

DOCTORAL THESIS

MICROSCOPIC THEORY OF TRANSPORT
PHENOMENA IN THE LONG-RANGE
INTERACTING SPIN SYSTEMS

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Declaration

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MICROSCOPIC THEORY OF TRANSPORT PHENOMENA IN THE LONG-RANGE INTERACTING SPIN SYSTEMS

HIDEAKI NISHIKAWA

ABSTRACT

LONG-RANGE INTERACTIONS are ubiquitous, exemplified by gravity, the Coulomb interaction, and the dipole interaction. Here, “long-range” means that the potential follows a power-law form $V(r) \sim r^{-\alpha}$ with the distance r between the sites. Recent experimental advancements have remarkably enabled precise control over the interaction regime and the power-law exponent α . Notable examples include Rydberg atoms, cold atomic, and trapped ionic systems. These platforms not only open pathways for quantum information processing but also serve as versatile laboratories for exploring fundamental phenomena such as thermalization, information propagation, and beyond. Long-range interactions give rise to distinctive phenomena that are absent in systems governed solely by short-range interactions, from static to dynamical aspects. As important dynamical properties, energy or spin propagation should be a crucial subject in the spin systems. We focus on spin models, motivated by their direct relevance to the experimental systems mentioned above. However, thus far, the impact of long-range interaction on energy transport in spin systems have not been systematically investigated. On the other hand, the effect of long-range interaction on spin transport has been numerically and phenomenologically studied, but so far, a rigorous theoretical foundation has not existed.

The primary aim of this thesis is to understand the general properties of energy and spin diffusion in spin systems with long-range interactions. We address the following questions: Is there a transition of normal and anomalous diffusion of energy and spin by changing the exponent α ? If so, what is the universal mechanism and systematic method to analyze the class of energy and spin diffusion?

Firstly, we investigate energy diffusion in long-range interacting spin systems. We consider prototypical spin systems, the transverse Ising model, and the XYZ model in the D -dimensional lattice with a finite exponent $\alpha > D$, which guarantees the thermodynamic extensivity. In one dimension, both normal and anomalous diffusion are observed, where the anomalous diffusion is attributed to anomalous enhancement of the amplitude of the equilibrium current correlation. We prove the power-law clustering property of arbitrary orders of joint cumulants in general dimensions. Applying this theorem to equal-time current correla-

tions, we further prove several theorems leading to the statement that the sufficient condition for normal diffusion in one dimension is $\alpha > 3/2$ regardless of the models. The fluctuating hydrodynamics approach consistently explains Levy diffusion for $\alpha < 3/2$, which implies the condition is optimal. In higher dimensions of $D \geq 2$, normal diffusion is indicated as long as $\alpha > D$.

Next, we study spin diffusion in long-range interacting XXZ model, with a power-law $\alpha > D$. We obtain the same scaling behavior for spin diffusion, both in one dimension and higher dimensions.

Moreover, we also investigate energy and spin diffusion in the same model above, with a power-law exponent $\alpha < D$ and the Kac factor. We generalize the cumulant power-law clustering theorem to the regime with the Kac factor, and obtain the upper bound for equal-time current correlations. However, the physical significance of this upper bound remains unclear.

We expect that this theoretical framework is broadly applicable to transport phenomena in a wide class of long-range interacting systems.

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The central question posed by Professor Saito in the summer of my first year of the master's program – "Does hydrodynamics exist in long-range interacting systems?" – has guided my research ever since. By continuously engaging with this question, I believe I have arrived at a certain level of understanding for long-range interacting spin systems. Nevertheless, many open problems remain, including an incomplete understanding of the strong interaction regime and the absence of a theoretical framework beyond spin systems. However, through this research, I was able to glimpse an essential aspect of statistical physics: the emergence of macroscopic universality from microscopic interactions. I would like to express my deepest gratitude to Professor Saito for this invaluable experience.

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List of publications

Papers related to the thesis

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Chapter 1

Introduction

1.1 Physics of long-range interactions

Long-range interactions, exemplified by gravitational, Coulomb, and dipole–dipole interactions, are ubiquitous in nature [1, 2]. In this context, long-range interactions refer to those whose potential follows a power-law form $V(r) \sim r^{-\alpha}$ as a function of distance r . The exponent α controls the interaction range: the smaller the value of α , the longer the effective range of the interaction. In recent years, advances in experimental techniques have made it possible to precisely control both the form of the interaction and the value of the exponent α . Representative examples include Rydberg-atom platforms, cold atomic systems, and trapped-ion experiments [3–16].

Electrically neutral atoms arranged in optical tweezers array are typically separated by several micrometers and thus do not interact directly. To induce interactions, one exploits the Rydberg state, a highly excited electronic state in which the outermost electron is promoted to a large principal quantum number, creating an extremely extended electronic orbital. Between Rydberg atoms, long-range dipole–dipole interactions naturally arise [4, 17]. In these systems, experimental parameters such as laser frequency and intensity, as well as the spacing between optical tweezers, can be tuned with high precision, allowing for versatile control of the Hamiltonian [5, 6].

Experiments based on Paul traps and Penning traps offer another promising platform [12–15]: by confining ions in one and two dimensions, respectively, one can engineer spin models with tunable interaction ranges mediated by phonon modes in the ion crystal. These experimental systems achieve an exceptional degree of parameter control and enable the realization of time-dependent dynamics in highly isolated quantum systems. Beyond their applications to quantum information processing, they have played a central role as experimental testbeds for fundamental problems such as thermalization and information propagation.

Long-range interactions are known to induce novel physical phenomena that do not arise in systems with short-range interactions. When the exponent α is smaller than the spatial

dimension, extensivity breaks down [1, 2, 18–21], and this has profound consequences, particularly for static properties. The breakdown of extensivity gives rise to numerous anomalous features, including ensemble inequivalence [21], negative specific heat [22–24], long-lived quasi-stationary states [25, 26], anomalous diffusion [24, 27, 28], suppression of chaos [26, 27, 29–31], and agreement with mean-field theory [32–34]. Even in the extensive regime where α exceeds the spatial dimension, long-range interactions can substantially influence equilibrium phase transitions [35–39] and the structure of the ground state [40–42].

On the dynamical side, long-range interactions are equally important, with one notable example being the persistence of correlations over long distances [27–30, 43]. In recent years, considerable effort has been devoted to understanding their effects on thermalization and information propagation, particularly in the context of the Lieb–Robinson bound [44–47], out-of-time-ordered correlators (OTOCs) [48], and fast scrambling [49]. Many of these researches have focused on spin systems, reflecting the fact that the aforementioned experimental platforms are well described by spin models [3]. A detailed overview of previous research on information propagation in the long-range interacting systems will be presented in the next section.

Another key dynamical property in long-range interacting spin systems is the transport of energy and spin. However, the influence of long-range interactions on transport properties has remained largely unexplored. Existing studies of energy transport have predominantly focused on spring-type models [50–56], often treating them as extensions of anomalous heat conduction in low-dimensional systems arising from long-time tails [57–61]. In contrast, energy transport in long-range interacting spin systems has not been investigated at all, even at the numerical level. Regarding spin diffusion in long-range interacting spin models, only a handful of numerical and phenomenological studies exist [62, 63], and no systematic theoretical framework has yet been established. A detailed overview of prior research on transport phenomena in long-range interacting systems will be presented in the following section.

1.2 Information propagation in the long-range interacting systems

In this section, as one of the important lines of prior research on the dynamical properties of long-range interacting systems, we discuss in detail the dynamics of information propagation in long-range interacting systems. In particular, we provide an overview of rigorous results concerning the Lieb–Robinson bounds and out-of-time-ordered correlators (OTOCs) in the presence of long-range interactions.

We begin with a detailed exposition of the Lieb–Robinson bounds in long-range interacting systems. A fundamental question in the study of quantum many-body dynamics concerns the maximal speed at which information can propagate through the system. In relativistic

theories, this question is answered by causality: as shown by Einstein more than a century ago, the propagation of signals is strictly constrained by the light cone, and information transfer outside the light cone is strictly forbidden.

In contrast, nonrelativistic quantum many-body systems do not possess an *a priori* light cone. In their seminal work of 1972, Lieb and Robinson proved that an “effective light cone” nevertheless emerges [64]: the commutator of two observables O_X and O_Y , initially supported on spatially separated regions X and Y , remains exponentially suppressed outside a region bounded by a characteristic velocity. More precisely, one has

$$\|[O_X(t), O_Y]\| \leq c \|O_X\| \|O_Y\| |X| |Y| e^{vt-r}, \quad (1.2.1)$$

where $\|\cdot\|$ denotes the operator norm, $r = d_{X,Y}$ is the distance between X and Y , and v is known as the Lieb–Robinson velocity.

The Lieb–Robinson bound provides one of the most fundamental constraints on quantum dynamics [65–73], and has been refined extensively in subsequent research [74–80]. Following Hastings’ work on the Lieb–Schultz–Mattis theorem [81], the Lieb–Robinson bound has become an indispensable analytical tool for the study of universal properties of quantum many-body systems. Applications include quasi-adiabatic continuation [82], entanglement area laws [83–85], thermalization [86–89], quantization of Hall conductance [90, 91], stability of topological order [92–94], clustering of correlations [95–97], construction of effective Hamiltonians [98, 99], and classical simulability of quantum systems [100–102]. More recently, Lieb–Robinson bounds have been applied to digital quantum simulation [103, 104] and quantum information scrambling [105–117]. Owing to recent experimental advances, direct observations of Lieb–Robinson light-cone structures have also become possible [13, 118–122].

For short-range interacting spin systems, the effective light cone is characterized by a finite velocity v , which restricts information propagation to a linear cone. In contrast, long-range interacting systems feature algebraically decaying interactions of the form $r^{-\alpha}$, allowing for instantaneous communication between arbitrarily distant sites. As a result, the existence of a linear light cone is not obvious *a priori*. Depending on the exponent α , one may observe either linear or nonlinear light-cone structures. As discussed in the previous section, systems with a wide range of α values have recently been realized experimentally, stimulating renewed interest in their universal dynamical properties. Against this backdrop, one of the most important and intriguing open problems in the context of information propagation is the *linear light-cone problem*: Does a linear light cone exist in long-range interacting systems, and if so, under what general conditions?

Despite its conceptual simplicity and practical importance for future quantum technologies, establishing upper bounds on information propagation in long-range interacting systems has long been regarded as a notoriously difficult mathematical challenge. Hastings and Koma showed that for $\alpha > D$ (where D is the spatial dimension), communication over distance r requires a time $t \gtrsim \log r$ [75], leading to what is sometimes called a logarithmic light cone. However, this upper bound is believed to be far from optimal. Subsequently, it was shown

that for $\alpha > 2D$ [103, 123, 124], one has $t \gtrsim r^\gamma$ for $0 < \gamma < 1$ [125]. A genuine linear light cone was first proven for one-dimensional systems with $\alpha > 3$, and later generalized to higher dimensions, yielding the condition $\alpha > 2D + 1$ [45]. These results demonstrate that long-range interacting systems with sufficiently large α behave effectively as short-range systems. Further refinements have been obtained through constructions of fast state-transfer and fast entanglement-generation protocols. These works established that $t \lesssim r^{\min(\alpha-2D, 1)}$ for $\alpha > 2D$, and $t \lesssim r^{o(1)}$ for $\alpha < D$, where $o(1)$ denotes an arbitrarily small exponent [47]. It was also shown that for any $\varepsilon > 0$, quantum information must lie within a light cone obeying $t \gtrsim r^{\min(\alpha-2D-\varepsilon, 1)}$, leading to polynomial light cones for $\alpha > 2D$. See the schematic phase diagram in Fig.1.1.

Taken together, these developments imply that for fermionic systems with bounded local operator norms, the conditions under which a linear light cone emerges in long-range interacting systems are now essentially fully understood.

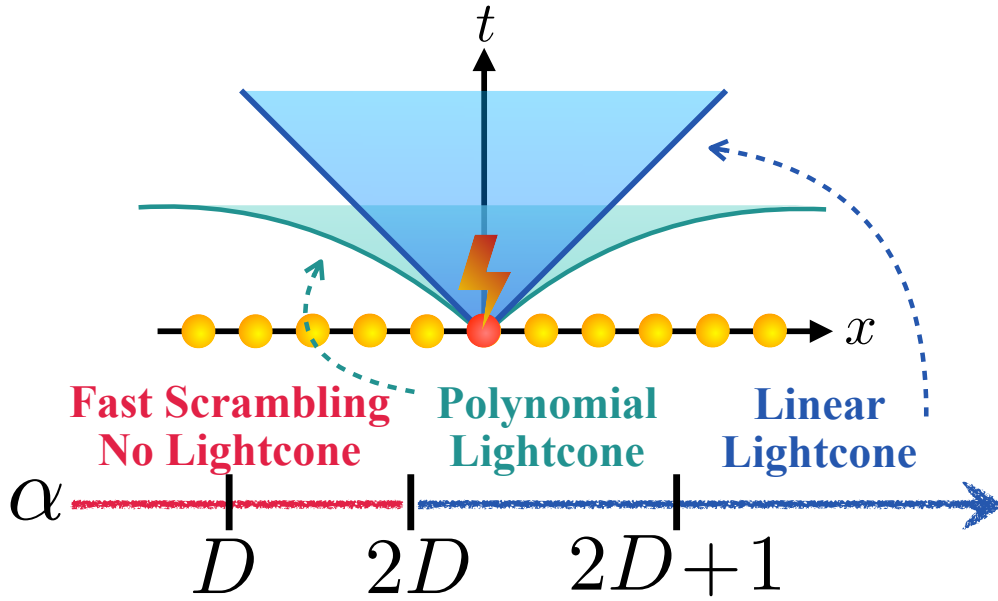


Figure 1.1: The schematic phase diagram of Lieb-Robinson bound in the long-range interacting systems.

In parallel with the Lieb–Robinson bound, information scrambling provides a complementary perspective on how initially localized quantum information becomes inaccessible after time evolution. Scrambling plays a central role in a wide range of interdisciplinary problems, spanning from thermalization in quantum many-body systems [126–129] to the black hole information paradox [130–133]. Recent developments at the interface of quantum chaos and information theory have revealed that the out-of-time-order correlator (OTOC) serves as a powerful quantitative tool for characterizing scrambling dynamics [106, 134]. In quantum

lattice models, the OTOC is defined as

$$C(R, t) := \frac{1}{\text{tr } \hat{\mathbf{1}}} \text{tr} \left([W_i(t), V_{i_0}]^\dagger [W_i(t), V_{i_0}] \right), \quad (1.2.2)$$

where $W_i(t) = e^{iHt} W_i e^{-iHt}$, H denotes the Hamiltonian of the system, and the operators W_i and V_{i_0} are supported on lattice sites i and i_0 , respectively, which are separated by a distance R .

When the Hamiltonian H contains only short-range interactions, the OTOC typically exhibits an exponential growth of the form

$$C(r, t) \sim e^{\lambda_L(t-r/v_B)}, \quad (1.2.3)$$

where λ_L is the quantum analogue of the Lyapunov exponent and v_B is the so-called butterfly velocity. The Lieb–Robinson velocity provides the simplest and most general upper bound for v_B in arbitrary quantum many-body systems [105]. Understanding the universal features of OTOC dynamics has become one of the central topics in modern theoretical physics [110, 111, 135–139]. Alongside theoretical progress, experimental schemes for measuring OTOCs have also been proposed and demonstrated in several physical setups [114, 140–143].

In systems with strictly short-range interactions, OTOCs exhibit a ballistic spreading of the wavefront characterized by the butterfly velocity v_B [105, 107, 113, 144, 145]. However, when the Hamiltonian contains long-range interactions of the form $r^{-\alpha}$, the wavefront can exhibit superlinear spreading in time [46, 146–151]. By analogy with short-range systems, one might expect an exponential growth of the form

$$C(r, t) \sim e^{\lambda_L t / r^\alpha}. \quad (1.2.4)$$

Such a behavior corresponds to so-called “fast scrambling”, meaning that the time scale over which initially localized quantum information spreads across the entire system becomes $t_s \approx \lambda_L \log N$, where N is the system size. Indeed, the widely used Lieb–Robinson-type bounds for long-range interacting systems provide upper bounds consistent with the form (1.2.4). In recent years, universal aspects of fast scrambling have attracted considerable interest, particularly in the context of black hole physics [49, 131, 132]. Since the discovery of the exact solution of the Sachdev–Ye–Kitaev model [152, 153], significant efforts have been made to clarify which quantum many-body systems permit or forbid fast scrambling [154–157].

