (S_Y(\mathbf{x}, \mathbf{z}) S_Y(\mathbf{z}, 0) - S(\mathbf{x}, 0)) . \quad (3.94)$$

Now we parametrize  $S_Y(\mathbf{x}, \mathbf{y}) = S_Y(d^2(\mathbf{x}, \mathbf{y})) = S_Y(d^2(\mathbf{x}', 0)) = g_Y(|\mathbf{x}'|)$ , where  $\mathbf{x}'^2 = d^2(\mathbf{x}, \mathbf{y}) R^2 / (1 - d^2(\mathbf{x}, \mathbf{y}))$ . From

$$S_Y(\mathbf{x}, \mathbf{z}) = g_Y \left( \sqrt{\frac{R^4 (\mathbf{x} - \mathbf{z})^2}{R^4 + \mathbf{x}^2 \mathbf{z}^2 + 2R^2 \mathbf{x} \cdot \mathbf{z}}} \right) , \quad (3.95)$$

the BK equation for  $g_Y$  is

$$\begin{aligned} \partial_Y g_Y(|\mathbf{x}|) = & \frac{\alpha_s N_c}{\pi} \int \frac{d^2 \mathbf{z}}{2\pi} \frac{\mathbf{x}^2}{(\mathbf{x} - \mathbf{z})^2 \mathbf{z}^2} \\ & \times \left[ g_Y \left( \sqrt{\frac{R^4 (\mathbf{x} - \mathbf{z})^2}{R^4 + \mathbf{x}^2 \mathbf{z}^2 + 2R^2 \mathbf{x} \cdot \mathbf{z}}} \right) g_Y(|\mathbf{z}|) - g_Y(|\mathbf{x}|) \right] . \end{aligned} \quad (3.96)$$

### 3.7. SOLVING THE IMPACT PARAMETER DEPENDENT BK EQUATION 37

In this parametrization, we can neglect the angular part of  $\mathbf{x}$  in the left part of the equation. Then the equation becomes

$$\begin{aligned} \partial_Y g_Y(x) = & \frac{\alpha_s N_c}{\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^\infty \frac{dz}{z} \frac{x^2}{x^2 + z^2 - 2xz \cos \phi} \\ & \times \left[ g_Y \left( \sqrt{\frac{R^4(x^2 + z^2 - 2xz \cos \phi)}{R^4 + x^2 z^2 + 2R^2 xz \cos \phi}} \right) g_Y(z) - g_Y(x) \right] . \end{aligned} \quad (3.97)$$

Using the above parameterizations, we take initial conditions for the BK equation as

$$g_{Y=0}(x) = e^{-d^2(\mathbf{x},0)} = \exp \left( \frac{x^2}{R^2 + x^2} \right) \quad (3.98)$$

$$h_{Y=0}(x) = e^{-d^2(\mathbf{x},-\mathbf{x})} = \exp \left( \frac{4R^2 x^2}{(R^2 + x^2)^2} \right) . \quad (3.99)$$

Note that  $S_{Y=0} \sim e^{-r^2/R^2}$  when  $r, b \ll R$ . It is the same form of the Golec-Biernat Wusthoff model[70].

As expected from the SO(3) symmetry, the peak position of the S matrix is always at  $r = 2\sqrt{b^2 + R^2}$ .



## Chapter 4

# Wigner distribution

In this chapter, we introduce the parton Wigner distributions of the nucleon/nucleus at high energy. At small Björken  $x$ , we show that the gluon Wigner distributions are represented by functions of the dipole-nucleon/nucleus scattering amplitude. Using the solution of the BK equation derived in the previous chapter, we investigate the properties of the gluon Wigner distribution. This chapter is based on the paper[65].

### 4.1 The Wigner distribution in Quantum Mechanics

The Wigner distribution has been introduced by Wigner as a quantum phase space distribution[71]. The Wigner distribution of a pure state in quantum mechanics is defined as

$$\begin{aligned} W(q, p, t) &= \int d\xi e^{ip\xi/\hbar} \psi^*(q - \xi/2, t) \psi(q + \xi/2, t) \\ &= \int d\xi e^{ip\xi/\hbar} \langle q - \xi/2 | \rho(t) | q + \xi/2 \rangle . \end{aligned} \quad (4.1)$$

Where  $\psi$  is a wave function of the system and  $\rho(t)$  is the density matrix. This function satisfies the following condition:

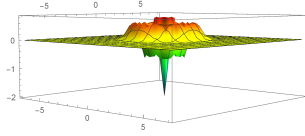
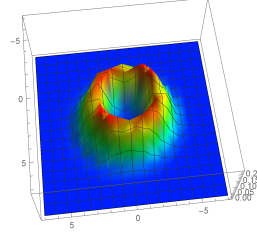
$$\int \frac{dq}{2\pi\hbar} W(q, p, t) = |\psi(p, t)|^2 , \quad \int \frac{dp}{2\pi\hbar} W(q, p, t) = |\psi(q, t)|^2 . \quad (4.2)$$

From the above formulas, we may think the Wigner distribution can be interpreted as the probability distribution, but from the uncertainty principle, it can have negative values. To calculate the expectation value of general observables, we just average over the phase space

$$\langle \hat{O}(q, p, t) \rangle = \int dq dp W(q, p, t) O(q, p, t) . \quad (4.3)$$

In quantum mechanics, there is a positive definite distribution called the Husimi distribution which is known to have the same information as the Wigner distribution[72]. This function is obtained by Gaussian smearing the Wigner distribution in the minimal uncertainty region  $\delta q \delta p = \hbar/2$ .

$$f_H(q, p, t) = \frac{1}{\pi\hbar} \int dq' dp' e^{-m\omega(q'-q)^2/\hbar - (p'-p)^2/(m\omega\hbar)} W(q', p', t) , \quad (4.4)$$

Figure 4.1: Wigner distribution( $n = 3$ )Figure 4.2: Husimi distribution( $n = 3$ )

The Husimi distribution can be rewritten as the coherent states  $|\lambda\rangle$  which is the eigenstates of the annihilation operator  $\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$ .

$$f_H(q, p, t) = \langle \lambda | \rho(t) | \lambda \rangle = |\langle \psi | \lambda \rangle|^2 \geq 0 \quad . \quad (4.5)$$

Let us show the example of the Wigner distribution fig. 4.1 and the Husimi distribution fig. 4.2 of the one-dimensional harmonic oscillator in quantum mechanics. The analytical form of these functions are

$$W(q, p)^{(n)} = 2(-1)^n e^{-\frac{2H}{\hbar\omega}} L_n \left( \frac{4H}{\hbar\omega} \right) \quad (4.6)$$

$$f_H^{(n)}(q, p) = \frac{1}{n!} e^{-\frac{H}{\hbar\omega}} \left( \frac{H}{\hbar\omega} \right)^n, \quad (4.7)$$

where  $L_n$  are the Laguerre polynomials,  $H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$  is the Hamiltonian of the harmonic oscillator and show the  $n$ -th excited state. We see that the Wigner distribution is oscillating and not positive definite distribution and the Husimi distribution is positive definite. The peak position of the Husimi distribution represents the classical orbit of the harmonic oscillator. The Wigner distributions are measured in experiments directly[73, 74, 75].

## 4.2 The phase space distribution in the small $x$ region

In this section, we explain how to calculate the matrix element in the saturation region. We show the concrete examples which will be used at the following section[76]. We can transform the gluon distribution at small- $x$  as

$$\begin{aligned} x f_g(x, \mathbf{k}_\perp, \Delta_\perp) &= \frac{1}{P^+} \int \frac{dz^- d^2 \mathbf{z}}{(2\pi)^3} e^{i\mathbf{k}_\perp \mathbf{z} - ixP^+ z^-} \\ &\quad \times \langle P + \Delta_\perp / 2 | \text{tr}(G^{+j}(0, z^-, \mathbf{z}) \mathcal{U}[z, 0] \\ &\quad \times G^{+j}(0) \mathcal{U}^\dagger[0, z]) | P - \Delta_\perp / 2 \rangle \\ &= \frac{2}{\langle P | P \rangle} \int \frac{d^3 z d^3 z'}{(2\pi)^3} e^{i\mathbf{k}_\perp (\mathbf{z} - \mathbf{z}') + i\frac{\Delta}{2} (z + z')} \\ &\quad \times \langle P | \text{tr}(G^{+j}(0, z^-, \mathbf{z}) \mathcal{U}[z, z'] \\ &\quad \times G^{+j}(0, z'^-, \mathbf{z}') \mathcal{U}^\dagger[z', z]) | P \rangle \quad . \end{aligned} \quad (4.8)$$

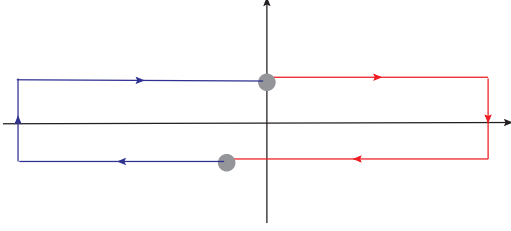


Figure 4.3: The trajectory of the dipole type Wilson line

We use the approximation  $e^{-ixP^+(z^- - z'^-)} \simeq 1$ . We identify

$$\begin{aligned} \frac{\langle P|O(z, z')|P \rangle}{\langle P|P \rangle} &= \langle O(z, z') \rangle_{Y=\log(1/x)} \\ &= \int \mathcal{D}[A] O(z, z') W_Y[A] . \end{aligned} \quad (4.9)$$

In the following, we show the parton distribution function with the saturation effects.