Fast scrambling implies that the system can relax arbitrarily quickly under local perturbations. However, it is highly unlikely that such extremely rapid information spreading is generic in physical systems. For sufficiently large α , the system effectively behaves as a short-range interacting system, and the exponential form (1.2.4) cannot describe the OTOC dynamics over the entire parameter range of α . Instead, a more accurate description for long-range interacting systems may lead to a polynomial growth of the form

$$C(r, t) \leq \left(\lambda_L t / r^\zeta \right)^{\bar{\alpha}}, \quad (1.2.5)$$

with exponents satisfying $\zeta \leq 1$ and $\bar{\alpha} \leq \alpha$. This inequality implies that the scrambling time scales algebraically with the system size as $t_s \sim n^{\zeta/D}/\lambda_L$. For sufficiently large α , both numerical and analytical studies support the polynomial growth form (1.2.5) [45, 47, 117, 123]. Against this background, a natural and fundamental question arises: what is the optimal condition on α that forbids the exponential form (1.2.4) of fast scrambling?

Existing numerical studies suggest that the polynomial growth of the OTOC breaks down for $\alpha \leq D$ [108, 117]. Thus, one expects at least the condition $\alpha > D$. Moreover, this requirement coincides with the physically natural stability condition for long-range interacting systems, ensuring that the total energy remains extensive and that the thermodynamic limit is well-defined.

As reviewed in the discussion of Lieb–Robinson bounds, most prior theoretical analyses have been limited to the regime $\alpha > 2D$ [45, 47, 103, 123, 124]. In this regime, Foss–Feig *et al.* proved that the exponent ζ in (1.2.5) is bounded from below by $(\alpha - 2D)/(\alpha - D + 1)$ [123], which was later improved to $(\alpha - 2D)/(\alpha - D)$ [103]. Furthermore, for $\alpha > 2D + 1$, it was shown that a finite butterfly velocity (i.e., $\zeta = 1$) emerges in general long-range interacting systems [45, 124, 125]. Collectively, these results imply that once α exceeds the threshold $\alpha = 2D$, fast scrambling of the form (1.2.4) cannot occur in long-range interacting systems.

The intermediate regime $D < \alpha \leq 2D$, however, remains far more subtle. In this region, long-range interactions can in principle enable quantum state transfer at subexponential speeds faster than any exponential but still slower than ballistic propagation [46]. As α approaches D , the system effectively behaves as if it has infinite dimensionality, giving rise to novel physical phenomena. For example, in one-dimensional systems, it is known that long-range interactions with $\alpha \leq 2$ qualitatively alter both static [35, 158] and dynamical properties [159, 160]. Consequently, the physical behavior induced by long-range interactions in this regime is highly nontrivial and may even produce unexpected outcomes.

Nevertheless, various empirical observations suggest that fast scrambling is likely forbidden in this parameter region as well. As a partial result, Tran *et al.* demonstrated that fast scrambling is impossible in one-dimensional systems for $\alpha > 3/2$ [124]. More recently, Kuwahara and Saito proved that fast scrambling cannot occur in *any* long-range interacting system provided that $\alpha > D$ [48]. Combined with explicit counterexamples for $\alpha \leq D$ [108, 117], this result identifies $\alpha > D$ as the optimal condition under which the polynomial bound (1.2.5) holds. As a general upper bound, they derived that the OTOC obeys a polynomial growth characterized by the exponent

$$\zeta = (2\alpha - 2D)/(2\alpha - D + 1). \quad (1.2.6)$$

This result supports the empirical hypothesis that physically natural classes of long-range interactions do not give rise to fast scrambling.

1.3 Anomalous heat transport in the low-dimensional nonlinear systems

The study of transport phenomena in long-range interacting systems has, in previous literature, primarily focused on heat transport, where the problem has often been treated as an extension of anomalous heat transport arising from long-time tails in low-dimensional systems [57–61]. However, in the non-integrable spin systems considered in this thesis, it is numerically well established that long-time tails are generally absent [161]. As will be demonstrated later, numerical evidence likewise shows that long-time tails do not arise in long-range interacting spin systems. For this reason, although a detailed discussion of long-time tails is somewhat tangential to the main line of inquiry of this thesis, understanding their role remains important for appreciating the broader context of heat transport phenomena in long-range interacting systems. We therefore begin with an overview of anomalous heat transport in short-range interacting systems, which serves as the fundamental reference point.

1.3.1 Breakdown of Fourier’s law

The phenomenon of heat transport has a long and rich history of investigation. In general, the propagation of heat in matter is commonly described by Fourier’s law. However, the fundamental question of why Fourier’s law holds in the first place remains unsolved. The macroscopic heat transport observed in bulk systems is a many-body effect arising from microscopic particle interactions, and, as will be discussed later, nonlinear particle interactions play a crucial role in Fourier’s law. Therefore, a first-principles derivation is extremely challenging.

Even more intriguingly, investigations into the dynamical origin of Fourier’s law have revealed that the law can break down in low-dimensional systems. More specifically, it has been found that the thermal conductivity diverges in the thermodynamic limit for one- and two-dimensional systems, a phenomenon referred to as anomalous heat transport. Characteristic features of anomalous heat transport include the divergence of thermal conductivity with system size, the emergence of nonlinear temperature profiles, and anomalous diffusion such as superdiffusion in the spreading of energy.

Before examining anomalous heat transport in detail, we first provide an overview of conventional heat transport phenomena. Heat transport can be broadly classified into transport mediated by particle motion, as in fluids or conduction electrons in metals, and transport mediated by particle vibrations (lattice vibrations), as in insulators. In this section, we focus on the heat transport arising from lattice vibrations.

Let us consider a lattice system consisting of particles arranged regularly and connected by springs. We take a one-dimensional system of N particles whose two ends are attached to

heat baths at temperatures T_L and T_R ($T_L > T_R$). If the temperature difference $\Delta T := T_L - T_R$ is sufficiently small, the heat current j is expressed as

$$j = \kappa \frac{\Delta T}{N}. \quad (1.3.1)$$

The widely accepted Fourier's law states that, for sufficiently large N , the thermal conductivity κ becomes independent of N . Taking both anomalous and normal heat transport into account, heat transport in low-dimensional systems may be classified as

$$\kappa \propto N^\delta, \quad \begin{cases} \delta = 1 & : \text{ballistic transport,} \\ \delta < 0 & : \text{localization,} \\ \delta = 0 & : \text{diffusive transport,} \\ 0 < \delta < 1 & : \text{anomalous transport.} \end{cases} \quad (1.3.2)$$

The case $\delta = 1$ corresponds to ballistic transport, in which the heat current is independent of N and propagates without attenuation. This behavior can be observed, for example, in the Debye model

$$H = \sum_{n=1}^N \frac{p_n^2}{2m} + \frac{k_0}{2} (q_{n+1} - q_n)^2. \quad (1.3.3)$$

The Debye model effectively describes the behavior of specific heat in solids at equilibrium; however, in the context of nonequilibrium heat transport, although it is exactly solvable [162], it is not a realistic model. Because the motion of particles can be decomposed into independent harmonic modes, heat transport proceeds without any scattering, resulting in ballistic propagation and the absence of temperature gradients. Conversely, this implies that nonlinear interactions are essential for scattering processes that prevent ballistic heat flow.

The case $\delta < 0$ indicates vanishing thermal conductivity in the thermodynamic limit, implying that heat current does not flow and energy becomes localized. A finite temperature gradient is formed in such a situation. This behavior is realized, for instance, in disordered lattice models [163, 164],

$$H = \sum_{n=1}^N \frac{p_n^2}{2\tilde{m}_n} + \frac{k_0}{2} (q_{n+1} - q_n)^2, \quad (1.3.4)$$

where each mass $\tilde{m}_n$ independently takes the value $1 + \Delta$ or $1 - \Delta$ with probability $1/2$. In this case, strong scattering induced by disorder leads to heat localization.

For $\delta = 0$, Fourier's law holds. Combining Fourier's law with energy conservation shows that energy spreads diffusively (diffusive transport). Heat current flows according to the temperature gradient, and the temperature profile becomes linear.

The case $0 < \delta < 1$ corresponds to anomalous heat transport. Although a finite temperature gradient is established, the thermal conductivity diverges algebraically. This

phenomenon is observed in the Fermi–Pasta–Ulam–Tsingou model with nonlinear interactions [165–169],

$$H = \sum_{n=1}^N \frac{p_n^2}{2m} + \frac{k_0}{2}(q_{n+1} - q_n)^2 + \frac{\nu}{4}(q_{n+1} - q_n)^4. \quad (1.3.5)$$

It is known that even in the presence of nonlinear interactions, transport becomes diffusive in three dimensions [169], which is attributed to the diminishing effect of wave localization and faster relaxation in higher dimensions. In contrast, for low-dimensional systems with nonlinear interactions, anomalous heat transport with $0 < \delta < 1$ appears ubiquitously.

These phenomena have been verified through extensive numerical simulations, and recent experiments on real materials have also produced results consistent with theoretical predictions. For example, in one-dimensional carbon nanotubes, the exponent has been estimated as $\delta \approx 0.3\text{--}0.5$ [170, 171], consistent with theoretical values $\delta = 1/2, 1/3$, which will be discussed in the next section. In two-dimensional graphene, experiments have observed logarithmic divergence $\kappa \propto \ln N$ [172].

Other important characteristics of anomalous heat transport should also be mentioned. In the linear response regime, the thermal conductivity can be related to equilibrium fluctuations of the heat current via the Green–Kubo formula [173, 174],

$$\tilde{\kappa} = \lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \beta^2 \int_0^t ds \frac{1}{N} \langle j_{\text{tot}}(s) j_{\text{tot}}(0) \rangle_{\text{eq}}. \quad (1.3.6)$$

Here, β denotes the inverse temperature of the heat bath. Note that, unlike the previous setup involving nonequilibrium steady states with heat baths at the boundaries, the above formula considers an isolated system with periodic boundary conditions. Thus, $\tilde{\kappa}$ represents the thermal conductivity around equilibrium under periodic boundary conditions and should be distinguished from the nonequilibrium steady-state conductivity κ . The total heat current j_{tot} is defined as the sum of local heat currents j_n flowing between sites n and $n + 1$, i.e., $j_{\text{tot}} := \sum_{n=1}^N j_n$.

In anomalous heat transport, the divergence of thermal conductivity corresponds to anomalous behavior in the heat-current autocorrelation function $C(t) = \langle j_{\text{tot}}(t) j_{\text{tot}}(0) \rangle_{\text{eq}} / N$. If $C(t)$ decays exponentially, $C(t) \propto e^{-t}$, or sufficiently fast as a power law $C(t) \propto t^{-\tilde{\beta}}$ with $\tilde{\beta} > 1$, the integral converges in the long-time limit and one finds $\tilde{\kappa} = \kappa$. However, in low-dimensional systems exhibiting anomalous heat transport, the autocorrelation function of the heat current $C(t)$ decays slowly as

$$C(t) \propto t^{-\tilde{\beta}}, \quad (0 < \tilde{\beta} < 1), \quad (1.3.7)$$

leading to a divergent integral [59, 60, 175–180].

Energy diffusion also deviates from the standard diffusion equation. The spatiotemporal correlation function of local energy ϵ_n ,

$$S(n, t) := \langle \delta \epsilon_n(t) \delta \epsilon_0(0) \rangle_{\text{eq}}, \quad (\delta \epsilon_n := \epsilon_n - \langle \epsilon_n \rangle_{\text{eq}}), \quad (1.3.8)$$

typically exhibits the scaling form

$$S(n, t) \sim \frac{a_h}{t^{\gamma_h}} f_h\left(\frac{n}{t^{\gamma_h}}\right) + \frac{a_s}{t^{\gamma_s}} \left\{ f_s\left(\frac{n+ct}{t^{\gamma_s}}\right) + f_s\left(\frac{n-ct}{t^{\gamma_s}}\right) \right\}. \quad (1.3.9)$$

[61, 181–184]. Here we assume $1/2 < \gamma_h, \gamma_s < 1$. The first term represents the heat mode, which is a locally diffusing mode, while the second and third terms represent the sound modes that propagate to the left and right with sound velocity c . Each mode exhibits a characteristic scaling form determined by the functions f_h, f_s and the exponents γ_h, γ_s .

Up to this point, we have reviewed the power-law divergence of the thermal conductivity, the slow decay of the heat-current autocorrelation function, and the anomalous diffusion of spatiotemporal energy correlations. However, the precise relationships among these phenomena are not yet fully understood. Here, we summarize an empirically known relation between the exponent δ characterizing the power-law divergence of the thermal conductivity in a nonequilibrium steady state and the exponent $\tilde{\beta}$ describing the slow relaxation of the heat-current autocorrelation function in a periodic isolated system.

In a periodic isolated system, the heat-current autocorrelation function $C(t)$ is related to the thermal conductivity $\tilde{\kappa}$ via the Green-Kubo formula,

$$\tilde{\kappa} = \lim_{t \rightarrow \infty} \beta^2 \int_0^t ds \, C(s). \quad (1.3.10)$$

If one takes the upper limit of the integral to be $t = N/c$, it is believed that the asymptotic behavior satisfies

$$\tilde{\kappa} \sim \kappa, \quad (1.3.11)$$

thus agreeing with the thermal conductivity κ obtained in a nonequilibrium steady state [176, 177, 185]. Here, c denotes the sound velocity of the sound mode arising in energy diffusion. The underlying assumption is that an energy fluctuation created at one end of the system reaches the opposite end at time $t = N/c$ via the sound mode, and that contributions to the Green-Kubo integral beyond this timescale are negligible. Under this hypothesis, one obtains the relation

$$\delta = 1 - \tilde{\beta}. \quad (1.3.12)$$

We now turn to the theoretical background underlying anomalous heat transport. The currently accepted viewpoint is that anomalous heat transport arises from the interplay between nonlinear interactions and the presence of conserved quantities. As we shall discuss later, any isolated one-dimensional Hamiltonian system possesses conservation of stretch (system length), momentum, and energy. In low-dimensional systems, the presence of these three conserved quantities is believed to be essential for the emergence of anomalous heat transport.

As a concrete setup, let us consider the Hamiltonian

$$H = \sum_{n=1}^N \frac{1}{2} p_n^2 + V(q_{n+1} - q_n), \quad (1.3.13)$$

where the mass is uniform and, for simplicity, set to $m = 1$ (a convention adopted throughout this thesis). With periodic boundary conditions $p_{n+N} = p_n$ and $q_{n+N} = q_n$, the system generally possesses at least the following three conserved quantities:

$$\text{Stretch : } N := \sum_{n=1}^N r_n, \quad (r_n := q_{n+1} - q_n), \quad (1.3.14)$$

$$\text{Momentum : } P := \sum_{n=1}^N p_n, \quad (1.3.15)$$

$$\text{Energy : } E := \sum_{n=1}^N \epsilon_n, \quad (\epsilon_n := p_n^2/2 + V(q_{n+1} - q_n)). \quad (1.3.16)$$

The number of conserved quantities is closely related to the nature of transport. For instance, in the Debye model, the Hamiltonian decomposes into N independent harmonic oscillators, so that each oscillator energy is conserved. Thus, there exist $\mathcal{O}(N)$ conserved quantities, leading to ballistic transport. A similar ballistic behavior appears in nonlinear lattice models such as the completely integrable Toda lattice with potential $V(r_n) \propto e^{r_n-1}$, which likewise possesses $\mathcal{O}(N)$ conserved quantities [181, 186, 187].

On the other hand, consider a system in which only stretch and energy are conserved, while momentum is not. For example, adding an onsite potential $\sum_{n=1}^N U(q_n)$ breaks translational symmetry, rendering momentum nonconserved. Alternatively, one may consider a velocity-flip model [188] in which p_n is probabilistically inverted ($p_n \rightarrow -p_n$), again violating momentum conservation. In such systems without momentum conservation, numerous numerical studies have shown that heat transport is typically diffusive and obeys Fourier's law [59, 60, 175, 181, 186, 189–191]. This is also consistent with the dispersion relation obtained from harmonic approximation: when translational symmetry is broken, a gap appears in the dispersion relation, eliminating the sound mode, and thereby precluding anomalous heat transport.

Given the above observations, one recognizes that the presence of three conserved quantities—stretch, momentum, and energy—is fundamental for anomalous heat transport in low-dimensional systems.

A phenomenological theoretical framework that emphasizes these conserved quantities is nonlinear fluctuating hydrodynamics [61]. This theory does not rely on any specific form of the microscopic potential and provides a powerful general framework applicable to broad classes of nonlinear, nonintegrable one-dimensional systems. Nevertheless, since nonlinear fluctuating hydrodynamics involves several phenomenological assumptions and approximations, its validity requires careful examination. In the next subsection, we review the basic structure of nonlinear fluctuating hydrodynamics.