#### 4.2.1 The Wigner distribution at small- $x$

In this section, we show the formation of the Wigner distribution at small- $x$ . The Wigner distribution functions are defined by the Fourier transformed GTMD.

$$xW(x, \mathbf{k}, \mathbf{b}) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\Delta_\perp \mathbf{b}} x f_g(x, \mathbf{k}, \Delta_\perp) . \quad (4.10)$$

Using the formula we have shown the previous chapter, the gluon Wigner distributions are

$$\begin{aligned} xW(x, \mathbf{k}, \mathbf{b}) &= 2 \int \frac{d^3z d^3z'}{(2\pi)^3} e^{i\mathbf{k}_\perp(\mathbf{z}-\mathbf{z}') + i\frac{\Delta}{2}(\mathbf{z}+\mathbf{z}')} \delta^{(2)}\left(\mathbf{b} - \frac{\mathbf{z} + \mathbf{z}'}{2}\right) \\ &\quad \times \langle G^{+j}(0, z^-, \mathbf{z}) \mathcal{U}[z, z'] G^{+j}(0, z'^-, \mathbf{z}') \mathcal{U}'^\dagger[z', z] \rangle_Y \\ &= 2 \int \frac{d^3z d^3z'}{(2\pi)^3} e^{i\mathbf{k}_\perp(\mathbf{z}-\mathbf{z}') + i\frac{\Delta}{2}(\mathbf{z}+\mathbf{z}')} \delta^{(2)}\left(\mathbf{b} - \frac{\mathbf{z} + \mathbf{z}'}{2}\right) \\ &\quad \times \left\langle \text{tr} \left[ U(\infty, z^-; \mathbf{z}_\perp) G^{+j}(0, z^-, ; \mathbf{z}_\perp) U(z^-, -\infty; \mathbf{z}_\perp) \right. \right. \\ &\quad \left. \left. \times U(-\infty, z'^-; \mathbf{z}'_\perp) G^{+j}(0, z'^-, \mathbf{z}') U(z'^-, \infty; \mathbf{z}'_\perp) \right] \right\rangle_Y . \end{aligned} \quad (4.11)$$

If we choose  $\mathcal{U}[x, y]$  as the future pointing Wilson lines  $\mathcal{U}[x, y] = U(x^-, -\infty, \mathbf{x}) U(-\infty, y^-, \mathbf{y})$  and  $\mathcal{U}'[x, y]$  as the past pointing Wilson lines, and the Wigner distribution is called the Dipole Wilson lines as pictured Fig. 4.3. We mainly consider this Wigner distribution. Using the formula

$$\partial_i U(\mathbf{x}) = ig_s \int_{-\infty}^{\infty} dx^- U(-\infty, x^-; \mathbf{x}) G^{i+}(x) U(x^-, \infty; \mathbf{x}) , \quad (4.12)$$

we have

$$\begin{aligned}
xW(x, \mathbf{k}, \mathbf{b}) &= \frac{2N_c}{\alpha_s} \int \frac{d^2\mathbf{r}}{(2\pi)^2} e^{i\mathbf{k}\mathbf{r}} \left( \frac{1}{4} \nabla_{\mathbf{b}}^2 + \mathbf{k}^2 \right) S(Y = \log(1/x), \mathbf{b} + \mathbf{r}/2, \mathbf{b} - \mathbf{r}/2) \\
&= -\frac{2N_c}{\alpha_s} \int \frac{d^2\mathbf{r}}{(2\pi)^2} e^{i\mathbf{k}\mathbf{r}} \left( \frac{1}{4} \nabla_{\mathbf{b}}^2 + \mathbf{k}^2 \right) N(Y = \log(1/x), \mathbf{b} + \mathbf{r}/2, \mathbf{b} - \mathbf{r}/2) ,
\end{aligned} \tag{4.13}$$

where  $S = 1 - N$  is the S matrix of the dipole-nucleon collision and using the formula  $\mathbf{k}^2 \delta^2(\mathbf{k}) = 0$ . Then the dipole Wigner distribution can be described using the dipole-nucleus scattering amplitude. In this formalism, the confinement effects are not fully included and at large  $r$  the scattering amplitude behave as  $N(r) \sim 1/r^{2\gamma}$  (We found  $\gamma \sim 0.79$  at  $Y = 5$  and  $\gamma \sim 0.79$  at  $Y = 10$ ) at large  $r$  region. After the angular integration in (4.13), The  $r$  integral becomes

$$\int_0^\infty dr \frac{r J_0(kr)}{r^{2\gamma}} . \tag{4.14}$$

Due to the oscillation of the Bessel function  $J_n$  in the integral, this integral is weakly converge. The large distance region of this integration is unphysical and cannot obtain the precise behavior of the large  $r$  region from the perturbative calculation. So we simply cut off the large distance by inserting the Gaussian factor  $e^{-\epsilon r^2}$  in the integration

$$\begin{aligned}
xW(x, \mathbf{k}, \mathbf{b}) &= -\frac{2N_c}{\alpha_s} \int \frac{d^2\mathbf{r}}{(2\pi)^2} e^{-\epsilon r^2} e^{i\mathbf{k}\mathbf{r}} \left( \frac{1}{4} \nabla_{\mathbf{b}}^2 + \mathbf{k}^2 \right) \\
&\quad \times N(Y = \log(1/x), \mathbf{b} + \mathbf{r}/2, \mathbf{b} - \mathbf{r}/2) .
\end{aligned} \tag{4.15}$$

### 4.3 The dipole type PDFs of QCD at small- $x$

As we have already seen that the parton distribution functions at small- $x$  are related to the dipole-nucleus scattering amplitude. In this section, we show the numerical analysis of these distribution functions and see the properties of the distributions. This section is one of the main parts of this thesis.

#### 4.3.1 The dipole Wigner distribution

Let us evaluate the angular dependent and independent part separately, and we expand the dipole Wigner distribution function in Fourier series

$$xW(x, \mathbf{k}, \mathbf{b}) = xW_0(x, k, b) + 2 \sum_{n=1}^{\infty} xW_n(x, k, b) \cos(2n\phi_{bk}) . \tag{4.16}$$

Let us concretely show the  $n = 0, 1$  part of the expansion.

$$\begin{aligned}
xW_0(x, k, b) &= -\frac{N_c}{2\alpha_s\pi^2} \left( \frac{1}{4} \frac{\partial^2}{\partial b^2} + \frac{1}{4b} \frac{\partial}{\partial b} + k^2 \right) \int_0^\infty dr r e^{-\epsilon r^2} J_0(kr) \\
&\quad \times \int_0^{2\pi} d\phi_{br} N_Y(r, b, \cos 2\phi_{br}) ,
\end{aligned} \tag{4.17}$$

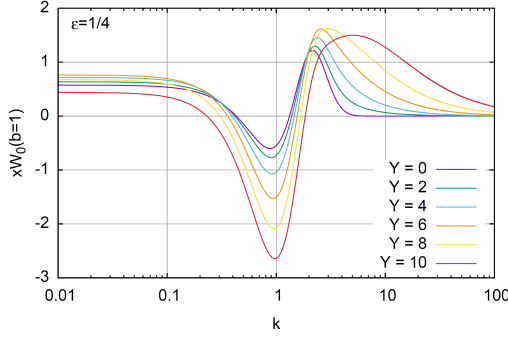


Figure 4.4: The angle independent part of the Wigner distribution at fixed impact parameter  $b = 1$

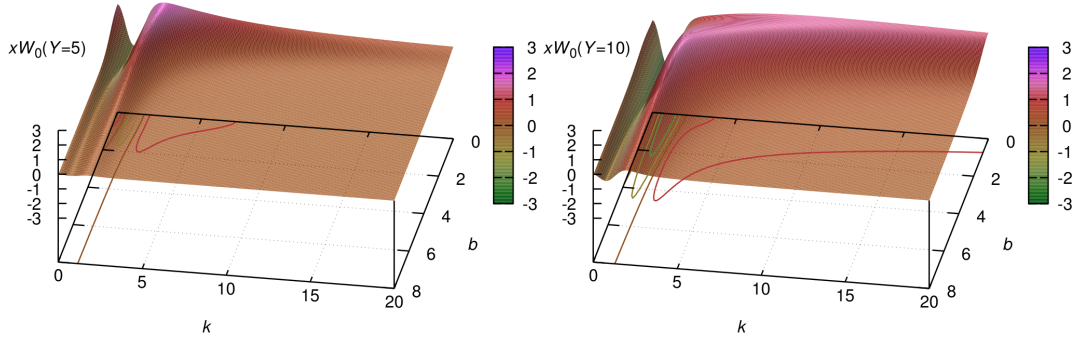


Figure 4.5: The angle independent part of the Wigner distribution at fixed rapidity  $Y = 5$ (left) and  $Y = 10$ (right)

and

$$\begin{aligned}
 xW_1(x, k, b) &= \frac{N_c}{2\alpha_s\pi^2} \left( \frac{1}{4} \frac{\partial^2}{\partial b^2} + \frac{1}{4b} \frac{\partial}{\partial b} - \frac{1}{b^2} + k^2 \right) \int_0^\infty dr \, r e^{-\epsilon r^2} J_2(kr) \\
 &\quad \times \int_0^{2\pi} d\phi_{br} \cos(2\phi_{br}) N_Y(r, b, \cos 2\phi_{br}) \quad . \quad (4.18)
 \end{aligned}$$

Later we will show the effects of the higher order terms in the Fourier expansion is not large. The impact parameter dependent BK equation does not have incorporated the full non-perturbative effects, so we introduce the Gaussian factor to mimic the confinement effects. We show the elliptic Wigner distribution which cannot be seen from GPDs or TMDs in Fig. 4.4. The results of the calculations are shown in Fig. 4.5 ~ 4.7. From these results, the positive peak position of the angular independent part moves towards large momentum as rapidity increases and can be identified with the saturation momentum which is an increasing function of rapidity and a decreasing function of the impact parameter. For the elliptic part, the magnitude of the peak is 3 ~ 5 % of that of the angular independent part. The positive peak positions have smaller dependence on the rapidity than that of the angular independent part. The intuitive explanation of this small rapidity dependence is explained as follows: the form of the solution of the BK equation in the geometric

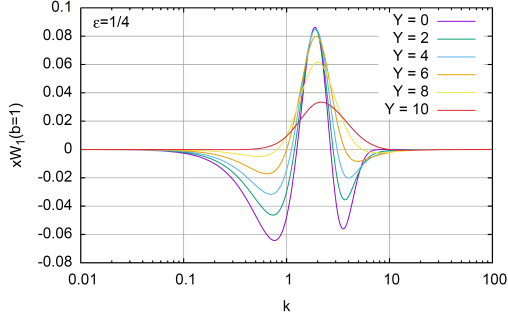


Figure 4.6: The elliptic part of the Wigner distribution at fixed impact parameter  $b = 1$

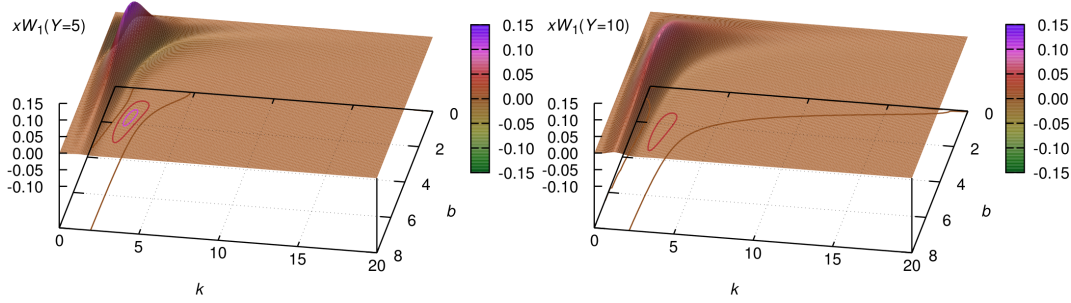


Figure 4.7: The elliptic part of the Wigner distribution at fixed rapidity  $Y = 5$ (left) and  $Y = 10$ (right)

scaling region are taking the form[66]

$$S_Y(\mathbf{x}, \mathbf{y}) = S_Y(d(\mathbf{x}, \mathbf{y})) \sim g(Q_s^2(Y)d^2(\mathbf{x}, \mathbf{y})) . \quad (4.19)$$

Expanding  $d^2$  in a small area of the relative distance  $r = |\mathbf{x} - \mathbf{y}| \ll R$  and in a not large area of the impact parameter  $b = |\mathbf{x} + \mathbf{y}|/2 \leq R$ , we have

$$d^2 \sim \frac{r^2}{R^2} \left( 1 + \frac{b^2 r^2}{2R^4} \cos 2\phi_{br} \right) . \quad (4.20)$$

Then the estimation of the peak position of the elliptic part is  $k_{peak} \sim 1/r \sim \sqrt{Q_s(Y)}$ , and it is slower rapidity evolution than that of the angular independent part  $k_{peak} \sim Q_s(Y)$ . The elliptic part has effects on the elliptic flow in  $pA$  collisions as we show in the last chapter.

### 4.3.2 The Husimi distribution

We define the Husimi distribution in QCD[77]:

$$H(x, \mathbf{k}, \mathbf{b}) = \frac{1}{\pi^2} \int d^2 \mathbf{b}' d^2 \mathbf{k}' e^{-\frac{1}{i^2}(\mathbf{b}-\mathbf{b}')-l^2(\mathbf{k}-\mathbf{k}')} W(x, \mathbf{k}', \mathbf{b}') . \quad (4.21)$$

This distribution is defined as the Gaussian smeared Wigner distribution in the area of the phase space which satisfy the minimal uncertainty principle  $\Delta k \Delta b = \frac{1}{2}$ .

Thus this distribution can be interpreted as the semi-classical representation of the Wigner distribution. We expect the Husimi distribution is positive definite as in quantum mechanics. Let us first evaluate the distribution functions in a Golec-Biernat-Wusthoff(GBW)-like model[70]

$$1 - N(x, \mathbf{b} + \mathbf{r}/2, \mathbf{b} - \mathbf{r}/2) = e^{-\frac{1}{4}\mathbf{r}^2 Q_s^2(x, \mathbf{b})} , \quad (4.22)$$

where  $Q_s^2(x, \mathbf{b})$  is the gluon saturation momentum. Then the dipole Wigner distribution is

$$xW(x, \mathbf{k}, \mathbf{b}) = \frac{2N_c}{\alpha_s(2\pi)^2} \left( \frac{\partial}{\partial \mathbf{b}^2} \mathbf{b}^2 \frac{\partial}{\partial \mathbf{b}^2} + \mathbf{k}^2 \right) \frac{e^{-\frac{\mathbf{k}^2}{Q_s^2}}}{\pi Q_s^2} . \quad (4.23)$$

This distribution is not positive definite for the realistic profile of  $Q_s$ . On the other hand, the Husimi distribution is written as

$$\begin{aligned} xH(x, \mathbf{k}, \mathbf{b}) &= \frac{2N_c}{l^4 \alpha_s \pi^2 (2\pi)^2} \int d^2 \mathbf{b}' e^{-(\mathbf{b}-\mathbf{b}')^2/l^2 - \frac{l^2}{1+l^2 Q_s^2} \mathbf{k}^2} \\ &\times \left[ \frac{(\mathbf{b}-\mathbf{b}')^2}{l^2} + \frac{(l^2 Q_s^2)^2}{(1+l^2 Q_s^2)^2} l^2 \mathbf{k}^2 - \frac{1}{1+l^2 Q_s^2} \right] \frac{l^2}{1+l^2 Q_s^2} . \end{aligned} \quad (4.24)$$

If we assume that  $Q_s^2(\mathbf{b}^2)$  is an arbitrary function which is monotonically decreasing with increasing  $\mathbf{b}^2$ , then the Husimi distribution is positive definite. To prove this, we note  $\frac{1}{1+l^2 Q_0^2} \leq \frac{1}{1+l^2 Q_s^2} < 1$  where  $Q_0^2 = Q_s(\mathbf{b}^2 = 0)$ . Then we find[78]

$$xH(x, \mathbf{k}, \mathbf{b}) > \frac{2N_c e^{-l^2 \mathbf{k}^2}}{l^2 \alpha_s \pi^2 (2\pi)^2 (1+l^2 Q_0^2)} \int d^2 \mathbf{b}' e^{-\frac{(\mathbf{b}-\mathbf{b}')^2}{l^2}} \left[ \frac{(\mathbf{b}-\mathbf{b}')^2}{l^2} - 1 \right] = 0 . \quad (4.25)$$

Now we proved that the Husimi distribution is positive definite for the phenomenological model in the small  $x$  region. It is natural because the gluon field is a coherent state in the saturation region.

Next, we evaluate the Husimi distribution by using the solution of the BK equation. Using (4.13), we get

$$\begin{aligned} xH(x, \mathbf{k}, \mathbf{b}) &= -\frac{2N_c}{l^4 \alpha_s \pi} \int d^2 \mathbf{b}' \frac{d^2 \mathbf{r}}{(2\pi)^2} e^{-(\mathbf{b}-\mathbf{b}')^2/l^2 - \frac{\mathbf{r}^2}{4l^2} + i\mathbf{k}\mathbf{r}} \\ &\times \left\{ \frac{(\mathbf{b}-\mathbf{b}')^2}{l^2} + l^2 \left( \mathbf{k} + \frac{i\mathbf{r}}{2l^2} \right) \right\} N(x, \mathbf{b} + \mathbf{r}/2, \mathbf{b} - \mathbf{r}/2) . \end{aligned} \quad (4.26)$$

As is the same for the Wigner distribution, we expand it in Fourier series

$$\begin{aligned} xH_0(x, k, b) &= \frac{2N_c}{l^4 \alpha_s \pi} \int b' db' \frac{r dr}{2\pi} e^{-(b^2+b'^2)/l^2 - \frac{r^2}{4l^2}} \\ &\times \left[ \left\{ \left( \frac{b^2+b'^2}{l^2} + l^2 k^2 - \frac{r^2}{4l^2} \right) I_0 \left( \frac{2bb'}{l^2} \right) - \frac{2bb'}{l^2} I_1 \left( \frac{2bb'}{l^2} \right) \right\} J_0(kr) \right. \\ &\left. - kr I_0 \left( \frac{2bb'}{l^2} \right) J_1(kr) \right] \int_0^{2\pi} d\phi_{b'r} N(x, r, b', \cos(2\phi_{b'r})) \end{aligned} \quad (4.27)$$