1.3.2 Nonlinear fluctuating hydrodynamics in one-dimensional anharmonic chains

Nonlinear Fluctuating Hydrodynamics (NLFH) [61, 179, 184, 192, 193] was proposed by Spohn in 2014 [61] as a mesoscopic effective theory describing anomalous heat transport in one-dimensional systems. It provides a phenomenological derivation of the fluctuating hydrodynamic picture for chains with nonlinear interactions.

We consider a one-dimensional Hamiltonian system with a nonlinear interaction potential $V(r)$,

$$H = \sum_{n=1}^N \frac{1}{2} p_n^2 + V(q_{n+1} - q_n), \quad (1.3.17)$$

together with periodic boundary conditions $p_{n+N} = p_n$ and $q_{n+N} = q_n$. In this setting, the stretch $r_n := q_{n+1} - q_n$, momentum p_n , and energy $\epsilon_n := p_n^2/2 + V(r_n)$ constitute three local conserved quantities. They satisfy the continuity equations

$$\partial_t \vec{g}_n = -(\vec{j}_{n+1} - \vec{j}_n), \quad (1.3.18)$$

$$\vec{g}_n = (r_n, p_n, \epsilon_n), \quad (1.3.19)$$

$$\vec{j}_n = (-p_n, -V'(r_{n-1}), -p_n V'(r_n)). \quad (1.3.20)$$

Our aim is to investigate the mesoscopic fluctuation dynamics of these local conserved quantities and, in particular, their space-time correlations. Let $\vec{u}(x, t)$ denote the deviations of the local equilibrium expectation values of $\vec{g}_n$ from their equilibrium means. Expanding the currents to first order in $\vec{u}(x, t)$ leads to the linearized hydrodynamic equation

$$\partial_t \vec{u}(x, t) + \partial_x A \vec{u}(x, t) = 0, \quad (1.3.21)$$

which can be diagonalized. Using a diagonalizing matrix R , one obtains

$$RAR^{-1} = \text{diag}(-c, 0, c), \quad (1.3.22)$$

implying that the three conserved quantities are converted into three normal modes, namely two sound modes and one heat mode. Writing the fluctuations in the normal-mode basis as $\vec{\phi} := R\vec{u}$, one obtains

$$\partial_t \phi_\alpha(x, t) + \alpha c \phi_\alpha(x, t) = 0, \quad (\alpha = -1, 0, +1), \quad (1.3.23)$$

where c denotes the sound velocity. This description, however, contains no nonlinear effects and thus corresponds to a macroscopic equation in which fluctuations vanish due to the law of large numbers.

To incorporate the interactions among hydrodynamic modes, we expand the currents to second order in $\vec{u}(x, t)$ and further introduce dissipative and stochastic contributions in a phenomenological manner. This yields the fluctuating hydrodynamic equation

$$\partial_t \vec{u}(x, t) + \partial_x \left(A \vec{u}(x, t) + \frac{1}{2} \vec{u}^\top(x, t) H \vec{u}(x, t) - \partial_x D \vec{u}(x, t) + B \vec{\xi}(x, t) \right) = 0, \quad (1.3.24)$$

where $\vec{\xi}(x, t)$ denotes standard Gaussian white noise. The matrices D and B are introduced so as to satisfy the fluctuation–dissipation relation appropriate for the underlying microscopic dynamics.

Transforming to the normal-mode basis $\vec{\phi} = R \vec{u}$ yields

$$\partial_t \phi_\alpha(x, t) + \partial_x \left[\alpha c \phi_\alpha + \sum_{\alpha', \alpha''} G_{\alpha' \alpha''}^\alpha \phi_{\alpha'} \phi_{\alpha''} - \partial_x (D \vec{\phi})_\alpha + (B \vec{\xi})_\alpha(x, t) \right] = 0, \quad (1.3.25)$$

for $\alpha, \alpha', \alpha'' = 0, \pm$. Here, $G_{\alpha' \alpha''}^\alpha$ are the mode-coupling coefficients obtained from the microscopic interaction.

Each of the above mode equations takes the form of a noisy Burgers-type equation, which can be mapped to the Kardar–Parisi–Zhang (KPZ) equation [194, 195], originally proposed as a model for stochastic interface growth. Beyond interface dynamics, the KPZ equation plays a key role in diverse nonequilibrium phenomena including the asymmetric simple exclusion process, oriented polymers and liquid crystals, and models of nucleation and growth. Its mathematical structure is deeply connected to random matrix theory and integrable systems, rendering KPZ an especially rich and universal class of nonequilibrium fluctuations [196, 197].

Since the connection between one-dimensional heat transport and the KPZ universality class was recognized [179], the theoretical understanding of anomalous transport phenomena has rapidly advanced. In particular, within the NLFH framework, the sound modes and the heat mode are predicted to belong to distinct universality classes depending on microscopic symmetries.

Furthermore, the spatiotemporal correlations of the hydrodynamic modes can be analyzed using mode-coupling theory, which allows the derivation of scaling laws for anomalous energy diffusion. Specifically, at sufficiently long times, the sound modes propagate far away from the origin, so their correlations with the heat mode remaining at the origin can be neglected. This approximation effectively truncates higher-order correlations. Under this scheme, the spatiotemporal correlation of energy fluctuations takes the form

$$\langle \delta \epsilon_n(t) \delta \epsilon_0(0) \rangle_{\text{eq}} \sim \frac{a_h}{t^{\gamma_h}} f_h\left(\frac{n}{t^{\gamma_h}}\right) + \frac{a_s}{t^{\gamma_s}} \left\{ f_s\left(\frac{n+ct}{t^{\gamma_s}}\right) + f_s\left(\frac{n-ct}{t^{\gamma_s}}\right) \right\}, \quad (1.3.26)$$

where two distinct universality classes are predicted. In general, the sound modes belong to the KPZ universality class, exhibiting KPZ scaling. Concretely, the scaling function f_s corresponds to the KPZ function, and the exponent is $\gamma_s = 2/3$. The heat mode, on the

other hand, displays superdiffusion following a Lévy distribution with exponent $5/3$, yielding $\gamma_h = 3/5$. In this case, the autocorrelation function exhibits anomalous relaxation with $\tilde{\beta} = 2/3$, and the divergence exponent of the thermal conductivity is $\alpha = 1/3$.

In contrast, in the special case where the interaction potential V is an even function and the pressure satisfies $-\langle V' \rangle_{\text{eq}} = 0$, the sound modes obey Gaussian diffusive scaling. Specifically, the scaling function f_s becomes Gaussian, with exponent $\gamma_s = 1/2$. Simultaneously, the heat mode exhibits superdiffusion following a Lévy distribution with exponent $3/2$, giving $\gamma_h = 2/3$. Consequently, the autocorrelation function shows anomalous relaxation with $\tilde{\beta} = 1/2$, and the divergence exponent of the thermal conductivity becomes $\alpha = 1/2$.

However, a recent study [198] has reported that even when the interaction potential V is an even function, both the heat mode and the sound modes exhibit the same dynamical exponent $\gamma = 2/3$. This conclusion is derived through analyses based on the dynamical renormalization group, and the corresponding scaling functions have been found to take values close to those of the KPZ universality class.

1.4 Anomalous heat transport in the long-range interacting systems

In this and the next section, we provide an overview of previous studies on transport phenomena in long-range interacting systems.

We begin by discussing energy transport. In long-range interacting systems, there had been no previous studies at all for energy transport in spin systems, even in the numerical approach. Instead, thermal transport has primarily been investigated in spring-particle systems, as a natural extension of the anomalous heat conduction observed in low-dimensional systems, where long-time tails play a crucial role. Moreover, establishing a general theoretical framework has proven challenging. Except for a limited class of exactly solvable models, current studies have largely relied on numerical simulations.

Among the most extensively studied setups are the α -Hamiltonian mean-field model (often referred to as the α -HMF or α -XY model) and the long-range interacting Fermi-Pasta-Ulam-Tsingou (α -FPUT) model. Their Hamiltonians are given respectively by

$$H_{\text{MF}}^{\alpha} := \sum_{i=1}^N \frac{L_i^2}{2I_i} + \frac{1}{\mathcal{N}_{\alpha}} \sum_{\substack{i,j=1 \\ (i \neq j)}}^N \frac{1 - \cos(\theta_i - \theta_j)}{|i - j|^{\alpha}}, \quad (1.4.1)$$

$$H_{\text{FPUT}}^{\alpha} := \sum_{i=1}^N \frac{p_i^2}{2} + \frac{1}{2} (q_{n+1} - q_n)^2 + \frac{1}{4\mathcal{N}_{\alpha}} \sum_{\substack{i,j=1 \\ (i \neq j)}}^N \frac{(q_i - q_j)^4}{|i - j|^{\alpha}}, \quad (1.4.2)$$

where $\mathcal{N}_{\alpha} := \sum_{r=1}^{N/2} r^{-\alpha}$ denotes the Kac normalization factor.

For the α -HMF model, numerical studies have suggested the existence of a critical value α_c such that energy transport is diffusive for $\alpha \leq \alpha_c$, while it becomes ballistic for $\alpha > \alpha_c$,

with $1 < \alpha_c < 2$ [50–52, 199]. Similarly, in the α -FPUT model, heat transport is generally anomalous, whereas ballistic transport is observed specifically at $\alpha = 2$, again supported by numerical evidence [54, 199, 200].

More recently, the long-range interacting ϕ^4 model,

$$H_{\phi^4}^\alpha := \sum_{i=1}^N \frac{p_i^2}{2} \pm \frac{1}{2} q_n^2 + \frac{1}{4} q_n^4 + \frac{1}{\mathcal{N}_\alpha} \sum_{\substack{i,j=1 \\ (i \neq j)}}^N \frac{q_i q_j}{|i-j|^\alpha}, \quad (1.4.3)$$

has also been explored. Anomalous heat transport has been shown for $1 < \alpha < 2$, while diffusive behavior emerges for $\alpha > 2$ [201].

Although a wide range of numerical studies have been conducted, finite-size effects become severely critical, making the extraction of unambiguous conclusions challenging. To circumvent these issues, analytical progress has been made by extending momentum-exchange models [202–204] to include long-range interactions [55].

The long-range interacting momentum-exchange model introduces long-range couplings into an otherwise exactly solvable momentum-exchange system, thereby providing a rare example of an analytically tractable long-range interacting model. The Hamiltonian incorporates harmonic oscillators with long-range interactions, and stochastic momentum exchanges are added to generate a hybrid deterministic–stochastic dynamics.

Concretely, the Hamiltonian is given by

$$H_{\text{ME}}^\alpha = \sum_{n=1}^N \frac{p_n^2}{2} + \frac{\nu}{2} q_n^2 + \sum_{n=1}^N \frac{1}{\mathcal{N}_\alpha} \sum_{r=1}^{N/2} \frac{1}{r^\alpha} \frac{1}{2} (q_{n+r} - q_n)^2, \quad (1.4.4)$$

where q_n and p_n denote the position and momentum of the n -th particle, respectively. The normalization factor is $\mathcal{N}_\alpha = \sum_{r=1}^{N/2} r^{-\alpha}$.

Incorporating stochastic momentum exchange, the full dynamics is described by

$$\begin{cases} dq_n = p_n dt, \\ dp_n = \left[-\nu q_n + \mathcal{N}_\alpha^{-1} \sum_{r=1}^{N/2} r^{-\alpha} (q_{n+r} - 2q_n + q_{n-r}) \right] dt \\ \quad + (p_{n+1} - p_n) d\mathcal{N}_n(t) + (p_{n-1} - p_n) d\mathcal{N}_{n-1}(t) \end{cases} \quad (1.4.5)$$

where $d\mathcal{N}_n$ is a Poisson noise that represents momentum exchange between particles n and $n+1$:

$$d\mathcal{N}_n(t) = \begin{cases} 1, & \text{with probability } \gamma dt, \\ 0, & \text{otherwise.} \end{cases} \quad (1.4.6)$$

We now summarize the thermal transport properties established for this model.

Unpinned case ($\nu = 0$). For $\alpha \leq 2$, the energy autocorrelation diverges, and no relaxation exponent can be defined. For $\alpha > 2$, anomalous heat transport is observed. The exponent of anomalous relaxation is given by $\tilde{\beta} = (\alpha - 2)/2$ for $2 < \alpha < 3$ and $\tilde{\beta} = 1/2$ for $\alpha > 3$ [55]. Energy diffusion is described by a fractional diffusion equation with exponent $3/(7 - \alpha)$ for $2 < \alpha \leq 3$, and exponent $3/4$ for $\alpha > 3$ [205].

Pinned case ($\nu \neq 0$). For the short-range interacting momentum-exchange model in the pinned case, the pinning potential suppresses anomalous transport entirely and exhibits normal diffusion [202–204]. In contrast, long-range interactions allow anomalous heat transport to persist depending on the interaction exponent. For $\alpha \leq 3/2$, the energy autocorrelation diverges and no relaxation exponent exists. For $3/2 < \alpha \leq 5/2$, anomalous heat transport emerges, whereas diffusive transport is restored for $\alpha > 5/2$. The anomalous relaxation exponent satisfies $\tilde{\beta} = (2\alpha - 3)/2$ for $3/2 < \alpha < 3$, and $\tilde{\beta} = 3/2$ for $\alpha > 3$ [55, 205].

Despite these analytical and numerical developments, a general theoretical framework applicable to generic long-range interacting systems is still lacking. Moreover, prior research has focused mainly on spring-particle systems, and, to the best of our knowledge, no studies have examined energy transport in long-range interacting spin systems, which are relevant for the recent experimental setup, such as Rydberg atoms, cold atomic gases and ionic traps. This motivates the theoretical development presented in the subsequent chapters.

Next, we discuss the particle and spin transport in the long-range interacting systems.

1.5 Anomalous spin and particle transport in the long-range interacting systems

In this section, we provide an overview of spin and particle transport in long-range interacting systems.

We begin by summarizing prior studies on spin transport in long-range spin models. Most investigations have focused on the one-dimensional long-range interacting XX model of the following form [63]

$$H_{\text{XX}}^\alpha = \frac{1}{\tilde{\mathcal{N}}_\alpha} \sum_{i,j=1}^N J \frac{S_i^x S_j^x + S_i^y S_j^y}{|i - j|^\alpha}, \quad (1.5.1)$$

or, on variants of this model supplemented with additional local interactions such as nearest-neighbor or next-nearest-neighbor couplings [62]. Here, $\hat{S}^\sigma$ denotes the spin-1/2 operator, and $\tilde{\mathcal{N}}_\alpha = \sqrt{\sum_{r=1}^{N/2} r^{-2\alpha}}$ is a normalization factor introduced to remove the N - and α -dependence of the time scale associated with the perturbative short-time dynamics of the central spin at $n = 0$. In this model, in addition to the total energy, the S^z magnetization constitutes a locally conserved quantity.

By examining the spatiotemporal correlations of z -spin $\langle S_n^z(t) S_0^z \rangle$, numerical simulations have revealed the following behavior: in one dimension, for $\alpha > 3/2$, the spin transport exhibits Gaussian diffusion, whereas for $1/2 < \alpha < 3/2$, the dynamics exhibits Lévy diffusion characterized by the exponent $2\alpha - 1$. In addition to numerical simulations, Ref. [63] has pursued a phenomenological approach in which the dynamics are mapped onto a classical master equation, allowing for a detailed discussion of the associated scaling behavior. However, this approach relies on phenomenological assumptions regarding the transition rates of the master equation, and no microscopic derivation has been established. Consequently, a systematic theoretical framework capable of accounting for these transport properties has been lacking.

Next, we discuss a noteworthy line of prior work in which long-range interactions are incorporated into a stochastic lattice field model of Ginzburg–Landau (GL) dynamics, and the resulting system is analyzed using the framework of Macroscopic Fluctuation Theory (MFT). In Ref. [206], they consider an one-dimensional long-range interacting GL dynamics possessing a single locally conserved field $\varphi(t)$:

$$\begin{aligned} \partial_t \varphi(x, t) = & \sum_y K(x - y) \beta(\varphi(x, t), \varphi(y, t)) \\ & + \sum_y \sqrt{2 K(x - y) \gamma(\varphi(x, t), \varphi(y, t))} \xi(x - y, t), \end{aligned} \quad (1.5.2)$$

where β represents the deterministic drift term and γ determines the strength of fluctuations through the mobility. The noise ξ satisfies an antisymmetry condition, $\xi(x - y, t) = -\xi(y - x, t)$, which ensures that $\varphi(t)$ is a locally conserved quantity. The long-range interaction is introduced via a kernel of the form $K(r) \sim r^{-1-\alpha}$.