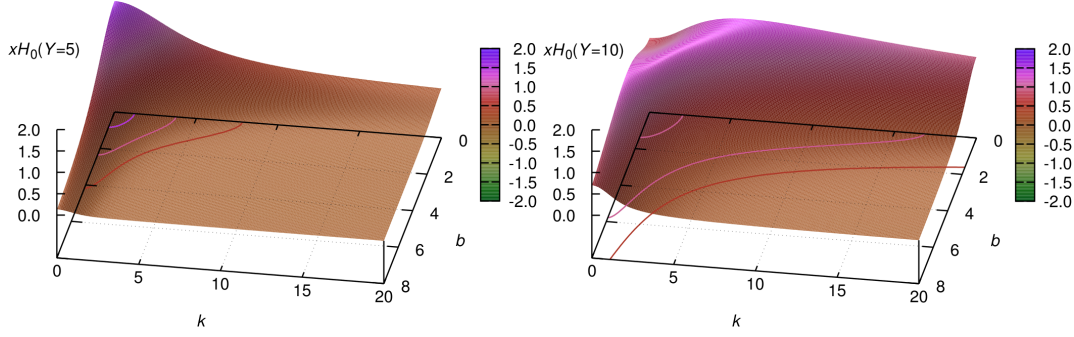


Figure 4.8: The angle independent part of the Husimi distribution at fixed rapidity  $Y = 5$ (left) and  $Y = 10$ (right)

$$\begin{aligned}
 xH_1(x, k, b) = & \frac{2N_c}{l^4 \alpha_s \pi} \int b' db' \frac{r dr}{2\pi} e^{-(b^2 + b'^2)/l^2 - \frac{r^2}{4l^2}} \\
 & \times \left[ \left\{ \left( \frac{b^2 + b'^2}{l^2} + l^2 k^2 - \frac{r^2}{4l^2} \right) I_2 \left( \frac{2bb'}{l^2} \right) - \frac{2bb'}{l^2} I_1 \left( \frac{2bb'}{l^2} \right) \right\} J_2(kr) \right. \\
 & \left. - kr I_2 \left( \frac{2bb'}{l^2} \right) J_1(kr) \right] \int_0^{2\pi} d\phi_{b'r} \cos(2\phi_{b'r}) N(x, r, b', \cos(2\phi_{b'r})) ,
 \end{aligned} \tag{4.28}$$

where  $I_n$  are the modified Bessel functions. Here the parameter  $l$  is arbitrary and we set  $l = R = 1$  in order that the Gaussian factor in the  $r$  integration becomes the same to the integration in the Wigner distribution.

From these figures (Fig. 4.8), the Husimi distribution is positive definite and can be interpreted as a probability distribution. As for the case of the Wigner distribution, we see that the gluons are mainly distributed around the saturation momentum.

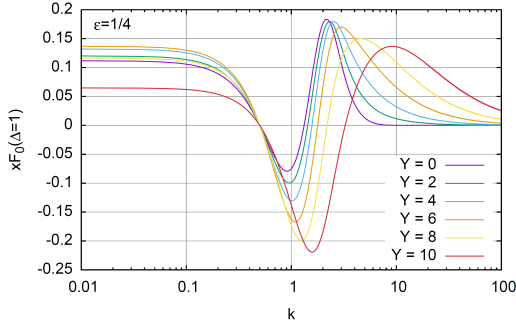
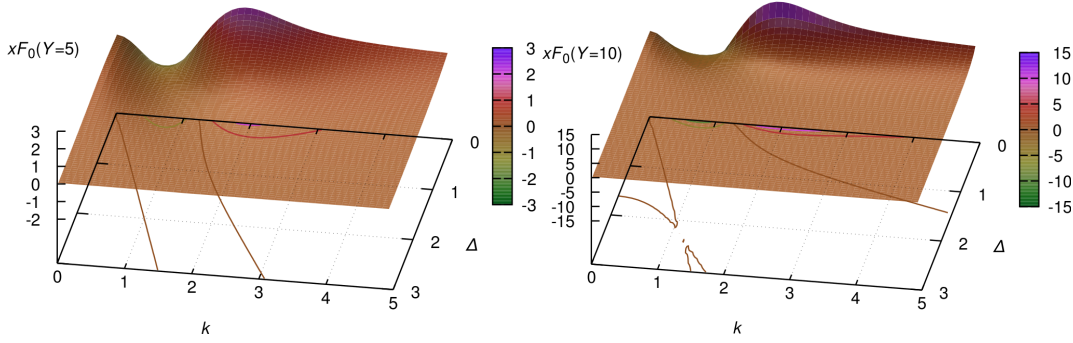
### 4.3.3 GTMD

At last in this chapter, we consider the gluon GTMD which is defined by

$$\begin{aligned}
 xF(x, \mathbf{k}, \Delta_\perp) &:= \int \frac{d^2 \mathbf{b}}{(2\pi)^2} e^{i\mathbf{b} \Delta_\perp} xW(x, \mathbf{k}, \mathbf{b}) \\
 &= \frac{2N_c}{\alpha_s} \left( \frac{\Delta_\perp^2}{4} - k^2 \right) \int \frac{d^2 \mathbf{r}}{(2\pi)^2} \int \frac{d^2 \mathbf{b}}{(2\pi)^2} e^{i\mathbf{k}\mathbf{r} + i\mathbf{b}\Delta_\perp} \\
 &\quad \times N(x, \mathbf{b} + \mathbf{r}/2, \mathbf{b} - \mathbf{r}/2) .
 \end{aligned} \tag{4.29}$$

As was done before, we expand the GTMD in the Fourier series

$$xF(x, \mathbf{k}, \Delta_\perp) = xF_0(x, k, \Delta) + 2 \sum_{n=1}^{\infty} xF_n(x, k, \Delta) \cos(2n\phi_{k\Delta}) . \tag{4.30}$$

Figure 4.9: The angler independent part of the GTMD at fixed  $\Delta = 1$ Figure 4.10: The angler independent part of the GTMD at fixed rapidity  $Y = 5$ (left) and  $Y = 10$ (right)

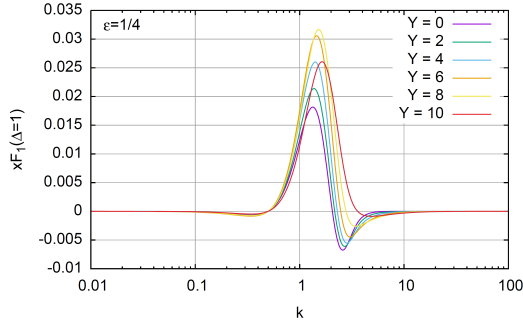
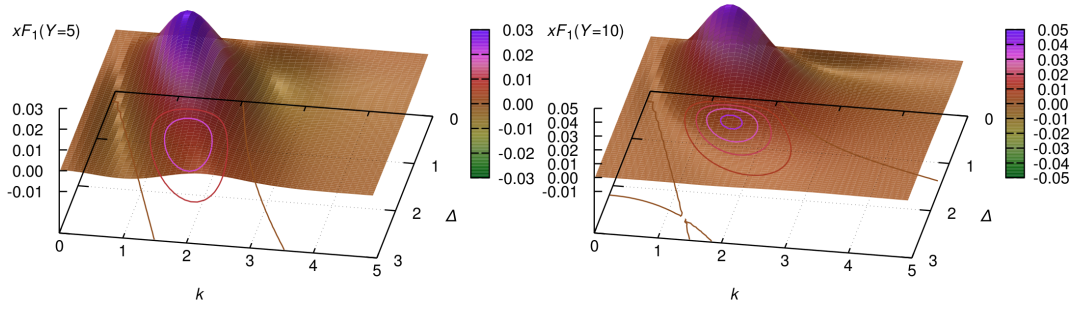
We show the first two terms

$$\begin{aligned}
 xF'_0(x, k, \Delta) &= \frac{N_c}{2\pi^2\alpha_s} \left( \frac{\Delta^2}{4} - k^2 \right) \int_0^\infty r J_0(kr) e^{-\epsilon r^2} dr \int_0^\infty b J_0(b\Delta) db \\
 &\quad \times \int \frac{d\phi_{br}}{2\pi} N(x, \mathbf{r}, \mathbf{b}) \quad , \quad (4.31)
 \end{aligned}$$

$$\begin{aligned}
 xF'_1(x, k, \Delta) &= -\frac{N_c}{2\pi^2\alpha_s} \left( \frac{\Delta^2}{4} - k^2 \right) \int_0^\infty r J_2(kr) e^{-\epsilon r^2} dr \int_0^\infty b J_2(b\Delta) db \\
 &\quad \times \int \frac{d\phi_{br}}{2\pi} \cos(2\phi_{br}) N(x, \mathbf{r}, \mathbf{b}) \quad . \quad (4.32)
 \end{aligned}$$

We introduce the Gaussian factor as in the Wigner distribution in the  $\mathbf{r}$  integration. It is not necessary to insert the Gaussian term in the  $\mathbf{b}$  integration, because of the  $N$  converge to zero quickly at  $|\mathbf{b}| \rightarrow \infty$ .

The results of the calculations are shown in Fig 4.9, 4.10, 4.11, 4.12. From the positive peak position, we can define the saturation momentum  $Q_s(Y, \Delta)$ . The angular independent part falls steeply with  $\Delta$  increasing but the elliptic part has a positive peak at finite  $\Delta \sim 1$ . The saturation momentum  $Q_s(Y, \Delta)$  is an increasing function of  $\Delta$ .

Figure 4.11: The elliptic part of the GTMD at fixed  $\Delta = 1$ Figure 4.12: The elliptic part of the GTMD at fixed rapidity  $Y = 5$ (left) and  $Y = 10$ (right)

Let us summarize this chapter, we have considered the phase space distributions in QCD. Using the BK equation with respect to the  $SU(3)$  symmetry, we can get the gluon phase space distributions. We will see at the next chapter that the GTMD is easier to access from the experiments.

## Chapter 5

# Accessing the dipole Wigner distribution

I have considered the properties of the dipole Wigner distribution so far. It is important to know the Wigner distribution, but is there any way to measure this function? In this chapter, we show that the dipole Wigner distribution can be obtained by the differential cross section of the ultraperipheral diffractive di-jet production process. This chapter is based on the paper[78].

### 5.1 diffractive di-jet production in DIS

In this section, we focus on the diffractive di-jet production in DIS at small- $x$ . We consider  $\gamma^* + p \rightarrow q + \bar{q} + p$  process[79, 80](see Fig. 5.1).