For this class of models, taking the hydrodynamic limit leads to a characteristic spacetime scaling $t \sim x^\alpha$, so that for $\alpha < 2$, one obtains a nonlinear fractional diffusion equation as the effective hydrodynamic description. Furthermore, using MFT, the large-deviation structure associated with this hydrodynamic equation has been analyzed.

1.6 Overview of the results

The primary aim of this thesis is to elucidate the general characteristics of energy and spin diffusion in spin systems with long-range interactions, with particular emphasis on classical spin systems as an initial step toward a broader theoretical understanding. Throughout this work, we investigate prototypical models, including the transverse Ising and XYZ models (the latter encompassing the XY model), under the condition $\alpha > D$ (with D denoting the spatial dimension). This regime guarantees the extensivity of equilibrium thermodynamics and enables the use of powerful analytical techniques. Within this framework, we address the following fundamental questions: Does a transition from normal to anomalous diffusion occur as the exponent α is varied, even when thermodynamic extensivity is preserved at

equilibrium? If such a transition exists, what are the universal mechanisms that govern it, and what systematic methods allow us to classify the corresponding diffusion behavior?

Energy (or spin) diffusion is quantitatively related to thermal (or spin) conductivity, which is formulated via the Green–Kubo relation for energy (or spin) current correlations. Through extensive numerical investigations, we find that anomalous transport does not originate from the long-time tail of the current correlation, as often discussed in short-range interacting systems. Instead, a distinct mechanism emerges: an anomalous enhancement of the amplitude of equal-time current correlations. Motivated by this observation, we focus on equal-time current correlations and analyze the scaling of their amplitudes. In this direction, we establish several theorems (Theorems 1, 2, 3, 4, and 5), all of which are formulated so as to remain valid even in quantum systems.

These theorems reveal a universal structural framework underlying energy and spin diffusion in long-range interacting systems. A central tool in our analysis is the cumulant power-law clustering theorem (Theorem 1), which asserts that joint cumulants of observables supported on spatially separated regions decay in a power-law fashion. By applying this theorem to equal-time energy and spin current correlations (Theorems 2 and 4), we demonstrate the following universal criterion: in one-dimensional systems, the condition $\alpha > \alpha_c = 3/2$ is sufficient to ensure normal diffusion of both energy and spin. Moreover, numerical simulations indicate that this bound is optimal, as systems with $\alpha < 3/2$ exhibit Lévy-type anomalous diffusion. Such Lévy diffusion can be naturally understood through the framework of fluctuating hydrodynamics. In higher-dimensional lattices, we prove that normal diffusion persists provided that $\alpha > D$, a condition coinciding with the requirement of thermodynamic extensivity at equilibrium (Theorems 3 and 5). These results collectively establish a rigorous and unified foundation for understanding the general features of energy and spin transport in long-range interacting spin systems. Furthermore, the analytical methods developed in this thesis provide a powerful and versatile framework for addressing a broad class of transport phenomena in systems with long-range interactions.

The remainder of this thesis is organized as follows.

In Chapter 2, we derive the cumulant power-law clustering theorem, which extends the conventional clustering property to higher-order cumulants and provides a rigorous foundation for analyzing transport properties in long-range interacting spin systems on D -dimensional lattices.

In Chapter 3, we develop a theoretical framework of the fluctuating hydrodynamics for long-range interacting systems based on the projection-operator formalism, establishing a systematic connection between microscopic Hamiltonian dynamics and mesoscopic stochastic descriptions of local conserved quantities.

In Chapter 4, we apply these theoretical tools to investigate energy diffusion in long-range interacting spin systems. By combining rigorous bounds derived from the clustering theorem with fluctuating-hydrodynamics arguments, we identify the conditions under which energy diffusion is normal or anomalous, and clarify their dependence on the interaction exponent

and spatial dimension.

In Chapter 5 we investigate spin diffusion in long-range interacting systems. Focusing on spin-conserving models, we show that the scaling behavior of spin diffusion closely parallels that of energy diffusion, suggesting a universal mechanism independent of the specific conserved quantity.

Finally, Chapter 6 explores energy and spin diffusion in the strongly long-range interacting regime $\alpha < D$ with the Kac normalization factor. While rigorous bounds on equal-time current correlations can still be established, the long-time transport behavior remains unresolved due to strong finite-size effects, highlighting open problems and directions for future research.

Chapter 2

Cumulant Power-law Clustering Theorem

To analyze the transport property of long-range interacting spin systems, we introduce what we refer to as the cumulant power-law clustering theorem. This theorem generalizes the standard clustering property [207–210], extending it to higher-order cumulants, and applies to spin systems with long-range interaction defined on D -dimensional lattices.

2.1 Mathematical preliminaries

In this chapter, we consider a D -dimensional long-range interacting spin system defined on a square lattice, where each spin occupies a lattice site. The set of lattice sites is denoted by Λ , and for any subset $X \subseteq \Lambda$, its cardinality is represented by $|X|$. The distance $d_{X,Y}$ between two subsets $X, Y \subseteq \Lambda$ is defined by the Manhattan distance of the shortest path connecting the two sets, and we set the lattice constant to unity. For computational convenience, we also define $d_{i,i} := 1$. Since the Euclidean distance is always shorter than the Manhattan distance, it is occasionally used to obtain convenient upper bounds. The complementary set of X is denoted by X^c , i.e., $X^c = \Lambda \setminus X$. We use the notation O_X to denote a quantity O (an operator in the quantum case) supported on the region X . The symbol $\|O\|$ denotes the maximal value of $\|O\|$, which corresponds to the spectral norm of O in the quantum case. Similarly, $\|O\|_1$ represents the L^1 norm of O , corresponding to the trace norm in the quantum setting. Throughout this section, we mainly employ the terminology of the quantum formalism, as all the results presented here remain valid in the quantum case.

We consider a k -local long-range spin model described by the Hamiltonian

$$H = \sum_{Z:|Z|\leq k} h_Z, \quad (2.1.1)$$

where Z denotes a subset of lattice sites that participate in an interaction, and h_Z represents the corresponding local Hamiltonian term. The cardinality $|Z|$ gives the number of interacting

sites, so that h_Z encodes a $|Z|$ -body interaction. As an illustrative example, in the long-range transverse Ising model, $H = -J \sum_{i,j} S_i^z S_j^z / d_{i,j}^\alpha - h \sum_i S_i^x$, one can identify the subsets as $Z = \{1, 2, 3, \dots, (1, 2), (1, 3), (1, 4), \dots, (2, 3), (2, 4), \dots\}$, with local Hamiltonians $h_{Z=(i)} = -h S_i^x$ and $h_{Z=(i,j)} = -J S_i^z S_j^z / d_{i,j}^\alpha$. To establish the cumulant power-law clustering theorem in a general setting, we assume that the interaction strength satisfies the following decay condition:

$$J_{i,i'} := \sum_{Z: Z \ni \{i,i'\}} \|h_Z\| \leq \frac{g}{d_{i,i'}^\alpha}, \quad (i \neq i'). \quad (2.1.2)$$

Here, $J_{i,i'}$ represents the maximal contribution from all interaction terms involving both sites i and i' . In quantum systems, this corresponds to the largest spectral norm of such terms. The coefficient g is a constant. We focus on D -dimensional lattices and restrict our attention to the regime $\alpha > D$, ensuring thermodynamic extensivity. We also note the useful relation

$$\sum_{Z \ni i} \|h_Z\| \leq \sum_{j \in \Lambda} \sum_{Z \ni (i,j)} \|h_Z\| \leq g u_0, \quad (2.1.3)$$

where $u_0 := \sum_{j \in \Lambda} 1/d_{i,j}^\alpha$.

2.2 Joint cumulant

We consider the joint cumulant defined through the cumulant generating function. Let O_X be an operator supported by the region X . Suppose that $X_1, X_2, \dots, X_n$ have no overlap between them, i.e., $X_i \cap X_j = \emptyset$ for any i and j ($1 \leq i, j \leq n$). The joint cumulant is defined as

$$\langle O_{X_1} O_{X_2} \cdots O_{X_n} \rangle_c = \frac{\partial^n}{\partial \lambda_1 \partial \lambda_2 \cdots \partial \lambda_n} \Big|_{\lambda=0} \mu(\lambda), \quad (2.2.1)$$

$$\mu(\lambda) = \ln \left\langle \exp \left(\sum_i \lambda_i O_{X_i} \right) \right\rangle, \quad (2.2.2)$$

where $\langle \dots \rangle$ is the average over an equilibrium state ρ with the inverse temperature $\beta (= (k_B T)^{-1})$ and $\langle \dots \rangle_c$ is the joint cumulant. Here, k_B is the Boltzmann constant and T is the temperature. The first order cumulant is identical to the first order moment, i.e., $\langle O_X \rangle_c = \langle O_X \rangle$. Other examples include the second order: $\langle O_{X_1} O_{X_2} \rangle_c = \langle O_{X_1} O_{X_2} \rangle - \langle O_{X_1} \rangle \langle O_{X_2} \rangle$ and the third order: $\langle O_{X_1} O_{X_2} O_{X_3} \rangle_c = \langle (O_{X_1} - \langle O_{X_1} \rangle)(O_{X_2} - \langle O_{X_2} \rangle)(O_{X_3} - \langle O_{X_3} \rangle) \rangle$. The crucial property is that the n th order mixed moment is expressed through all decomposition of cumulants, i.e.,

$$\langle O_{X_1} O_{X_2} \cdots O_{X_n} \rangle = \langle O_{X_1} O_{X_2} \cdots O_{X_n} \rangle_c + D_n, \quad (2.2.3)$$

where D_n is a sum of all decomposition with joint cumulants up to $(n-1)$ th order. Examples include the second and third-order expressions:

$$\langle O_{X_1} O_{X_2} \rangle = \langle O_{X_1} O_{X_2} \rangle_c + D_2, \quad (2.2.4)$$

$$D_2 := \langle O_{X_1} \rangle_c \langle O_{X_2} \rangle_c, \quad (2.2.5)$$

$$\langle O_{X_1} O_{X_2} O_{X_3} \rangle = \langle O_{X_1} O_{X_2} O_{X_3} \rangle_c + D_3, \quad (2.2.6)$$

$$D_3 := \langle O_{X_1} O_{X_2} \rangle_c \langle O_{X_3} \rangle_c + \langle O_{X_1} O_{X_3} \rangle_c \langle O_{X_2} \rangle_c + \langle O_{X_2} O_{X_3} \rangle_c \langle O_{X_1} \rangle_c + \langle O_{X_1} \rangle_c \langle O_{X_2} \rangle_c \langle O_{X_3} \rangle_c. \quad (2.2.7)$$

The following expression is also useful in the following analysis.

$$\langle O_{X_1} O_{X_2} \cdots O_{X_n} \rangle_c = \langle O_{X_1} O_{X_2} \cdots O_{X_n} \rangle - D_n. \quad (2.2.8)$$

2.3 Cumulant power-law clustering

Theorem 1. [*Cumulant Power-law Clustering Theorem*] Consider a k -local long-range spin system defined on a D -dimensional lattice. Let O_{X_j} be operators supported on mutually disjoint regions X_j satisfying $X_i \cap X_j = \emptyset$ for $i \neq j$. For any exponent $\alpha > D$, the n th-order joint cumulant $\langle O_{X_1} \cdots O_{X_n} \rangle_c$ in equilibrium obeys the inequality

$$\langle O_{X_1} \cdots O_{X_n} \rangle_c \leq c 2^n n! \prod_{j=1}^n \|O_{X_j}\| \sum_P \prod_{i=1}^{n-1} \frac{|X_{P_i}| |X_{P_{i+1}}| e^{(|X_{P_i}| + |X_{P_{i+1}}|)/k}}{(d_{X_{P_i}, X_{P_{i+1}}})^\alpha}, \quad (2.3.1)$$

for $\beta_c = (nguk)^{-1} W(1/2e^2)$, where $u = 2^\alpha \sum_{i \in \Lambda} d_{i,j}^{-\alpha} (< \infty)$, and W denotes the Lambert function. Using $W(1/2e^2) < 1/(4e)$, one further finds $\beta_c < (4enguk)^{-1}$. Here, c is a constant, $\|O_X\|$ denotes the operator norm of O_X , and P runs over all Hamiltonian paths connecting the vertices $X_1, \dots, X_n$. The quantity $d_{X,Y}$ represents the minimal distance between supports X and Y , while $|X|$ gives the number of sites within region X .

Theorem 1 generalizes the conventional clustering property, which is typically formulated for the second-order cumulant ($n = 2$). In general, the clustering property—valid for both short-range and long-range interacting systems—can be expressed as

$$|\langle O_{X_1} O_{X_2} \rangle_c| < c \|O_{X_1}\| \|O_{X_2}\| |X_1| |X_2| e^{-c' d_{X_1, X_2}} \quad (\text{short-range}), \quad (2.3.2)$$

$$|\langle O_{X_1} O_{X_2} \rangle_c| < c \|O_{X_1}\| \|O_{X_2}\| |X_1| |X_2| (d_{X_1, X_2})^{-\alpha} \quad (\text{long-range}), \quad (2.3.3)$$

where X_1 and X_2 denote disjoint regions (so that $X_1 \cap X_2 = \emptyset$), and the constants c and c' are independent of the choice of observables. Here, d_{X_1, X_2} quantifies the spatial separation between the two regions. Inequality (2.3.2) describes the exponential clustering property [207, 208], which holds across a broad range of temperatures and typically remains valid even at low temperatures, except in the vicinity of critical points associated with phase

transitions. The corresponding relation for long-range interactions, inequality (2.3.3), is likewise expected to be robust, as discussed in Ref. [210]. However, when one attempts to derive these relations rigorously from first principles using cluster expansion techniques, the convergence of the expansion series usually necessitates a high-temperature condition ($\beta < \beta_c$), reflecting the possible emergence of finite-temperature phase transitions [209–211]. Moreover, the cluster expansion approach provides explicit functional forms for the clustering properties of higher-order cumulants. From a physical standpoint, however, it is reasonable to expect that such functional behavior extends well beyond the strict high-temperature regime, persisting over a wider temperature range where the expansion itself may no longer be rigorously convergent.

2.4 Proof of Theorem 1

Proof of Theorem 1. From this point onward, we proceed to prove Theorem 1. To this end, we employ the cluster expansion [209, 211, 212] for the n th-order joint cumulant. The key technique used in our derivation is the multi-phase-space method, which can be straightforwardly extended to quantum systems. In the quantum case, the multi-phase space is naturally replaced by the multi-Hilbert space representation. Let $O_X^{(i)}$ denote the operator O_X supported on the region X in the i th Hilbert space ($i = 1, \dots, n$). We define the multi-copied density operator $\tilde{\rho}$ associated with the equilibrium state ρ as

$$\tilde{\rho} = \rho^{(1)} \otimes \rho^{(2)} \otimes \dots \otimes \rho^{(n)}, \quad (2.4.1)$$

where $\rho^{(i)}$ represents the copy of ρ acting on the i th Hilbert space. We also define the difference operator $O_X^{(i,j)}$ ($i < j$) by

$$O_X^{(i,j)} := O_X^{(i)} - O_X^{(j)}, \quad (i < j), \quad (2.4.2)$$

where, for simplicity, we denote $O_X^{(i)}$ as $O_X^{(i)} = \mathbf{1} \otimes \mathbf{1} \otimes \dots \otimes O_X^{(i)} \otimes \dots \otimes \mathbf{1}$. In addition, we define the following cumulant operator.

Definition 1. (*Cumulant operator*) We define the n th order cumulant operator $\tilde{O}_{X_1, X_2, \dots, X_n}^{(i_1, i_2, \dots, i_n)}$

$$\tilde{O}_{X_1, X_2, \dots, X_n}^{(i_1, i_2, \dots, i_n)} := O_{X_1}^{(i_1)} \prod_{j=2}^n \left(\sum_{k=1}^{j-1} O_{X_j}^{(i_k, i_j)} \right), \quad i_1 < i_2 < \dots < i_n. \quad (2.4.3)$$

The product is aligned from the left to right in the quantum case. When $n = 1$, $\tilde{O}_{X_1}^{(i_1)} = O_{X_1}^{(i_1)}$. Examples include the second order ($n = 2$): $\tilde{O}_{X_1, X_2}^{(1,2)} = O_{X_1}^{(1)} O_{X_2}^{(1,2)}$, and the third order ($n = 3$): $\tilde{O}_{X_1, X_2, X_3}^{(1,2,3)} = O_{X_1}^{(1)} O_{X_2}^{(1,2)} (O_{X_3}^{(1,3)} + O_{X_3}^{(2,3)})$.