Let us first show the kinematics of this process. We take the rapidity of the dijet as  $y_1, y_2$ . The invariant mass of the dijet is

$$m^2 = 2(k_1^+ + k_2^+)(k_1^- + k_2^-) - (\mathbf{k}_1 + \mathbf{k}_2)^2 = 2|\mathbf{k}_1||\mathbf{k}_2|(\cosh(y_1 - y_2) - \cos \phi_{12}) \quad , (5.1)$$

where  $\phi_{12}$  is the relative angle between the dijet. The value of the momentum fraction of the parton from the nucleon is determined from the condition  $(xp + q)^2 = m^2$ .

$$x = \frac{m^2}{4E_p\omega} = \frac{|\mathbf{k}_1||\mathbf{k}_2|(\cosh(y_1 - y_2) - \cos \phi_{12})}{2E_p\omega} \quad , \quad (5.2)$$

where  $q^\mu = \omega(1, 0, 0_\perp)$  is the momentum of the photon. Then the rapidity gap between the dijet and the nucleon is

$$\begin{aligned} \Delta Y &= \min\{y_1, y_2\} + \log \frac{\sqrt{2}p^-}{|\mathbf{k}_1 + \mathbf{k}_2|} \\ &= \min \left\{ \log \frac{\sqrt{2}k_1^+}{|\mathbf{k}_1|}, \log \frac{\sqrt{2}k_2^+}{|\mathbf{k}_2|} \right\} + \log \frac{2E_p}{|\mathbf{k}_1 + \mathbf{k}_2|} \\ &\sim \log \frac{E_p\omega}{\mathbf{k}^2} \sim \log \frac{1}{x} \quad , \end{aligned} \quad (5.3)$$

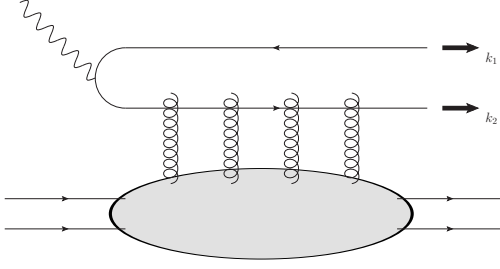


Figure 5.1: The di-jet production in deep inelastic scattering

where we use approximations  $y_1 \sim y_2$ ,  $\mathbf{k}_1 \sim \mathbf{k}_2$  and neglect some constant because we now focus on the small  $x$  region.

The total cross section is (see section 3.6)

$$\begin{aligned} \frac{d\sigma^{\gamma^*p}}{dy_1 dy_2 d^2\mathbf{k}_1 d^2\mathbf{k}_2}(x, Q^2) &= N_c \alpha_{em} (2\pi)^2 q^+ \delta(k_1^+ + k_2^+ - q^+) \\ &\times \sum_f e_f^2 2z(1-z)(z^2 + (1-z)^2) |\vec{M}|^2 \quad . \quad (5.4) \end{aligned}$$

The scattering amplitude is

$$\vec{M} = \int d^2\mathbf{k} S(x, \mathbf{k}, \Delta_\perp) \left[ \frac{P_\perp - \mathbf{k}}{(P_\perp - \mathbf{k})^2 + z(1-z)Q^2} \right] \quad , \quad (5.5)$$

where the  $P_\perp = (\mathbf{k}_2 - \mathbf{k}_1)/2$ ,  $\Delta_\perp = -(\mathbf{k}_1 + \mathbf{k}_2)$ ,  $z = \frac{|\mathbf{k}_1|e^{y_1}}{|\mathbf{k}_1|e^{y_1} + |\mathbf{k}_2|e^{y_2}}$  and  $S(x, \mathbf{k}, \Delta_\perp) = \int \frac{d^2\mathbf{r} d^2\mathbf{b}}{(2\pi)^4} e^{i\mathbf{r}\mathbf{k} + i\mathbf{b}\Delta_\perp} (1 - N(x, \mathbf{b}, \mathbf{r}))$ .

To get the Wigner distribution at small  $x$ , what we need to know is the dipole-nucleon S matrix. From the formula (5.4), (5.5), the cross section of the diffractive di-jet production is obtained by the complicated convolution of the dipole S matrix, so it is difficult to extract the dipole S matrix from the cross section. If the momentum transfer of the process is zero, and the convolution of the integration becomes simple. There is a process to match this condition; Ultraperipheral diffractive di-jet production. In the next section, we show that it is possible to extract the S matrix from the scattering amplitude of the ultraperipheral diffractive di-jet production.

## 5.2 Ultraperipheral collision

The ultraperipheral collision (UPC) are defined by the scattering in which the impact parameter  $b$  is larger than the sum of the radii of the target and projectile particles as shown in Fig. 5.2 (cf. [81]).

The UPC in  $pA$  is related to the  $\gamma p$  collisions

$$\frac{d\sigma^{Ap}}{dy_1 dy_2 d^2\mathbf{k}_1 d^2\mathbf{k}_2} = \int d\omega \frac{dN}{d\omega} \frac{\sigma^{\gamma p}}{dy_1 dy_2 d^2\mathbf{k}_1 d^2\mathbf{k}_2} \quad , \quad (5.6)$$

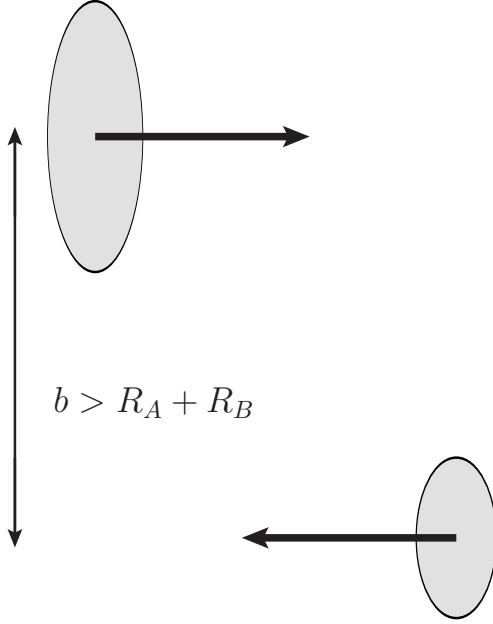


Figure 5.2: The schematic picture of the ultraperipheral collision

where  $\frac{dN}{d\omega}$  is the photon flux and given by

$$\frac{dN}{d\omega} = \frac{2Z^2\alpha_{em}}{\pi\omega} \left[ \xi K_0(\xi) K_1(\xi) - \frac{\xi^2}{2} (K_1^2(\xi) - K_0^2(\xi)) \right] , \quad (5.7)$$

with  $\xi = \omega \frac{R_p + R_A}{den}$  and  $\gamma = \frac{\sqrt{s}}{2m_p}$ . Where the  $R_p$  and  $R_A$  are the radii of the proton and the nucleus. The flux decreases exponentially when  $\xi \gg 1$ . In the p-Au collision in the RHIC at  $\sqrt{s_{NN}} = 200$  GeV, then we have  $Z = 79$ ,  $\gamma = 100$ ,  $R_A \sim 8$  fm and  $\omega \sim 2$  GeV. In the p-Pb collision in the LHC at  $\sqrt{s_{NN}} = 5.2$  TeV, we have  $Z = 82$ ,  $\gamma \sim 2500$  and  $\omega \sim 50$  GeV. The virtuality of the photon in the UPCs is small compared to hard scale so it is reasonable to set the momentum transfer  $Q \sim 0$ .

In the UPC, the scattering amplitude becomes

$$\vec{M} = \int d^2\mathbf{k} S(x, \mathbf{k}, \Delta_\perp) \frac{P_\perp - \mathbf{k}}{(P_\perp - \mathbf{k})^2} . \quad (5.8)$$

From the results of the previous chapter, we have known that the elliptic part of the dipole S matrix is much smaller than the angular independent part. Now we expand the S matrix in Fourier series and approximate

$$S(x, \mathbf{k}, \Delta_\perp) = S_0(x, |\mathbf{k}|, |\Delta_\perp|) + 2 \cos(2(\phi_k - \phi_\Delta)) S_1(x, |\mathbf{k}|, |\Delta_\perp|) , \quad (5.9)$$

This approximation is similar to the dipole Wigner distribution as already stated. Then we can perform the angular integration in the amplitude

$$\int_0^\infty k dk \int_0^{2\pi} d\phi_k \frac{P_\perp - \mathbf{k}}{(P_\perp - \mathbf{k})^2} S_0(x, |\mathbf{k}|, |\Delta_\perp|) = \frac{2\pi P_\perp}{P_\perp^2} \int_0^{P_\perp} dk k S_0(x, |\mathbf{k}|, |\Delta_\perp|) , \quad (5.10)$$

$$\begin{aligned}
& \int_0^{2\pi} d\phi_k \frac{P_\perp - \mathbf{k}}{(P_\perp - \mathbf{k})^2} \cos(2(\phi_k - \phi_\Delta)) S_1(x, |\mathbf{k}|, |\Delta_\perp|) \\
&= \frac{\pi P_\perp}{P_\perp^2} \cos(2(\phi_P - \phi_\Delta)) \left( \frac{\mathbf{k}^2}{P_\perp^2} \Theta(|P_\perp| - |\mathbf{k}|) - \frac{P_\perp^2}{\mathbf{k}^2} \Theta(|\mathbf{k}| - |P_\perp|) \right) .
\end{aligned} \tag{5.11}$$

In this scattering process,  $q^+ = \sqrt{2}\omega$  in the formula (5.4), then we can perform the  $\omega$  integration and we get

$$\begin{aligned}
\frac{d\sigma^{Ap}}{dy_1 dy_2 d^2\mathbf{k}_1 d^2\mathbf{k}_2} &\simeq \omega \frac{dN}{d\omega} \frac{2(2\pi)^4 N_c \alpha_{em}}{P_\perp^2} \\
&\times \sum_f e_f^2 z(1-z)(z^2 + (1-z)^2)(A^2 + 2\cos(2(\phi_P - \phi_\Delta))AB) ,
\end{aligned} \tag{5.12}$$

where the  $A$  and  $B$  are defined as

$$A(x, |P_\perp|, |\Delta_\perp|) := - \int_0^{|P_\perp|} dk k S_0(x, k, |\Delta_\perp|) , \tag{5.13}$$

$$B(x, |P_\perp|, |\Delta_\perp|) := - \int_0^{|P_\perp|} dk \frac{k^3}{P_\perp^2} S_1(x, k, |\Delta_\perp|) + \int_{|P_\perp|}^\infty dk \frac{P_\perp^2}{k} S_1(x, k, |\Delta_\perp|) . \tag{5.14}$$

The photon energy is

$$\omega = \frac{1}{\sqrt{2}}(k_1^+ + k_2^+) . \tag{5.15}$$

In this formula, we neglect the  $S_1^2$  term because it is a small quantity compared to others. From this formula, we can access  $A$  and  $B$  from the experiments and what we want to have is whether we can derive  $S_0$  and  $S_1$  from  $A$  and  $B$ .  $S_0$  can be easily derived

$$S_0(x, P, \Delta) = -\frac{1}{P} \frac{\partial}{\partial P} A(x, P, \Delta) . \tag{5.16}$$

It is more complicated for the  $S_1$  case. First we change the variable  $P_\perp^2 = e^v, k^2 = e^u$ . Then the formula becomes

$$B(v) = \int_{-\infty}^\infty du (-e^{u-v} \Theta(v-u) + e^{v-u} \Theta(u-v)) C(u) , \tag{5.17}$$

where we set  $C(u) = \frac{k^2}{2} S_1(x, k, \Delta)$ . And change the variable further

$$B(v) = \int_0^\infty dt e^{-t} (C(v+t) - C(v-t)) . \tag{5.18}$$

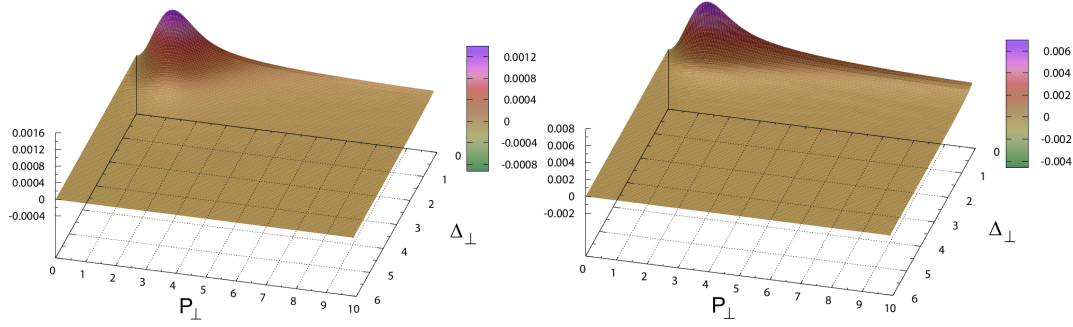


Figure 5.3: The function A at fixed rapidity  $Y = 5$ (left) and  $Y = 10$ (right) calculated in the BK model

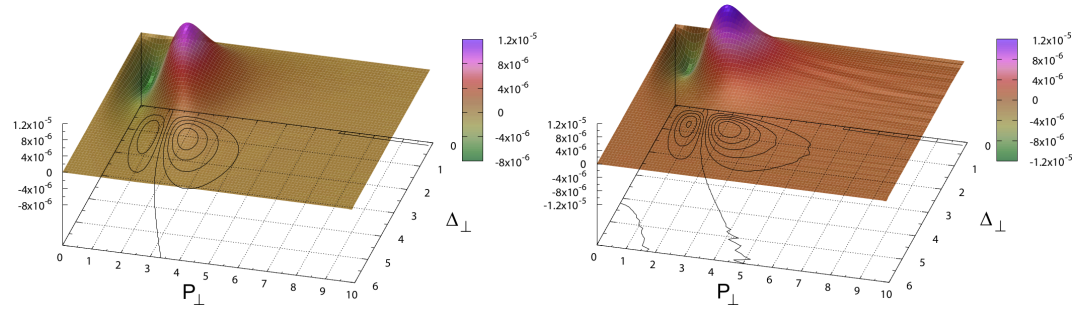


Figure 5.4: The function B at fixed rapidity  $Y = 5$ (left) and  $Y = 10$ (right) calculated in the BK model

Then we expand C around the symmetric point in the bracket, we get

$$B(v) = 2 \sum_{n=0}^{\infty} \frac{d^{2n+1}}{dv^{2n+1}} C(v) \quad . \quad (5.19)$$

Finally, we get

$$S_1(x, P, \Delta) = -\frac{\partial B(x, P, \Delta)}{\partial P^2} + \frac{2}{P^2} \int_0^{P^2} \frac{dP'^2}{P'^2} B(x, P', \Delta) \quad . \quad (5.20)$$

We have seen that there is the one to one correspondence between the differential cross section and the dipole Wigner distribution. From the rapidity gap  $\Delta Y$ (5.3), we can measure the  $x$  dependence.

### 5.2.1 Numerical implementation

We make a numerical prediction of the differential cross section using the BK equation and the MV model.

We first see the cross section using the solution of the BK equation. We need the impact parameter dependent solution of the BK equation, so we use the one got in the previous chapter.

We show the numerical results in Fig. 5.3, 5.4. In this calculations, we set  $R = 0.4\text{fm}$  for a realistic initial condition and we then can interpret  $P_{\perp} = 0.5\text{GeV}$

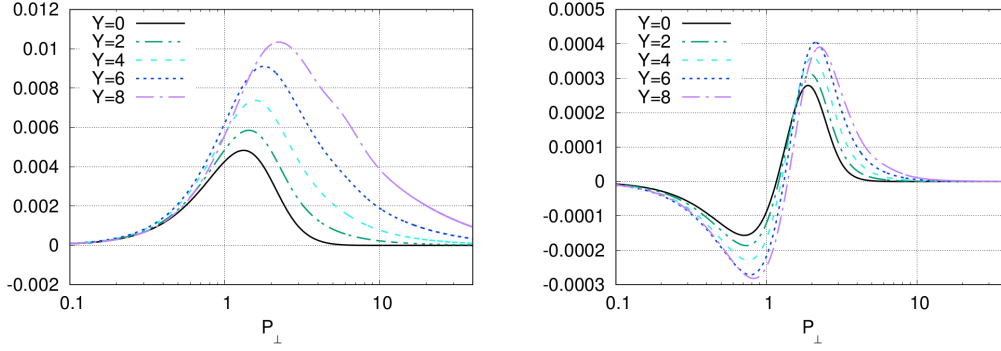


Figure 5.5: The function  $A$ (left) and  $B$ (right) at fixed momentum  $\Delta_{\perp} = 1$  calculated in the BK model

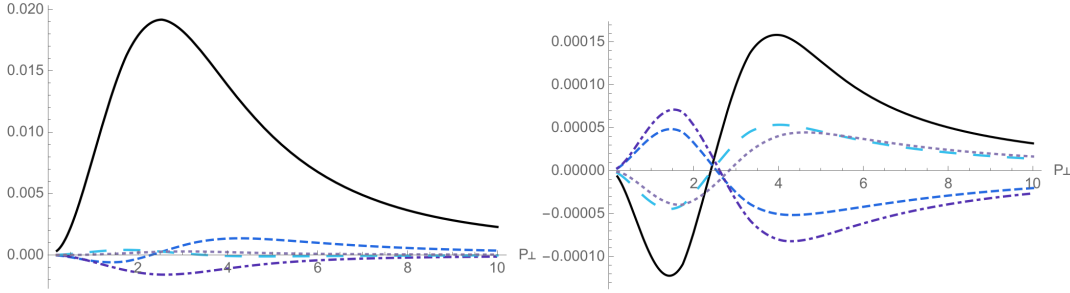


Figure 5.6: The function  $A$ (left) and  $B$ (right) in the MV model for a nucleus target at  $\Delta = 0.2$ (solid lines),  $\Delta = 0.25$ (dashed lines),  $\Delta = 0.3$ (dotted-dashed lines),  $\Delta = 0.4$ (long-dashed lines), and  $\Delta = 0.5$ (dotted lines).

in these figures. In Fig. 5.5, we plot the rapidity dependence of  $A$  and  $B$  functions at fixed  $\Delta_{\perp} = 1$ . It is expected from Chap 4.3 that the position of the positive peak in  $A$  function moves towards the large momentum region. But for  $B$  function, it depends on rapidity slightly.

Next, we focus on the MV model calculations of  $A$  and  $B$  functions. We have not included the  $x$  dependence in this model. We set the saturation momentum scale as  $Q_{s,A}^2 = A^{1/3}Q_s^2$ , where proton saturation momentum  $Q_s$  is taken to be  $Q_s^2 = R^{-2}$ . We set the Gaussian factor in the Fourier transformation of the S matrix as the same form in the BK model to cut off the large  $\mathbf{r}$  and  $\mathbf{b}$ .