Proposition 1. The n th order cumulant operator yields the n th order joint cumulant

$$\text{tr} \left(\tilde{\rho} \tilde{O}_{X_1, X_2, \dots, X_n}^{(i_1, i_2, \dots, i_n)} \right) = \langle O_{X_1} O_{X_2} \dots O_{X_n} \rangle_c, \quad (i_1 < i_2 < \dots < i_n), \quad (2.4.4)$$

where tr means the phase-space integral over the multi-phase space, and in the quantum case, it reads the trace over the multi-Hilbert space.

Here, the choice of space $(i_1, i_2, \dots, i_n)$ is arbitrary as long as $i_1 < i_2 < \dots < i_n$. Practically, we use $(i_1, i_2, \dots, i_n) = (1, 2, \dots, n)$.

Proof of Proposition 1. We now prove Proposition 1 by induction on n . For the base case $n = 2$, let us choose $i_1 = 1$ and $i_2 = 2$. Then, we define

$$\tilde{O}_{X_1, X_2}^{(1,2)} = O_{X_1}^{(1)} O_{X_2}^{(1)} - \mathcal{D}_2, \quad (2.4.5)$$

$$\mathcal{D}_2 = \tilde{O}_{X_1}^{(1)} \tilde{O}_{X_2}^{(2)}. \quad (2.4.6)$$

It is straightforward to verify that

$$\text{tr}(\tilde{\rho} \tilde{O}_{X_1, X_2}^{(1,2)}) = \langle O_{X_1} O_{X_2} \rangle_c. \quad (2.4.7)$$

Moreover, the space indices (1) and (2) can be replaced by arbitrary (i_1) and (i_2) as long as $i_1 < i_2$. The operator $\mathcal{D}_2$ simply represents the product of first-order cumulants. In general, $\mathcal{D}_n$ is defined as the sum over all decompositions of n th-order correlations into products of cumulant operators up to order $(n-1)$, involving the operators $O_{X_1}, \dots, O_{X_n}$ acting on the spaces $(i_1, \dots, i_n)$. This construction serves as the operator analogue of the c -number expansion of D_n in Eq. (2.2.8). Next, we consider the case $n = 3$, taking $(i_1, i_2, i_3) = (1, 2, 3)$:

$$\tilde{O}_{X_1, X_2, X_3}^{(1,2,3)} = O_{X_1}^{(1)} O_{X_2}^{(1)} (O_{X_3}^{(1)} + O_{X_3}^{(2,3)}) - \mathcal{D}_3, \quad (2.4.8)$$

$$\begin{aligned} \mathcal{D}_3 &= \tilde{O}_{X_1}^{(1)} \tilde{O}_{X_2, X_3}^{(2,3)} + \tilde{O}_{X_1, X_3}^{(1,3)} \tilde{O}_{X_2}^{(2)} + \tilde{O}_{X_1, X_2}^{(1,2)} \tilde{O}_{X_3}^{(3)} + \tilde{O}_{X_1}^{(1)} \tilde{O}_{X_2}^{(2)} \tilde{O}_{X_3}^{(3)} \\ &= O_{X_1}^{(1)} O_{X_2}^{(2)} O_{X_3}^{(2,3)} + O_{X_1}^{(1)} O_{X_3}^{(1,3)} O_{X_2}^{(2)} + O_{X_1}^{(1)} O_{X_2}^{(1,2)} O_{X_3}^{(3)} + O_{X_1}^{(1)} O_{X_2}^{(2)} O_{X_3}^{(3)}. \end{aligned} \quad (2.4.9)$$

Again, one can verify that

$$\text{tr}(\tilde{\rho} \tilde{O}_{X_1, X_2, X_3}^{(1,2,3)}) = \langle O_{X_1} O_{X_2} O_{X_3} \rangle_c. \quad (2.4.10)$$

Note that this argument closely parallels the structure of Eq. (2.2.8). Furthermore, we observe that $\mathcal{D}_3$ can be recursively expressed in terms of $\mathcal{D}_2$ as

$$\mathcal{D}_3 = \mathcal{D}_2 (O_{X_3}^{(1,3)} + O_{X_3}^{(2,3)}) + (\tilde{O}_{X_1, X_2}^{(1,2)} + \mathcal{D}_2) \tilde{O}_{X_3}^{(3)}. \quad (2.4.11)$$

It is easy to check that the space indices (1), (2), and (3) can be replaced by arbitrary (i_1) , (i_2) , and (i_3) as long as $i_1 < i_2 < i_3$.

We now assume that the following relations hold for all orders up to n :

$$\tilde{O}_{X_1, X_2, \dots, X_n}^{(1,2,\dots,n)} = \tilde{O}_{X_1, X_2, \dots, X_n}^u - \mathcal{D}_n, \quad (2.4.12)$$

$$\tilde{O}_{X_1, X_2, \dots, X_n}^u := O_{X_1}^{(1)} O_{X_2}^{(1)} \prod_{j=3}^n \left(O_{X_j}^{(1)} + \sum_{k=2}^{j-1} O_{X_j}^{(k,j)} \right). \quad (2.4.13)$$

The operator $\tilde{O}_{X_1, X_2, \dots, X_n}^u$ reproduces the n th-order mixed moment as

$$\begin{aligned} \text{tr}(\tilde{\rho} \tilde{O}_{X_1, X_2, \dots, X_n}^u) &= \text{tr}(\tilde{\rho} O_{X_1}^{(1)} O_{X_2}^{(1)} \cdots O_{X_n}^{(1)}) \\ &= \text{tr}(\rho O_{X_1} O_{X_2} \cdots O_{X_n}) = \langle O_{X_1} O_{X_2} \cdots O_{X_n} \rangle. \end{aligned} \quad (2.4.14)$$

Recall that the operator $\mathcal{D}_n$ is defined as the sum of all n th-order representations that are decomposed into products of cumulant operators up to order $(n-1)$, with the operators $O_{X_1}, \dots, O_{X_n}$ acting on the space $(1, \dots, n)$. By construction we have $\text{tr}(\tilde{\rho} \mathcal{D}_n) = D_n$, where D_n denotes the sum of all decompositions involving joint cumulants of order at most $(n-1)$ (see again the relation (2.2.8)). Together with (2.4.14), this yields

$$\text{tr}(\tilde{\rho} \tilde{O}_{X_1, X_2, \dots, X_n}^{(i_1, i_2, \dots, i_n)}) = \langle O_{X_1} O_{X_2} \cdots O_{X_n} \rangle_c.$$

Assuming (2.4.12) and (2.4.13) to hold for all orders up to n , we now show that the same structure persists for order $(n+1)$. To this end, we employ Lemma 2 for $\mathcal{D}_{n+1}$ (proved below). We compute

$$\begin{aligned} \tilde{O}_{X_1, X_2, \dots, X_{n+1}}^u - \mathcal{D}_{n+1} &= \tilde{O}_{X_1, X_2, \dots, X_{n+1}}^u - \mathcal{D}_n \sum_{\ell=1}^n O_{X_{n+1}}^{(\ell, n+1)} - (\tilde{O}_{X_1, X_2, \dots, X_n}^{(1, 2, \dots, n)} + \mathcal{D}_n) O_{X_{n+1}}^{(n+1)} \\ &= \tilde{O}_{X_1, X_2, \dots, X_{n+1}}^u - \mathcal{D}_n \sum_{\ell=1}^n O_{X_{n+1}}^{(\ell, n+1)} - \tilde{O}_{X_1, X_2, \dots, X_n}^u O_{X_{n+1}}^{(n+1)} \\ &= (\tilde{O}_{X_1, X_2, \dots, X_n}^u - \mathcal{D}_n) \sum_{\ell=1}^n O_{X_{n+1}}^{(\ell, n+1)} \\ &= \tilde{O}_{X_1, X_2, \dots, X_n}^{(1, 2, \dots, n)} \sum_{\ell=1}^n O_{X_{n+1}}^{(\ell, n+1)} = \tilde{O}_{X_1, X_2, \dots, X_{n+1}}^{(1, 2, \dots, n+1)}. \end{aligned} \quad (2.4.15)$$

In the first line we used Lemma 2, while the second and third lines follow from the induction hypothesis, namely the relations (2.4.12) and (2.4.13). This completes the inductive step and establishes the desired expression for order $(n+1)$. $\square$

Lemma 2. *The operator $\mathcal{D}_{n+1}$ admits the following decomposition in terms of $\mathcal{D}_n$:*

$$\mathcal{D}_{n+1} = \mathcal{D}_n \sum_{\ell=1}^n O_{X_{n+1}}^{(\ell, n+1)} + (\tilde{O}_{X_1, X_2, \dots, X_n}^{(1, 2, \dots, n)} + \mathcal{D}_n) O_{X_{n+1}}^{(n+1)}. \quad (2.4.16)$$

Proof of Lemma 2. We begin by analyzing the second term on the right-hand side of Eq. (2.4.16). Since this term contains precisely one instance of $O_{X_{n+1}}^{(n+1)}$, it generates all contributions in which $O_{X_{n+1}}$ appears only through the first-order cumulant operator $\tilde{O}_{X_{n+1}}^{(n+1)}$. An explicit example of this mechanism is illustrated in Eq. (2.4.11). Therefore, the second term in Eq. (2.4.16) exhausts all decompositions that involve cumulant operators containing exactly one factor of $O_{X_{n+1}}^{(n+1)}$.

We next examine the first term in Eq. (2.4.16). The operator $\mathcal{D}_n$ already comprises all decompositions of products of the operators $O_{X_1}, \dots, O_{X_n}$ into cumulant operators of order up to $(n-1)$ within the replicated operator space $(1, \dots, n)$. To analyze its contribution, consider an arbitrary term within $\mathcal{D}_n$ consisting of m cumulant operators ($m \leq n$), which we write as

$$\tilde{O}_{X_{i(1)}, \dots, X_{i(\ell_1)}}^{(i(1), \dots, i(\ell_1))} \tilde{O}_{X_{i(\ell_1+1)}, \dots, X_{i(\ell_2)}}^{(i(\ell_1+1), \dots, i(\ell_2))} \dots \tilde{O}_{X_{i(\ell_{m-1}+1)}, \dots, X_{i(\ell_m)}}^{(i(\ell_{m-1}+1), \dots, i(\ell_m))}, \quad (2.4.17)$$

where the indices satisfy

$$i(\ell_k + j) < i(\ell_k + j + 1), \quad 1 \leq j \leq \ell_{k+1} - \ell_k - 1, \quad k = 0, \dots, m-1,$$

with $\ell_0 = 0$ and, due to operator-number conservation, $\ell_m = n$.

Each cumulant operator in (2.4.17) can be elevated to the next-order cumulant by incorporating the operator $O_{X_{n+1}}$. For instance, the first cumulant in (2.4.17) is upgraded according to

$$\begin{aligned} \tilde{O}_{X_{i(1)}, \dots, X_{i(\ell_1)}}^{(i(1), \dots, i(\ell_1))} &\longrightarrow \tilde{O}_{X_{i(1)}, \dots, X_{i(\ell_1)}}^{(i(1), \dots, i(\ell_1))} \left(O_{X_{n+1}}^{(i(1), n+1)} + \dots + O_{X_{n+1}}^{(i(\ell_1), n+1)} \right) \\ &= \tilde{O}_{X_{i(1)}, \dots, X_{i(\ell_1)}, X_{n+1}}^{(i(1), \dots, i(\ell_1), n+1)}. \end{aligned} \quad (2.4.18)$$

In total, the m cumulant operators in (2.4.17) produce exactly m such upgrades. Summing over the contributions from all possible upgrades yields the expression

$$(2.4.17) \times \sum_{\ell=1}^n O_{X_{n+1}}^{(\ell, n+1)}.$$

Consequently, the first term on the right-hand side of Eq. (2.4.16) accounts for all decompositions in which $O_{X_{n+1}}$ participates in cumulant operators of order two or higher.

Collecting the contributions of the two terms completes the proof. $\square$

From the proposition 1, the n th order joint cumulant in the equilibrium state can be obtained as

$$\begin{aligned} \langle O_{X_1} O_{X_2} \dots O_{X_n} \rangle_c &= \text{tr} \left(\tilde{\rho} \tilde{O}_{X_1, X_2, \dots, X_n}^{(1, 2, \dots, n)} \right) \\ &= \text{tr} \left(\tilde{\rho} O_{X_1}^{(1)} O_{X_2}^{(1, 2)} (O_{X_3}^{(1, 3)} + O_{X_3}^{(2, 3)}) \dots (O_{X_n}^{(1, n)} + \dots + O_{X_n}^{(n-1, n)}) \right), \end{aligned} \quad (2.4.19)$$

where $\tilde{\rho}$ is the multi-copied equilibrium density operator defined as

$$\tilde{\rho} = \rho^{(1)} \otimes \rho^{(2)} \otimes \dots \otimes \rho^{(n)} = \frac{1}{\mathcal{Z}^n} \exp(-\beta H^{\text{sym}}), \quad (2.4.20)$$

$$H^{\text{sym}} = H \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1} + \mathbf{1} \otimes H \otimes \dots \otimes \mathbf{1} + \dots + \mathbf{1} \otimes \mathbf{1} \otimes \dots \otimes H. \quad (2.4.21)$$

Here, $\mathcal{Z}$ is the partition function for only one space, i.e., $\mathcal{Z} = \text{tr}_{(1)} e^{-\beta H}$ where $\text{tr}_{(1)}$ means the trace only for one space. In general, it is convenient to use the symmetrized operators over all spaces for an arbitrary operator O :

$$O^{\text{sym}} = O \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1} + \mathbf{1} \otimes O \otimes \dots \otimes \mathbf{1} + \dots + \mathbf{1} \otimes \mathbf{1} \otimes \dots \otimes O. \quad (2.4.22)$$

We now consider the case where the sets $X_1, X_2, \dots, X_n$ are mutually disjoint. In the cluster expansion, the Hamiltonian is expanded via a Taylor series (that is, a high-temperature expansion). A crucial step in this procedure is to identify the local Hamiltonians associated with the subsets $Z_1, Z_2, \dots$ that form connected clusters linking the regions $X_1, X_2, \dots, X_n$. Here, “connected” means that any region X_k can be reached from any other region $X_{k'}$ by traversing a sequence of overlapping subsets among $Z_1, Z_2, \dots$ and $X_1, X_2, \dots, X_n$. Here, we establish the lemma for the disconnected clusters:

Lemma 3. *If the sets $Z_1, Z_2, \dots, Z_m$ do not connect all of the regions $X_1, X_2, \dots, X_n$, then*

$$\text{tr} \left(O_{Z_1}^{\text{sym}} O_{Z_2}^{\text{sym}} \dots O_{Z_m}^{\text{sym}} \tilde{O}_{X_1, X_2, \dots, X_n}^{(1,2, \dots, n)} \right) = 0. \quad (2.4.23)$$

Proof of Lemma 3. By definition, we obtain

$$\begin{aligned} & \text{tr} \left(O_{Z_1}^{\text{sym}} O_{Z_2}^{\text{sym}} \dots O_{Z_m}^{\text{sym}} \tilde{O}_{X_1, X_2, \dots, X_n}^{(1,2, \dots, n)} \right) \\ &= \text{tr} \left(O_{Z_1}^{\text{sym}} O_{Z_2}^{\text{sym}} \dots O_{Z_m}^{\text{sym}} O_{X_1}^{(1)} O_{X_2}^{(1,2)} (O_{X_3}^{(1,3)} + O_{X_3}^{(2,3)}) \dots (O_{X_n}^{(1,n)} + \dots + O_{X_n}^{(n-1,n)}) \right). \end{aligned} \quad (2.4.24)$$

The condition that $\{Z_1, \dots, Z_m\}$ is not connected to $\{X_1, \dots, X_n\}$ implies that

$$\begin{aligned} & \exists \{i_1, \dots, i_n\} = \{1, \dots, n\}, \text{ s.t. } \forall \{j_1, \dots, j_m\} = \{1, \dots, m\} \\ & (X_{i_1} \cup \dots \cup X_{i_s} \cup Z_{j_1} \cup \dots \cup Z_{j_{s'}}) \cap (X_{i_{s+1}} \cup \dots \cup X_{i_n} \cup Z_{j_{s'+1}} \cup \dots \cup Z_{j_m}) = \emptyset. \end{aligned} \quad (2.4.25)$$

Without loss of generality, we take $(i_1, \dots, i_n) = (1, \dots, n)$. Then, we define

$$W_1 = O_{Z_{j_1}}^{\text{sym}} \dots O_{Z_{j_{s'}}}^{\text{sym}}, \quad W_2 = O_{Z_{j_{s'+1}}}^{\text{sym}} \dots O_{Z_{j_m}}^{\text{sym}}, \quad (2.4.26)$$

which satisfy

$$(X_1 \cup \dots \cup X_s \cup \text{supp}(W_1)) \cap (X_{s+1} \cup \dots \cup X_n \cup \text{supp}(W_2)) = \emptyset. \quad (2.4.27)$$