In Fig. 5.6, we show the results of the calculations. While at small  $\Delta < 0.3\text{GeV}$  the function  $A$  is positive definite at small  $P_{\perp}$ , it decrease at large  $\Delta$  and  $P_{\perp}$  and change its sign and a single node appears. The elliptic part  $B$  also has a single node and slower dependence on  $\Delta$  than the function  $A$ . The node position in  $P_{\perp}$  dimension is independent of  $\Delta$ . Similar to the function  $A$ ,  $B$  decrease as  $P_{\perp}$  becomes large. In the case of the large nuclear target, the magnitude of the positive peak of  $A$  and  $B$  at  $\Delta < 0.3\text{GeV}$  are 1 to 2 order larger than those of the results from the BK model. Interestingly, at  $\Delta > 0.7\text{ GeV}$  same facts also holds. From these results, it is better to measure nucleus-nucleus UPCs for studying the sophisticated properties of

the Wigner distributions. From the both of the results of the two model, the Wigner distribution has the non-trivial structure in the small  $P_{\perp} < 4\text{GeV}$  region, and it is challenging to measure in this region from experiments at the LHC.



## Chapter 6

# The elliptic flow in small system from the Wigner distribution

In this thesis, we have studied the Wigner distribution in the small  $x$  region. Does the Wigner distribution affect the observed quantity? In this chapter, we show the phenomenological application of the Wigner distribution function. There are some fully uncovered phenomenon in the high energy collisions. One of the interesting experimental observations from the proton-proton and proton-nucleus collisions at the Large Hadron Collider (LHC) and Relativistic Heavy Ion Collider (RHIC) is the long range rapidity and azimuthal angle correlations between produced particles. The long range rapidity correlation is that the produced particles have correlations even if they are largely separated in rapidity space [82, 83, 84]. Another one is the azimuthal angle correlations in the small system. It is known that many models have been proposed to explain the experimental results, including (but not limited to) hydrodynamics [85, 86, 87, 88, 89]. For a small system like the pp or pA collisions, it seems there are no local equilibrium and cannot validate the hydrodynamical picture, so we should seek some other sources which generate the elliptic flow. In this chapter, we show that the elliptic part of the Wigner distribution can affect the elliptic flow in pA collisions. This chapter is based on the paper [90].

### 6.1 The elliptic flow from two particle angular correlations

At first, to get the elliptic flow from the two particle correlations, we introduce the double parton scattering (DPS) frameworks [91, 92]. The DPS processes are not power suppressed and we need to consider. To factorize the cross sections for two particle production processes, we need to know double parton distributions which govern the non-perturbative part in the cross sections.

It was first pointed out in [93] that the DPS plays an important role in two particle production in forward pA and dA collisions at RHIC. This idea was followed up in the saturation formalism in [94] to estimate the so-called pedestal contribution in the correlation measurements. Further study in [95] also confirmed the importance of these contributions in the two particle production in pA collisions. However, all

these studies assumed that the two hard scatterings are essentially uncorrelated. In the following, we will extend the DPS mechanism to include the impact parameter dependence which naturally encodes the correlation between the two scatterings.

Using the double parton distribution, the cross section can be written as

$$\left. \frac{d\sigma}{dy_1 d^2\mathbf{k}_1 dy_2 d^2\mathbf{k}_2} \right|_{DPS} = \int d^2\mathbf{r}_1 d^2\mathbf{r}_2 d^2\mathbf{b}_1 d^2\mathbf{b}_2 e^{i\mathbf{k}_1 \cdot \mathbf{r}_1} e^{i\mathbf{k}_2 \cdot \mathbf{r}_2} \times F_p(x_p, x'_p; \mathbf{z}) F_A(x_A, x'_A; \mathbf{b}_1, \mathbf{b}_2; \mathbf{r}_1, \mathbf{r}_2) \quad , \quad (6.1)$$

where  $\mathbf{z} = |\mathbf{b}_1 - \mathbf{b}_2|$ , and  $\mathbf{b}_1$  and  $\mathbf{b}_2$  are the positions of the two produced particles from the center of the target nucleus.  $\mathbf{r}_1$  and  $\mathbf{r}_2$  are the Fourier conjugate variables to the transverse momentum of the outgoing particles and can be interpreted as the dipole size.  $x_p, x'_p, x_A$  and  $x'_A$  are the longitudinal momentum fractions. We assume that the projectile is dilute and the target is dense, hence we use the color glass condensate formalism to calculate the scattering of the partons with target nucleus. We assume the factorized form of the distribution for the large target nucleus.

$$F_A \sim S(Y_A, \mathbf{b}_1, \mathbf{r}_1) S(Y'_A, \mathbf{b}_2, \mathbf{r}_2) \quad , \quad (6.2)$$

where  $Y_A = \log(1/x_A)$  and  $Y'_A = \log(1/x'_A)$ . This factorization is not valid in the finite  $N_c$  or the small target. This violation is another source of the elliptic flow which is studied recently[96, 97, 98, 99]. We take the double parton distribution  $F_p$  as the Gaussian form[100]

$$F_p(\mathbf{z}) \sim e^{-\Lambda^2 \mathbf{z}^2} \quad , \quad (6.3)$$

where  $\Lambda$  is the confinement scale.

Now we approximate the S matrix

$$\begin{aligned} S(x_A, \mathbf{b}, \mathbf{k}) &= \int d^2\mathbf{r} e^{i\mathbf{r} \cdot \mathbf{k}} S(x, \mathbf{b}, \mathbf{r}) \\ &= G_0(x, |\mathbf{b}|, |\mathbf{k}|) + 2 \cos(2(\phi_b - \phi_k)) G_1(x, |\mathbf{b}|, |\mathbf{k}|) \quad . \end{aligned} \quad (6.4)$$

Then the angular integration in the cross section can be performed

$$\begin{aligned} &\int^{1/\Lambda} d^2\mathbf{b}_1 d^2\mathbf{b}_2 e^{-\Lambda^2 |\mathbf{b}_1 - \mathbf{b}_2|^2} S(x_A, \mathbf{b}_1, \mathbf{k}_1) S(x'_A, \mathbf{b}_2, \mathbf{k}_2) \\ &= 4\pi^2 \int_0^{1/\Lambda} b_1 db_1 b_2 db_2 e^{-\Lambda^2 (b_1^2 + b_2^2)} \\ &\quad \times [I_0(2\Lambda^2 b_1 b_2) G_0(x_A, b_1, |\mathbf{k}_1|) G_0(x_A, b_1, |\mathbf{k}_1|) G_0(x'_A, b_2, |\mathbf{k}_2|) \\ &\quad + 2 \cos 2(\phi_{k_1 - k_2}) I_2(2\Lambda^2 b_1 b_2) G_1(x_A, b_1, |\mathbf{k}_1|) G_1(x'_A, b_2, |\mathbf{k}_2|)] \quad , \end{aligned} \quad (6.5)$$

where  $I_n$  is the modified Bessel function. Then we define

$$\begin{aligned} &V_2(\mathbf{k}_1, \mathbf{k}_2) \\ &= \frac{\int_0^{R_{cut}} b_1 db_1 b_2 db_2 e^{-\Lambda^2 (b_1^2 + b_2^2)} I_2(2\Lambda^2 b_1 b_2) G_1(x_A, b_1, |\mathbf{k}_1|) G_1(x'_A, b_2, |\mathbf{k}_2|)}{\int_0^{R_{cut}} b_1 db_1 b_2 db_2 e^{-\Lambda^2 (b_1^2 + b_2^2)} I_0(2\Lambda^2 b_1 b_2) G_0(x_A, b_1, |\mathbf{k}_1|) G_0(x_A, b_1, |\mathbf{k}_1|) G_0(x'_A, b_2, |\mathbf{k}_2|)} \quad . \end{aligned} \quad (6.6)$$

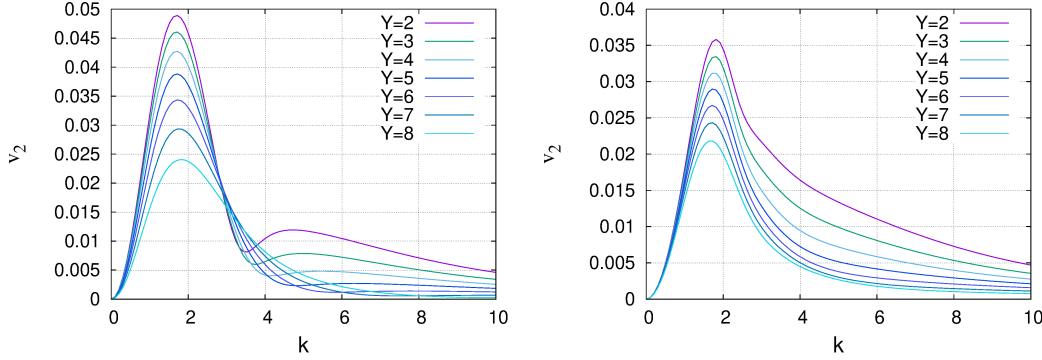


Figure 6.1:  $k$  dependence of  $v_2$ . The cutoff in the impact parameter integration is  $R_{cut} = 2$ (left) and  $R_{cut} = 5$ .

The elliptic flow  $v_2$  is obtained by

$$v_2(\mathbf{k}, \mathbf{k}^{ref}) = \frac{V_2(\mathbf{k}, \mathbf{k}^{ref})}{\sqrt{V_2(\mathbf{k}^{ref}, \mathbf{k}^{ref})}} \quad , \quad (6.7)$$

where  $\mathbf{k}^{ref}$  is a reference momentum. If  $V_2$  factorize  $V_2(\mathbf{k}_1, \mathbf{k}_2) \sim v_2(\mathbf{k}_1)v_2(\mathbf{k}_2)$ , then  $v_2$  is independent to the reference momentum. This factorization is assumed in experiments, but we find that the factorization is violated in our model calculation and it might be true in more small  $x$  region in the experiment.

## 6.2 The BK model calculation

To evaluate the effects of the elliptic Wigner distribution and the double parton distribution function, we use the solutions of the BK equation obtained in Chap. 4. The only difference is that we use the more realistic initial condition:  $S(Y=0) = e^{-d^2} \rightarrow e^{-cd^2}$  with  $c = 6$ . When the dipole size is small and the impact parameter is small then the initial conditions become  $S(Y=0) \sim e^{-cr^2/R^2}$  where  $R$  is the target size. For the nucleon, we use  $R = 1\text{fm}$  in order that the initial condition  $S \sim e^{-r^2/(0.4\text{fm})^2}$  is the same as the GBW model[70]. We use  $R$  as the unit length then  $R_{cut} = 2$  means  $R_{cut} = 2R$ .

The results of the calculations are shown in Fig. 6.1 for different rapidity values. The elliptic part  $G_1$  changes its sign as the impact parameter becomes large, so if the cut off of the  $\mathbf{b}$  integration is small then the elliptic flow has two peaks. On the other hand, if the cut off is large enough, the elliptic flow has a single peak. Note that the positions of the largest peak do not change as the rapidity changes. This is one of the characteristic properties of the elliptic Wigner distributions which we have already shown. We expect that the properties are measured at the LHC or future collider facilities. We also note that the magnitude of the positive peak decrease as the rapidity increasing. It is already observed in the elliptic Wigner distribution function. The magnitude of  $v_2$  generated in the present formalism is somewhat smaller than one obtained from the experimental data, but this has non-negligible contribution to the elliptic flow.

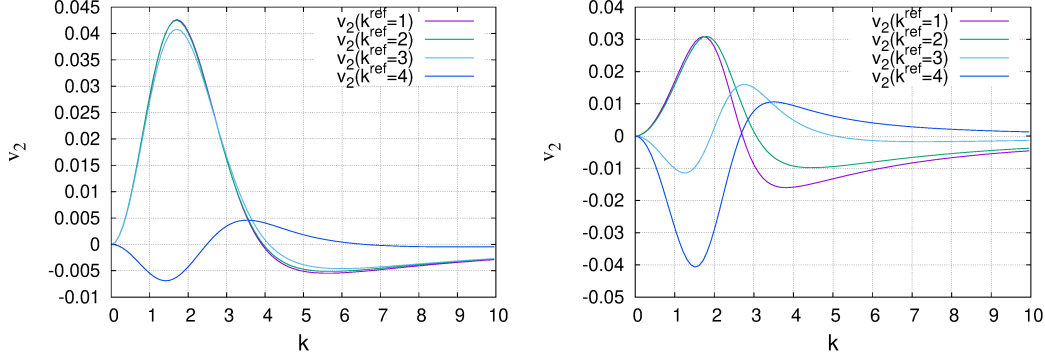


Figure 6.2: The reference momentum dependance of  $v_2$  at  $Y = 4$ ;  $R_{cut} = 2$ (left),  $R_{cut} = 5$ .

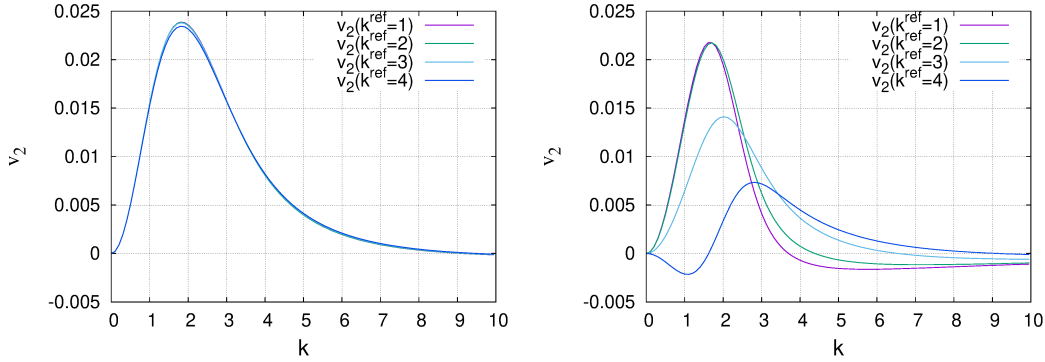


Figure 6.3: The reference momentum dependance of  $v_2$  at  $Y = 8$ ;  $R_{cut} = 2$ (left),  $R_{cut} = 5$ .

Next we check the factorization of  $V_2(k, k') \sim v_2(k)v_2(k')$ . We change  $k^{ref}$  and see the  $k^{ref}$ -dependence of the elliptic flow. Because of the approximation of the modified Bessel function  $I_1(2\Lambda^2 b_1 b_2) \sim (b_1 b_2)^2$  at small  $b_i$ ,  $V_2$  is factorize if the cut off is small which is seen in the left panel of Fig. 6.2( $Y = 4$ ), 6.3( $Y = 8$ ). We also deduce the fact from the figure that the factorization holds better for large rapidity values.  $v_2$  is not positive definite function if  $k \neq k^{ref}$  because  $G_1$  is negative in the large  $k^{ref}$  region.

Finally we show the rapidity correlation between two produced particles. We fix the momentum  $k = k^{ref} = 2$  and take different  $x_A$  and  $x_{A'}$  values. We show the results of the calculations in Fig. 6.4. From the results, we recognize that there is a long range rapidity correlation from this DPS and the elliptic Wigner distribution mechanism, because  $v_2$  decrease slowly with increasing rapidity separation between  $Y = \log(1/x_A)$  and  $Y' = \log(1/x_{A'})$ .

To summarize this chapter, we have investigated the elliptic flow generated from the double parton distributions and the elliptic Wigner distribution in small system. The double parton distribution generate the correlations of the produced two particles and the elliptic Wigner distribution makes the correlations of the momentum and

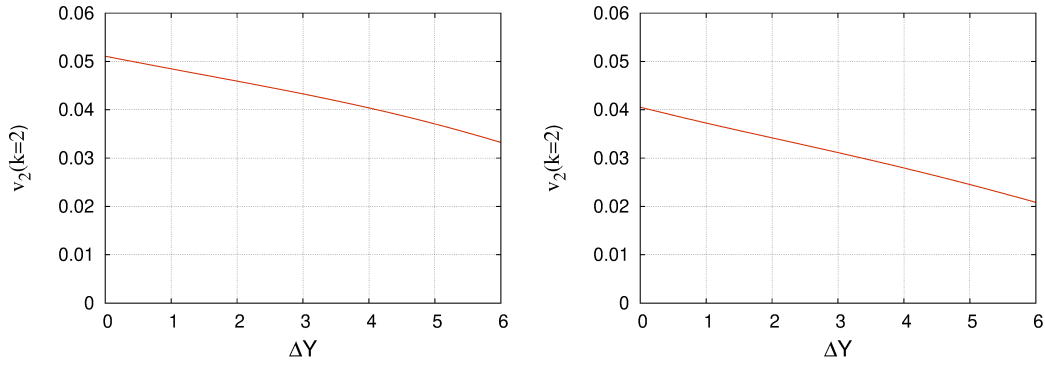


Figure 6.4: The rapidity separation dependence of  $v_2(K = k^{ref} = 2):R_{cut} = 2$ (left) and  $R_{cut} = 4$ (right)

the impact parameter of the particles. This is the key ingredients which makes the correlation of the angles between the produced particles. The long range rapidity correlations are also generated and we can say that the elliptic Wigner distribution have non-negligible contributions.



## Chapter 7

# Conclusion

We have seen the many aspects of the parton phase space distributions in QCD at small Björken  $x$ . From the Wigner distributions, we see that we can understand the structure of the nucleon in more detail compared to the GPDs or the TMDs. The angular correlations of the impact parameter and the momentum of the partons are one of the ingredients of the elliptic flow in  $pp$  collision or the angular momentum of the proton. In my research, we mainly focus on the dipole gluon Wigner distribution function without polarization at small  $x$ . In the small  $x$  region, the number of the gluon becomes large and the nucleon is filled with the gluons, then the number of the gluons becomes saturated because of the recombination process of the gluons. The gluon saturation effects have important effects on the study in the small  $x$  region, and we include these effects to the parton distribution functions by the solutions of the BK equation.

In this works, we have studied the phase space distribution(Wigner, Husimi, GTMD) with gluon saturation effects. In order to have a good precision calculation, we assume that the solution of the BK equation has the  $SO(3)$  conformal symmetry and we solve the one parameter differential equation. We include the Gaussian factor in the Wigner distribution in order to include the confinement effects of QCD which cannot be fully included by the BK equation. For the Husimi distribution, the confinement effects are effectively included in the Gaussian factor in the definition. We have seen that the Wigner distribution has a positive peak and it is the saturation momentum of the system. The phase space distributions in the saturation region are characterized by the saturation momentum.

We have also shown that the dipole Wigner distribution can be directly measured by the ultraperipheral diffractive di-jet production process. This makes it possible to constrain the gluon distribution from the experiment. In this research, we have shown that there is a one-to-one correspondence between the dipole Wigner distribution function and the differential cross section of the di-jet production. We have seen that the specific feature of the gluon distribution can be measured around  $p_{\perp} < 10$  GeV. It is now difficult to measure but it may be measured at the LHC in the near future. We also find the elliptic Wigner distribution has an effect on the elliptic flow. This contributions are not fully considered in the literature and we show that they have non-negligible effects on the measured quantities.

We have only considered the future-past pointing Wilson lines in the definition

of the Wigner distribution, we can consider the more involved path of the Wilson lines, so there are some other definitions of the Wigner distributions. These Wigner distributions may be studied or measured by some other scattering processes.

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