Using this decomposition, Eq. (2.4.24) can be expressed as

$$\begin{aligned} & \text{tr} \left(O_{Z_1}^{\text{sym}} O_{Z_2}^{\text{sym}} \dots O_{Z_m}^{\text{sym}} O_{X_1}^{(1)} O_{X_2}^{(1,2)} (O_{X_3}^{(1,3)} + O_{X_3}^{(2,3)}) \dots (O_{X_n}^{(1,n)} + \dots + O_{X_n}^{(n-1,n)}) \right) \\ &= \text{tr} \left(W_1 O_{X_1}^{(1)} O_{X_2}^{(1,2)} \dots (O_{X_s}^{(1,s)} + \dots + O_{X_s}^{(s-1,s)}) \right) \\ & \quad \times \text{tr} \left(W_2 (O_{X_{s+1}}^{(1,s+1)} + \dots + O_{X_{s+1}}^{(s,s+1)}) \dots (O_{X_n}^{(1,n)} + \dots + O_{X_n}^{(n-1,n)}) \right). \end{aligned} \quad (2.4.28)$$

The second factor above can be rewritten as

$$\begin{aligned} & \text{tr} \left(W_2 (O_{X_{s+1}}^{(1,s+1)} + \dots + O_{X_{s+1}}^{(s,s+1)}) \dots (O_{X_n}^{(1,n)} + \dots + O_{X_n}^{(n-1,n)}) \right) \\ &= \sum_{j=1}^s \text{tr} \left(W_2 O_{X_{s+1}}^{(j,s+1)} \prod_{i=s+2}^n \left(\sum_{k=1}^{i-1} O_{X_i}^{(k,i)} \right) \right) \\ &= \sum_{j=1}^s \text{tr} \left(W_2 O_{X_{s+1}}^{(s+1,j)} \prod_{i=s+2}^n \left(\sum_{k=1}^{i-1} O_{X_i}^{(k,i)} \right) \right) \\ &= \sum_{j=1}^s \text{tr} \left(W_2 (-O_{X_{s+1}}^{(j,s+1)}) \prod_{i=s+2}^n \left(\sum_{k=1}^{i-1} O_{X_i}^{(k,i)} \right) \right) \\ &= -\text{tr} \left(W_2 (O_{X_{s+1}}^{(1,s+1)} + \dots + O_{X_{s+1}}^{(s,s+1)}) \dots (O_{X_n}^{(1,n)} + \dots + O_{X_n}^{(n-1,n)}) \right), \end{aligned} \quad (2.4.29)$$

where in the second equality we have used the invariance of $\sum_{k=1}^{i-1} O_{X_i}^{(k,i)}$ under the exchange of the labels j and $s+1$, and in the third inequality we have used the anti-symmetry of $O_X^{(j,s+1)}$ under the exchange of j and $s+1$. This immediately leads that

$$\text{tr} \left(W_2(O_{X_{s+1}}^{(1,s+1)} + \dots + O_{X_{s+1}}^{(s,s+1)}) \dots (O_{X_n}^{(1,n)} + \dots + O_{X_n}^{(n-1,n)}) \right) = 0, \quad (2.4.30)$$

and therefore

$$\text{tr} \left(O_{Z_1}^{\text{sym}} O_{Z_2}^{\text{sym}} \dots O_{Z_m}^{\text{sym}} \tilde{O}_{X_1, X_2, \dots, X_n}^{(1,2, \dots, n)} \right) = 0. \quad (2.4.31)$$

□

This lemma immediately implies that only connected clusters yield nonvanishing contributions in the cluster expansion. Consequently, the n th-order joint cumulant in equilibrium can be formally written as

$$\begin{aligned} \langle O_{X_1} O_{X_2} \dots O_{X_n} \rangle_c &= \text{tr} \left(\tilde{\rho} \tilde{O}_{X_1, X_2, \dots, X_n}^{(1,2, \dots, n)} \right) \\ &= \frac{1}{\mathcal{Z}^n} \sum_{m=0}^{\infty} \sum_{\substack{Z_1, Z_2, \dots, Z_m: \\ \text{connected}}} \frac{(-\beta)^m}{m!} \text{tr} \left(h_{Z_1}^{\text{sym}} h_{Z_2}^{\text{sym}} \dots h_{Z_m}^{\text{sym}} \tilde{O}_{X_1, X_2, \dots, X_n}^{(1,2, \dots, n)} \right), \end{aligned} \quad (2.4.32)$$

where the summation $\sum_{\substack{Z_1, Z_2, \dots, Z_m: \\ \text{connected}}}$ runs over all connected collections of interaction subsets. Note that different Z_s are allowed to coincide, e.g., $Z_1 = Z_2$. We now introduce the distribution operator ρ^{con} that collects all connected contributions:

$$\rho^{\text{con}} = \frac{1}{\mathcal{Z}^n} \sum_{m=0}^{\infty} \sum_{\substack{Z_1, Z_2, \dots, Z_m: \\ \text{connected}}} \frac{(-\beta)^m}{m!} h_{Z_1}^{\text{sym}} h_{Z_2}^{\text{sym}} \dots h_{Z_m}^{\text{sym}}. \quad (2.4.33)$$

Using this operator, the n th-order joint cumulant is bounded from above as

$$\langle O_{X_1} O_{X_2} \dots O_{X_n} \rangle_c \leq 2^n n! \|O_{X_1}\| \|O_{X_2}\| \dots \|O_{X_n}\| \|\rho^{\text{con}}\|_1. \quad (2.4.34)$$

Here, $\|\cdot\|$ and $\|\cdot\|_1$ respectively denote the spectral norm and trace norm in the quantum case, while in the classical case they correspond to the maximum absolute value and the L^1 norm. It is important to note that, within a connected cluster $\{Z_1, Z_2, \dots, Z_m\}$, not all subsets necessarily contribute to connecting the regions $X_1, X_2, \dots, X_n$. Some subsets may be disjoint from the connected portion. Hence, we classify the collection $\{Z_1, Z_2, \dots, Z_m\}$ into two disjoint parts: the subsets w_{cl} that actually participate in the connection to $\{X_1, X_2, \dots, X_n\}$, and the complementary subsets w_{cl}^c , i.e.,

$$\{Z_1, Z_2, \dots, Z_m\} = w_{\text{cl}} \oplus w_{\text{cl}}^c, \quad w_{\text{cl}}^c := \{Z_1, Z_2, \dots, Z_m\} \setminus w_{\text{cl}}. \quad (2.4.35)$$

In other words, the connected subset w_{cl} is taken such that $w_{\text{cl}} \oplus \{X_1, \dots, X_n\}$ are connected to each other,

$$Z_i \cap Z_j = \emptyset \quad \text{for} \quad Z_i \in w_{\text{cl}}, Z_j \in w_{\text{cl}}^c. \quad (2.4.36)$$

We show a schematic example of w_{cl} and w_{cl}^c for the case $n = 3$ in Fig. 2.1. By construction, the subsets in w_{cl} do not overlap with any subsets in w_{cl}^c . Only those in w_{cl} form the minimal connected cluster linking $X_1, X_2, \dots, X_n$. We denote the family of all such connected subsets by $\mathcal{G}_{\text{cl}}$, and we define $\mathcal{G}_{\text{cl}}^c$ as the set of all remaining interaction subsets excluding w_{cl} . Note that $\mathcal{G}_{\text{cl}}^c$ depends on the specific choice of w_{cl} , and $w_{\text{cl}}^c \in \mathcal{G}_{\text{cl}}^c$. Let H_c^{sym} denote the Hamiltonian constructed from the local Hamiltonians supported on $\mathcal{G}_{\text{cl}}^c$. Clearly, the total Hamiltonian can be decomposed as $H_c = H - H_{w_{\text{cl}}}$, where $H_{w_{\text{cl}}} = \sum_{Z \in w_{\text{cl}}} h_Z$.

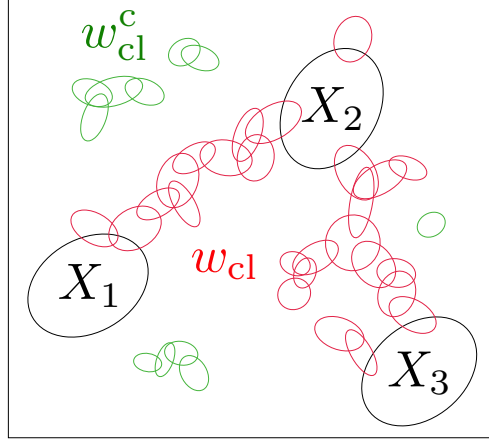


Figure 2.1: The schematic example of the connected subsets w_{cl} and the unconnected subsets w_{cl}^c for the case $n = 3$.

We can then express ρ^{con} as

$$\begin{aligned} \rho^{\text{con}} &= \frac{1}{\mathcal{Z}^n} \sum_{m=0}^{\infty} \frac{(-\beta)^m}{m!} \sum_{s=0}^m \frac{m!}{(m-s)!s!} \sum_{\substack{\{Z_1, \dots, Z_s\} \\ = w_{\text{cl}} \in \mathcal{G}_{\text{cl}}}} \tilde{h}(Z_1, \dots, Z_s) \sum_{\substack{w_{\text{cl}}^c \in \mathcal{G}_{\text{cl}}^c \\ Z_{s+1}, \dots, Z_m \\ \in w_{\text{cl}}^c}} h_{Z_{s+1}}^{\text{sym}} h_{Z_{s+2}}^{\text{sym}} \dots h_{Z_m}^{\text{sym}} \\ &= \frac{1}{\mathcal{Z}^n} \sum_{s=0}^{\infty} \frac{(-\beta)^s}{s!} \sum_{w_{\text{cl}} \in \mathcal{G}_{\text{cl}}} \tilde{h}(Z_1, \dots, Z_s) e^{-\beta H_c^{\text{sym}}}. \end{aligned} \quad (2.4.37)$$

Here, we have defined

$$\tilde{h}(Z_1, \dots, Z_s) := \sum_{\tilde{P}} h^{\text{sym}}(Z_{\tilde{P}_1}) \dots h^{\text{sym}}(Z_{\tilde{P}_s}), \quad (2.4.38)$$

where the summation $\sum_{\tilde{P}}$ runs over all possible combinations of distinct elements among $\{Z_1, \dots, Z_s\}$. In particular, when all Z_i are distinct, this reduces to a sum over all permutations.

We next employ the Golden–Thompson inequality to relate the above expression to an upper bound on ρ^{con} .

$$\begin{aligned} \|e^{-\beta H_c^{\text{sym}}}\|_1 &= \text{tr}(e^{-\beta H_c^{\text{sym}}}) = \text{tr}(e^{-\beta(H^{\text{sym}} - H_{w_{\text{cl}}}^{\text{sym}})}) \\ &\leq \text{tr}(e^{-\beta H^{\text{sym}}} e^{\beta H_{w_{\text{cl}}}^{\text{sym}}}) \leq \mathcal{Z}^n e^{n\beta g_{\text{uks}}}, \end{aligned} \quad (2.4.39)$$

where we use

$$\begin{aligned} \text{tr}\left(e^{-\beta H^{\text{sym}}} e^{\beta H_{w_{\text{cl}}}^{\text{sym}}}\right) &= \|e^{-\beta H^{\text{sym}}} e^{\beta H_{w_{\text{cl}}}^{\text{sym}}}\|_1 \\ &\leq \|e^{-\beta H^{\text{sym}}}\|_1 \|e^{\beta H_{w_{\text{cl}}}^{\text{sym}}}\| = \text{tr}(e^{-\beta H^{\text{sym}}}) \|e^{\beta H_{w_{\text{cl}}}^{\text{sym}}}\|. \end{aligned} \quad (2.4.40)$$

We also have used $\|H_{w_{\text{cl}}}^{\text{sym}}\| \leq nguks$. Using the inequality (2.4.39), we obtain the following relation

$$\|\rho^{\text{con}}\|_1 \leq \sum_{s=0}^{\infty} (n\beta e^{n\beta guk})^s \sum_{\{Z_1, \dots, Z_s\}=w_{\text{cl}} \in \mathcal{G}_{\text{cl}}} \|h_{Z_1}\| \|h_{Z_2}\| \cdots \|h_{Z_s}\|, \quad (2.4.41)$$

where we have used the relation $\|\tilde{h}(Z_1, \dots, Z_s)\| \leq s!n^s \|h_{Z_1}\| \|h_{Z_2}\| \cdots \|h_{Z_s}\|$.

We now classify the set w_{cl} in more detail. Recall that the subsets in w_{cl} serve to connect the regions $X_1, X_2, \dots, X_n$. There are multiple possible connection patterns depending on how these regions are linked. For example, when $n = 2$, there is only one possible connection pattern. For $n = 3$, there are three distinct patterns. In general, for n regions, there exist $n!/2$ possible connection patterns. See Fig. 2.2(a) for an illustration. Identifying these connection patterns is equivalent to finding all possible Hamilton paths in a graph whose vertices correspond to $X_1, X_2, \dots, X_n$.

Let P denote the index of a particular pattern (Hamilton path), and let P_i ($i = 1, \dots, n$) be the vertex at the i -th step of the path, corresponding to the region X_{P_i} . The edge connecting P_i and P_{i+1} is formed by a cluster composed of subsets belonging to w_{cl} . We denote by $w_{\text{cl}}^{P_i} (\in w_{\text{cl}})$ the collection of subsets that contribute to the edge between P_i and P_{i+1} . See Fig. 2.2(b) for a schematic representation of these structures.

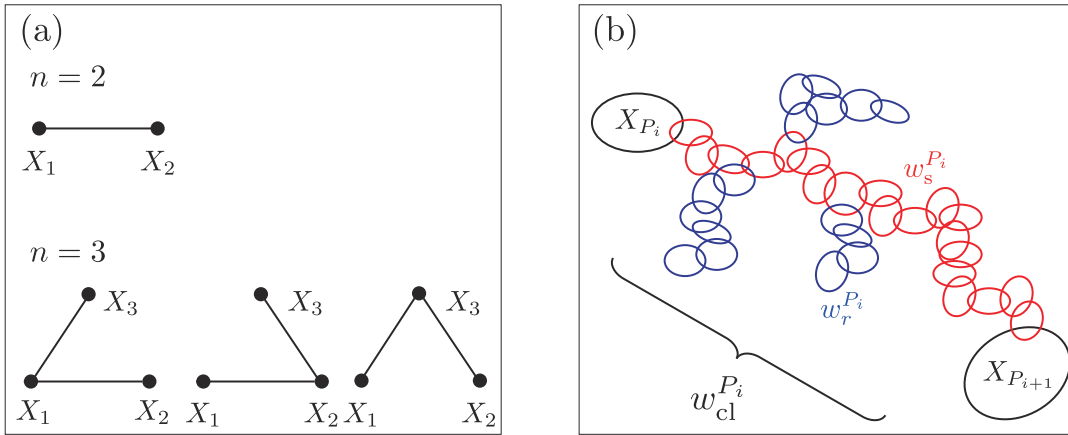


Figure 2.2: (a): Patterns of the Hamilton Path. (b): Subsets of the shortest path $w_s^{P_i}$ and the subsets of the remaining part of the subsets $w_r^{P_i}$. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

We next focus on a single edge $P_i - P_{i+1}$ and its corresponding subset $w_{\text{cl}}^{P_i}$. The set $w_{\text{cl}}^{P_i}$ is further decomposed into two disjoint subsets,

$$w_{\text{cl}}^{P_i} = w_s^{P_i} \oplus w_r^{P_i}, \quad (2.4.42)$$

where $w_s^{P_i}$ represents the subset corresponding to the shortest connecting path, while $w_r^{P_i}$ contains the remaining subsets that provide longer or redundant connections. We denote the lengths associated with $w_s^{P_i}$ and $w_r^{P_i}$ by ℓ_s and ℓ_r , respectively. We now analyze the contributions from these subsets. To this end, we employ the following lemma.

Lemma 4. *Assuming that $\alpha > D$, the following inequalities hold:*

$$\sum_{m \in \Lambda} \frac{1}{d_{i,m}^\alpha} \frac{1}{d_{m,j}^\alpha} \leq \frac{u}{d_{i,j}^\alpha}, \quad (2.4.43)$$

$$\sum_{m \in \Lambda} J_{i,m} J_{m,j} \leq \frac{g^2 u}{d_{i,j}^\alpha}, \quad (2.4.44)$$

where $u = 2^\alpha \sum_m d_{i,m}^{-\alpha} (< \infty)$. The iterative application of inequality (2.4.44) leads to

$$[\mathbf{J}^\ell]_{i,k} \leq \frac{g^\ell u^{\ell-1}}{(d_{i,k})^\alpha}. \quad (2.4.45)$$

Proof of Lemma 4. Proving (2.4.43) is enough.

$$\begin{aligned} \sum_k \frac{1}{d_{i,k}^\alpha} \frac{1}{d_{k,j}^\alpha} &= \frac{1}{d_{i,j}^\alpha} \sum_k \frac{d_{i,j}^\alpha}{d_{i,k}^\alpha d_{k,j}^\alpha} \leq \frac{1}{d_{i,j}^\alpha} \sum_k \frac{(d_{i,k} + d_{k,j})^\alpha}{d_{i,k}^\alpha d_{k,j}^\alpha} \\ &\leq 2^{\alpha-1} \frac{1}{d_{i,j}^\alpha} \sum_k \frac{d_{i,k}^\alpha + d_{k,j}^\alpha}{d_{i,k}^\alpha d_{k,j}^\alpha} = 2^{\alpha-1} \frac{1}{d_{i,j}^\alpha} \sum_k \left(\frac{1}{d_{i,k}^\alpha} + \frac{1}{d_{k,j}^\alpha} \right) = \frac{u}{d_{i,j}^\alpha} \end{aligned} \quad (2.4.46)$$

Here, we use $\{(x+y)/2\}^\alpha \leq (x^\alpha + y^\alpha)/2$, following from the convexity of $f(x) = x^\alpha$ and the convexity inequality. Note that we define $d_{i,i} = 1$ throughout this thesis. $\square$

We first consider the contribution from the shortest path, i.e., $w_s^{P_i}$ is computed as follows.

$$\sum_{\substack{\{Z_1, \dots, Z_{\ell_s}\} = w_s^{P_i} \\ \in w_{\text{cl}}^{P_i} \in w_{\text{cl}} \in \mathcal{G}_{\text{cl}}}} \|h_{Z_1}\| \cdots \|h_{Z_{\ell_s}}\| \quad (2.4.47)$$

$$\begin{aligned} &\leq \sum_{i_1 \in X_{P_i}} \sum_{i_2 \in \Lambda} \sum_{Z_1 \ni \{i_1, i_2\}} \sum_{i_3 \in \Lambda} \sum_{Z_2 \ni \{i_2, i_3\}} \cdots \sum_{i_{\ell_s} \in \Lambda} \sum_{i_{\ell_s+1} \in X_{P_{i+1}}} \sum_{Z_{\ell_s} \ni \{i_{\ell_s}, i_{\ell_s+1}\}} \|h_{Z_1}\| \cdots \|h_{Z_{\ell_s}}\| \\ &= \sum_{i_1 \in X_{P_i}} \sum_{i_2 \in \Lambda} \cdots \sum_{i_{\ell_s} \in \Lambda} \sum_{i_{\ell_s+1} \in X_{P_{i+1}}} J_{i_1, i_2} \cdots J_{i_{\ell_s}, i_{\ell_s+1}} \\ &\leq u^{-1} (gu)^{\ell_s} \frac{|X_{P_i}| |X_{P_{i+1}}|}{(d_{X_{P_i}, X_{P_{i+1}}})^\alpha}. \end{aligned} \quad (2.4.48)$$

We next consider the contribution from the remaining part of the subsets, i.e., $w_r^{P_i}$. Let us define the set $L := X_{P_i} \cup X_{P_{i+1}} \cup Z_1 \cup \cdots \cup Z_{\ell_s}$, where $Z_1, \cdots, Z_{\ell_s}$ are the subsets in $w_s^{P_i}$. To consider the summation over the remaining part of the subsets, we compute the following summation of the tuples

$$\sum_{\substack{\{Z'_1, \dots, Z'_{\ell_r}\} = w_r^{P_i} \\ \in w_{\text{cl}}^{P_i} \in w_{\text{cl}} \in \mathcal{G}_{\text{cl}}}} \|h_{Z'_1}\| \cdots \|h_{Z'_{\ell_r}}\| = \frac{1}{\ell_r!} \sum_{(Z'_1, \dots, Z'_{\ell_r}) : w_r^{P_i}} \|h_{Z'_1}\| \cdots \|h_{Z'_{\ell_r}}\|. \quad (2.4.49)$$

Here, $(Z'_1, \dots, Z'_{\ell_r})$ is the summation over possible tuples. Below, for the notational simplicity, we use the notation ξ_i instead of Z'_i . We define the set $w_L := \{L, \xi_1, \dots, \xi_{\ell_r}\}$. In addition, following Ref. [211], we introduce the layered structure:

$$w_L = L \oplus \xi^{(1)} \oplus \xi^{(2)} \oplus \cdots \oplus \xi^{(v)} \quad 1 \leq v \leq \ell_r, \quad (2.4.50)$$

where $\xi^{(j)}$ has an overlap with the set $\xi^{(j-1)}$ ($\xi^{(0)} := L$). $\xi^{(j)}$ is a set achievable from the set L with the j th step. This layered strature is depicted in Fig.2.3. Note that $\xi^{(j)}$ is a set

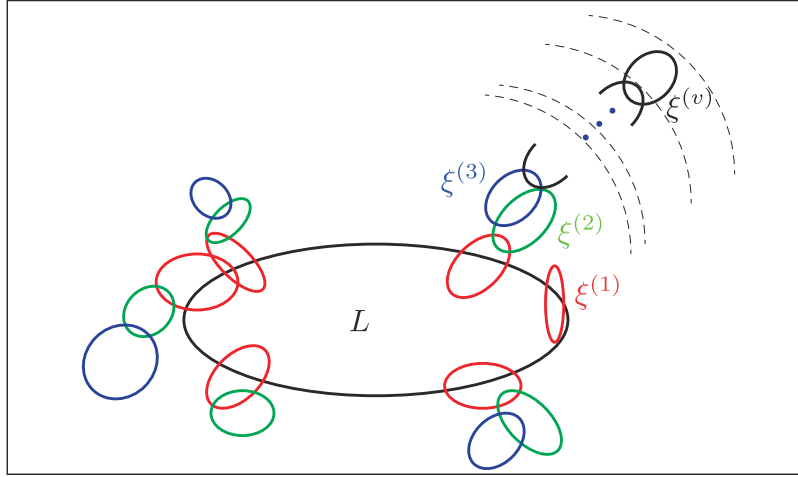


Figure 2.3: Schematic picture of the layered structure. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

consisting of multiple sets. We denote $|\xi^{(j)}|$ by n_j (≥ 1). Then, (2.4.49) is written as follows

$$\begin{aligned} & \frac{1}{\ell_r!} \sum_{(Z'_1, \dots, Z'_{\ell_r}) : w_r^{P_i}} \|h_{Z'_1}\| \cdots \|h_{Z'_{\ell_r}}\| \\ &= \frac{1}{\ell_r!} \sum_{v=1}^{\ell_r} \sum_{\substack{n_1 + \dots + n_v = \ell_r \\ n_j \geq 1}} \frac{\ell_r!}{n_1! \cdots n_v!} \\ & \times \left(\sum_{\xi_i^{(1)} | L} \|h_{\xi_i^{(1)}}\| \right)^{n_1} \left(\sum_{\xi_i^{(2)} | \{L, \xi^{(1)}\}} \|h_{\xi_i^{(2)}}\| \right)^{n_2} \cdots \left(\sum_{\xi_i^{(v)} | \{L, \xi^{(1)}, \dots, \xi^{(v-1)}\}} \|h_{\xi_i^{(v)}}\| \right)^{n_v}, \end{aligned} \quad (2.4.51)$$

where $\xi_i^{(j)}|\{L, \xi^{(1)}, \dots, \xi^{(j-1)}\}$ implies taking a summation over $\xi_i^{(j)}$ with a fixed set $\{L, \xi^{(1)}, \dots, \xi^{(j-1)}\}$. We now compute the most right contribution:

$$\left(\sum_{\xi_i^{(v)}|\{L, \xi^{(1)}, \dots, \xi^{(v-1)}\}} \|h_{\xi_i^{(v)}}\| \right)^{n_v} \leq \left(\sum_{i \in \xi^{(v-1)}} \sum_{Z \ni i} \|h_Z\| \right)^{n_v} \leq (n_{v-1} k g u)^{n_v}. \quad (2.4.52)$$

Iterating the similar computation down to the integration over $\xi_i^{(1)}$, we obtain the following bound

$$\frac{1}{\ell_r!} \sum_{(Z'_1, \dots, Z'_{\ell_r}): w_r^{P_i}} \|h_{Z'_1}\| \cdots \|h_{Z'_{\ell_r}}\| \leq \sum_{v=1}^{\ell_r} \sum_{\substack{n_1 + \dots + n_v = \ell_r \\ n_j \geq 1}} \frac{1}{n_1! \cdots n_v!} (|L| g u)^{n_1} (k n_1 g u)^{n_2} \cdots (k n_{v-1} g u)^{n_v}. \quad (2.4.53)$$

We here use the relations $n_j! \geq (n_j/e)^{n_j}$ and $(1+x/y)^y \leq e^x$ for $x > 0$ and $y > 0$ to get

$$\begin{aligned} & \frac{1}{\ell_r!} \sum_{(Z'_1, \dots, Z'_{\ell_r}): w_r^{P_i}} \|h_{Z'_1}\| \cdots \|h_{Z'_{\ell_r}}\| \\ & \leq \sum_{v=1}^{\ell_r} \sum_{\substack{n_1 + \dots + n_v = \ell_r \\ n_j \geq 1}} e^{n_1} (|L| g u / n_1)^{n_1} e^{n_2} (k n_1 g u / n_2)^{n_2} \cdots e^{n_v} (k n_{v-1} g u / n_v)^{n_v} \\ & \leq (g k u e)^{\ell_r} \sum_{v=1}^{\ell_r} \sum_{\substack{n_1 + \dots + n_v = \ell_r \\ n_j \geq 1}} (1 + (|L|/k)/n_1)^{n_1} (1 + n_1/n_2)^{n_2} \cdots (1 + n_{v-1}/n_v)^{n_v} \\ & \leq (g k u e^2)^{\ell_r} e^{|L|/k} \sum_{v=1}^{\ell_r} \sum_{\substack{n_1 + \dots + n_v = \ell_r \\ n_j \geq 1}} \\ & = 2^{-1} (2 g k u e^2)^{\ell_r} e^{|L|/k} \\ & \leq 2^{-1} (2 g k u e^2)^{\ell_r} e^{(|X_{P_i}| + |X_{P_{i+1}}|)/k + \ell_s}, \end{aligned} \quad (2.4.55)$$

where we use the relation $|L| < |X_{P_i}| + |X_{P_{i+1}}| + \ell_s k$.

We put all contributions together to get the final expression of the bound for $\|\rho^{\text{con}}\|_1$:

$$\begin{aligned} \|\rho^{\text{con}}\|_1 & \leq \sum_P \sum_{s=0}^{\infty} (n_\beta e^{n\beta g u k})^s \left[\prod_{i=1}^{n-1} \sum_{s_i=0}^{\infty} \delta(\sum_{i=1}^{n-1} s_i, s) \right] \prod_{i=1}^{n-1} \sum_{\ell_{s,i}=0}^{s_i} \sum_{\ell_{r,i}=0}^{s_i} \delta(\ell_{s,i} + \ell_{r,i}, s_i) \\ & \quad \times 2^{-1} u^{-1} (g u)^{\ell_{s,i}} \frac{|X_{P_i}| |X_{P_{i+1}}|}{(d_{X_{P_i}, X_{P_{i+1}}})^\alpha} (2 g u e^2 k)^{\ell_{r,i}} e^{\ell_{s,i}} e^{(|X_{P_i}| + |X_{P_{i+1}}|)/k} \\ & \leq \sum_P \prod_{i=1}^{n-1} 2^{-1} u^{-1} \sum_{s_i=0}^{\infty} (2 n \beta g u e^2 k e^{n\beta g u k})^{s_i} \sum_{\ell_{s,i}} (1/2 k e)^{\ell_{s,i}} \frac{|X_{P_i}| |X_{P_{i+1}}|}{(d_{X_{P_i}, X_{P_{i+1}}})^\alpha} e^{(|X_{P_i}| + |X_{P_{i+1}}|)/k} \\ & \leq \sum_P \prod_{i=1}^{n-1} u^{-1} \frac{|X_{P_i}| |X_{P_{i+1}}|}{(d_{X_{P_i}, X_{P_{i+1}}})^\alpha} e^{(|X_{P_i}| + |X_{P_{i+1}}|)/k} \frac{1}{1 - 2 n \beta g u e^2 k e^{n\beta g u k}}, \end{aligned} \quad (2.4.56)$$

where $\delta(x, y)$ in the first line is the Kronecker's delta function. We have used the convergence condition $\beta < \beta_c$. This completes the proof of Theorem 1.

□

Chapter 3

Fluctuating Hydrodynamics for the Long-range Interacting Systems

3.1 Fluctuating hydrodynamics for many-body systems

Hydrodynamics provides a universal theoretical framework for describing the flow of locally conserved quantities [213]. In physics, while the dynamics of individual particles in a microscopic system can be described with high precision by Newtonian or quantum mechanics, the collective behavior of physical quantities often necessitates the use of effective equations. The Navier–Stokes equations, a paradigmatic example, have successfully described a wide variety of macroscopic fluid phenomena, ranging from the streamlined flow around aircraft wings to fully developed turbulence. In recent years, alongside advances in numerical simulations of complex macroscopic flows [214], the concepts of hydrodynamics have been extended beyond their classical domain to systems where textbook hydrodynamics is not directly applicable, such as nano-fluids [215] and cold atomic gases [216–220], and have developed into a unified framework for understanding transport phenomena in solids. For example, direct evidence of viscous electron fluids has been reported [221–225], including electron flow resembling Poiseuille flow. Theoretically, substantial progress has been made in one-dimensional systems, where hydrodynamics for integrable models has been systematically derived [216, 218–220, 226].

Despite such broad developments, macroscopic hydrodynamic equations are inherently deterministic and ignore fluctuations rooted in the microscopic structure of fluids. However, these fluctuations play a crucial role in fluid phenomena at the mesoscopic scale, situated between microscopic and macroscopic regimes. Fluctuating hydrodynamics extends the traditional description by incorporating thermal fluctuations and provides a framework that captures transport phenomena at this mesoscopic level with greater fidelity. This theory has played an essential role in numerous applications [227–238]. Notable examples include predictions of long-range correlations in nonequilibrium steady states [239–242] and analy-

ses of dissipation in turbulent flows [243–246]. In low-dimensional fluids in particular, it is well known that long-time tails in correlation functions lead to divergences in macroscopic transport coefficients such as viscosity [59, 60, 175, 247–249].

As mentioned in Sec. 1.3.2, divergent macroscopic transport coefficients also occur in low-dimensional materials [250, 251], where thermal conductivity has been experimentally observed to diverge in systems such as carbon nanotubes and graphene. Fluctuating hydrodynamics has emerged as a powerful tool capable of making quantitative predictions for such anomalous heat transport [249, 252, 253]. In systems with strong covalent bonding, such as carbon nanotubes, heat is transported via lattice vibrations, and the effective Hamiltonian is described by a nonlinear lattice model. For one-dimensional non-integrable many-body Hamiltonians, the framework of nonlinear fluctuating hydrodynamics has been proposed [249, 253], revealing connections to the Kardar-Parisi-Zhang universality class [194] and attracting considerable attention. The dynamics of anomalous heat transport in a one-dimensional nonlinear lattice system is often described by the following Hamiltonian [249, 254]:

$$H = \sum_n \frac{p_n^2}{2} + V(q_{n+1} - q_n), \quad (3.1.1)$$

where q_n and p_n denote, respectively, the position and momentum of the n -th particle in the one-dimensional lattice, and the potential V is assumed to depend only on the stretch variable $r_n = q_{n+1} - q_n$. This system possesses three locally conserved quantities—stretch, momentum, and energy—so that the long-time, long-wavelength behavior of the nonlinear lattice is expected to be governed by the effective dynamics of the corresponding conserved densities $u_a(x)$, where the index a represents stretch ($a = r$), momentum ($a = p$), and energy ($a = \epsilon$). As discussed in Sec. 1.3.2, when the system is close to thermal equilibrium, the fluctuating hydrodynamic equations for the conserved fields $u_a(x)$ are effectively described by

$$\partial_t u_a(x, t) = -\nabla_x \left[\mathcal{J}_a^{\text{eq}}(x, t) + \nabla_x \sum_{a'} D_{a,a'} u_{a'}(x, t) + \xi_a(x, t) \right], \quad (3.1.2)$$

where $\mathcal{J}_a^{\text{eq}}$ denotes the local equilibrium current, determined from the local equilibrium distribution parameterized by (u_r, u_p, u_ϵ) . The terms $D_{a,a'}$ and ξ_a represent, respectively, the dissipative contribution and the stochastic noise, both of which are introduced so as to satisfy the fluctuation–dissipation relation required by equilibrium thermodynamics. Analyses of anomalous heat transport based on this set of equations proceed as outlined in Sec. 1.3.2.

For these anomalous transport phenomena as well, fluctuating hydrodynamics enables quantitative predictions of dynamical behavior, once an appropriate form of the equations and parameter values are specified.

Moreover, by employing the techniques of the dynamic renormalization group [248, 255], fluctuating hydrodynamics allows one to compute the nonlinear interactions among fluctuations over a wide range of time and length scales in the mesoscopic regime. This methodology

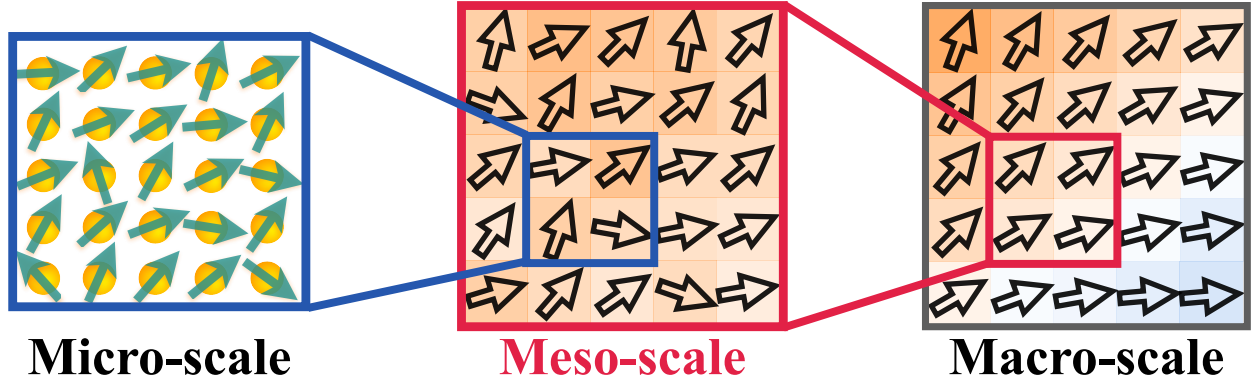


Figure 3.1: Schematics of fluid dynamics across scales: micro-scale (molecular), meso-scale (with fluctuations), and macro-scale (standard hydrodynamic equation). At the mesoscopic scale, fluctuations arising from the underlying atomic motion play a dominant role, necessitating the use of fluctuating hydrodynamics.

has significantly contributed to the analysis of dynamic critical phenomena and to the establishment of the divergence of transport coefficients in low-dimensional fluids. Incorporation of thermal noise introduces stochastic terms into hydrodynamic equations, modifying their canonical form. The transport coefficients in the hydrodynamic equations represent dissipative processes responsible for entropy production, and in the mesoscopic regime, they are accompanied by noise terms that obey the fluctuation–dissipation theorem. Once explicit noise terms are introduced, the notion of bare transport coefficients becomes essential.

More concretely, in the case of low-dimensional fluids, the dynamics are described by the fluctuating Navier–Stokes equation of the form

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla P = D \nabla^2 \mathbf{v} + \boldsymbol{\xi}(\mathbf{x}, t). \quad (3.1.3)$$

This equation is obtained by linearizing the macroscopic Navier–Stokes equation around thermal equilibrium and subsequently writing down the time evolution of the fluctuation fields. The second and third terms on the left-hand side represent the reversible Euler contributions. In contrast, the right-hand side consists of irreversible terms: the first term corresponds to dissipation, while the second term represents stochastic noise. The coefficient D is often referred to as the bare transport coefficient [254, 256–260]. In near equilibrium, the dissipation coefficient D and the noise field ξ satisfy the fluctuation–dissipation relation.

The macroscopic transport coefficients used in conventional hydrodynamics and the bare coefficients describe distinct scale regimes of transport. Whereas macroscopic coefficients characterize deterministic transport after fluctuations have been averaged out (Fig.3.1, right), the bare coefficients determine the time scale governing the crossover from particle-based descriptions (Fig.3.1, left) to hydrodynamic descriptions (Fig.3.1, center), and thus constitute more fundamental quantities that explicitly account for fluctuations. Importantly, bare transport coefficients do not, in general, coincide with their macroscopic counterparts (although,

in purely diffusive systems without equilibrium currents, they are expected to agree). To investigate this concretely, studies in the 1970s employed the dynamical renormalization group [255] to start from fluctuating hydrodynamic equations, integrate out small-scale fluctuations, and compute macroscopic transport coefficients. These analyses demonstrated, for example, that the transport coefficients of two-dimensional fluids diverge [248]. In this way, a methodology connecting mesoscopic fluctuating hydrodynamics to macroscopic hydrodynamics was established.

While the relationship between fluctuating and macroscopic hydrodynamics has been explored through methods such as dynamical renormalization group, a microscopic derivation of fluctuating hydrodynamics itself remained absent until recently. Fluctuating hydrodynamic equations incorporate noise terms satisfying the fluctuation–dissipation relation, and thus describe stochastic processes. There are two major approaches to deriving stochastic processes from microscopic Hamiltonian dynamics: (i) the system-size expansion [261, 262], in which a Langevin equation is derived via a singular perturbation of the Boltzmann–Lorentz equation obtained from the BBGKY (Born–Bogoliubov–Green–Kirkwood–Yvon) hierarchy; and (ii) the projection-operator method [254, 256–260] pioneered by Zwanzig [256]. Because the system-size expansion relies on the Boltzmann equation, it is justified only in the dilute limit and is not suitable for describing general fluid flows. The projection-operator method, on the other hand, assumes a separation of scales and projects dynamics onto the subspace of slowly varying variables, treating the remaining fast components as fluctuations. Although this method introduces an abstract projection and relies on a scale-separation assumption, it can be applied beyond the dilute regime. In hydrodynamic systems, local conserved quantities serve as slow variables. One then projects onto these slow variables by taking an ensemble over microscopic states that share the same values of the local conserved quantities—that is, by taking local microcanonical averages, which effectively project onto the space of slow variables. This constitutes the projection-operator method introduced by Zwanzig [256]. Building upon this framework, a method for deriving fluctuating hydrodynamics from microscopic lattice models has been constructed [254]. Using this method, fluctuating hydrodynamics was derived for one-dimensional nonlinear lattice models, allowing the bare transport coefficients to be computed from microscopic descriptions and demonstrating that the transport coefficients remain finite. However, this methodology has been formulated only for systems with short-range interactions.

In this chapter, our aim is to apply this approach to dynamical systems with long-range interactions and derive an effective fluctuating hydrodynamic equation for the long-range interacting systems.

3.2 Microscopic derivation of fluctuating hydrodynamics in the long-range interacting systems

Fluctuating hydrodynamics provides a systematic theoretical framework for describing the mesoscopic evolution of conserved quantities, in which stochastic fluctuations originating from microscopic degrees of freedom play an essential role, as seen in the previous section. A systematic derivation of such mesoscopic stochastic equations is most commonly achieved through Zwanzig's projection-operator formalism [256, 257]. This method provides a principled way of separating the slow dynamics associated with conserved quantities from the fast microscopic fluctuations and yields a closed equation of motion for the hydrodynamic fields. Within this formalism, the effective equations take the form of generalized Langevin dynamics, in which stochastic noise terms encode the residual effects of the eliminated microscopic degrees of freedom. Such equations constitute a bridge between microscopic Hamiltonian dynamics and coarse-grained stochastic descriptions.

In this thesis, we apply this framework to investigate the effective dynamics of a class of spin systems characterized by the conservation of energy (and, where applicable, spin). Our aim is to elucidate how the interplay between conservation laws, nonlinearity, and microscopic fluctuations gives rise to the emergent transport behavior observed at mesoscopic scales.

3.2.1 Setup

In the following, we derive the fluctuating hydrodynamics starting from a microscopic nonlinear lattice model [254]. As the microscopic foundation, we consider an N -spin Hamiltonian system defined on the phase space

$$\Gamma := (S_1^x, S_1^y, S_1^z, \dots, S_N^x, S_N^y, S_N^z), \quad (3.2.1)$$

The continuous variable S_n^σ denotes the σ component of the spin at site n and satisfies $\sum_\sigma (S_n^\sigma)^2 = 1$. The Hamiltonian is given by

$$\hat{H} = \sum_{n=1}^N \hat{h}_n, \quad (3.2.2)$$

where $\hat{h}_n$ is the local Hamiltonian at the site n . We impose periodic boundary conditions as $n + N \equiv n$.

We define $\hat{c}_{a,n}$ as the local conserved quantities at the site n . a is the index of local conserved quantities. For example, the long-range interacting transverse Ising model is believed to have only one local conserved quantity—energy $\hat{h}_n$, and the long-range interacting XXZ model is believed to have two local conserved quantities—energy $\hat{h}_n$ and z -spin S_n^z .

Since the total quantities $\sum_n \hat{c}_{a,n}$ are conserved, the local conserved quantities $\hat{c}_{a,n}$ satisfy the discrete continuity equation

$$\partial_t \hat{c}_{a,n}(t) = \left\{ \hat{c}_{a,n}(t), \hat{H} \right\} = -(\hat{j}_{a,n+1}(t) - \hat{j}_{a,n}(t)), \quad (3.2.3)$$

where $\{\cdot, \cdot\}$ is the Poisson bracket in the spin dynamics,

$$\{A, B\} = \sum_i \sum_{a,b,c} \varepsilon_{abc} \frac{\partial A}{\partial S_i^a} \frac{\partial B}{\partial S_i^b} S_i^c \quad (3.2.4)$$

with the Levi-Civita symbol ε_{abc} .

Let $\hat{\rho}_t(\Gamma)$ denote the probability distribution on phase space at time t generated by the Hamiltonian dynamics. The time evolution of $\hat{\rho}_t$ is governed by the Liouville equation,

$$\partial_t \hat{\rho}_t = \{\hat{H}, \hat{\rho}_t\} =: \mathcal{L} \hat{\rho}_t, \quad (3.2.5)$$

where we symbolically introduce the Liouville operator $\mathcal{L}$. Formally, the solution of this equation is expressed as

$$\hat{\rho}_t = e^{\mathcal{L}t} \hat{\rho}_0. \quad (3.2.6)$$

Likewise, the time evolution of a local observable $\hat{c}_{a,n}$ is given by

$$\hat{c}_{a,n}(t) = e^{\mathcal{L}^\dagger t} \hat{c}_{a,n}(0), \quad (3.2.7)$$

where $\mathcal{L}^\dagger$ is the adjoint of the Liouvillian operator.

3.2.2 Coarse-graining

Two essential techniques play a central role in the derivation of fluctuating hydrodynamics: *coarse-graining* and *projection*. Through coarse-graining, one specifies the spatiotemporal scales on which the hydrodynamic description becomes valid. Subsequently, by projecting the microscopic dynamics onto the space spanned by the coarse-grained conserved quantities, the fluctuating hydrodynamic equations can be systematically derived. In this subsection, we formulate the first step of this program—the coarse-graining procedure.

At the microscopic level, we consider ℓ particles labeled by the coordinate n . By coarse-graining over these ℓ sites, we introduce a new mesoscopic coordinate x , which labels a “fluid element” at the mesoscale. The corresponding coarse-grained conserved quantities are defined as

$$\hat{u}_{a,x} := \frac{1}{\ell} \sum_{n=(x-1)\ell+1}^{x\ell} \hat{c}_{a,n}, \quad (3.2.8)$$

Each coarse-grained quantity $\hat{u}_{a,x}$ satisfies a local continuity equation of the Euler type:

$$\partial_t \hat{u}_{a,x} = -\nabla_x \hat{\mathcal{J}}_{a,x}, \quad (3.2.9)$$

where the discrete gradient operator is defined by

$$\nabla_x A_x := \frac{1}{\ell} (A_{x+1} - A_x). \quad (3.2.10)$$

The corresponding coarse-grained currents are introduced as

$$\hat{\mathcal{J}}_{a,x} = \hat{j}_{a,(x-1)\ell+1}. \quad (3.2.11)$$

Fig. 3.2 shows the schematic definition of coarse-grained local conserved quantities and local currents, $\hat{u}_{a,x}$ and $\hat{\mathcal{J}}_{a,x}$. As seen in Fig. 3.2, the coarse-grained currents $\hat{\mathcal{J}}_{a,x}$ are located between the local sites $n = (x-1)\ell$ and $n = (x-1)\ell + 1$ and equivalent to the local current $\hat{j}_{a,(x-1)\ell+1}$. The coarse-grained fields $\hat{u}_{a,x}$ represent mesoscopic (i.e., globally slow) variables, while the associated currents $\hat{\mathcal{J}}_{a,x}$ remain microscopic and localized at the cell boundaries.

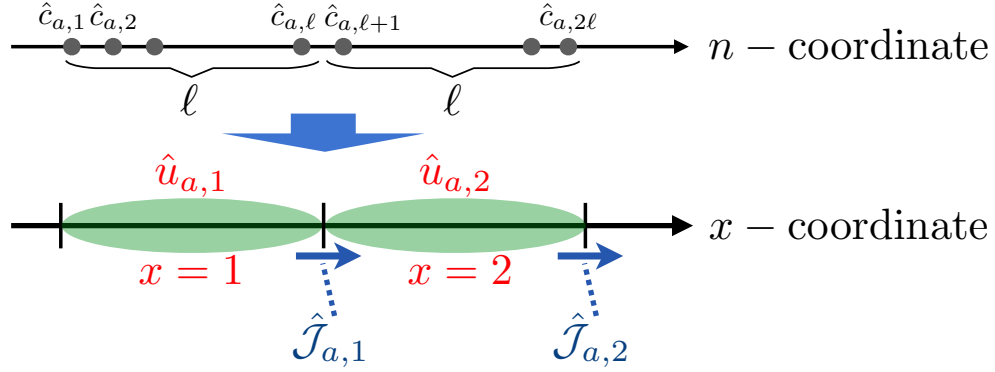


Figure 3.2: Schematic picture of coarse-grained local conserved quantities and local currents, $\hat{u}_{a,x}$ and $\hat{\mathcal{J}}_{a,x}$.

Provided that ℓ is sufficiently large, these local fluxes may be treated as Markovian processes with respect to the coarse-grained variables.

We define the distribution function for the coarse-grained fields $\hat{u}_{a,x}$ as

$$f_t(u) := \int d\Gamma \hat{\rho}_t(\Gamma) \delta(\hat{u}(\Gamma) - u), \quad (3.2.12)$$

where

$$\int d\Gamma \hat{A}(\Gamma) := \prod_{n=1}^N \int_{-\infty}^{\infty} dr_n \int_{-\infty}^{\infty} dp_n \hat{A}(r_1, p_1, \dots, r_N, p_N), \quad (3.2.13)$$

$$\delta(\hat{u}(\Gamma) - u) := \prod_{a=r,p,\epsilon} \prod_{x=1}^{N/\ell} \delta(\hat{u}_{a,x}(\Gamma) - u_{a,x}). \quad (3.2.14)$$

The time evolution of $f_t(u)$ follows from that of $\hat{\rho}_t(\Gamma)$ as

$$\begin{aligned}
\partial_t f_t(u) &= \partial_t \int d\Gamma \hat{\rho}_t(\Gamma) \delta(\hat{u}(\Gamma) - u) \\
&= \int d\Gamma \mathcal{L} e^{\mathcal{L}t} \hat{\rho}_0(\Gamma) \delta(\hat{u}(\Gamma) - u) \\
&= \int d\Gamma \hat{\rho}_t(\Gamma) \mathcal{L}^\dagger \delta(\hat{u}(\Gamma) - u) \\
&= \int d\Gamma \hat{\rho}_t(\Gamma) \sum_{a,x} \left(-\mathcal{L} \hat{u}_{a,x}(\Gamma) \right) \frac{\delta}{\delta \hat{u}_{a,x}} \delta(\hat{u}(\Gamma) - u) \\
&= \int d\Gamma \hat{\rho}_t(\Gamma) \sum_{a,x} \nabla_x \hat{\mathcal{J}}_{a,x} \frac{\delta}{\delta u_{a,x}} \delta(\hat{u}(\Gamma) - u), \tag{3.2.15}
\end{aligned}$$

where we use $\mathcal{L} \hat{u}_{a,x} = \nabla_x \hat{\mathcal{J}}_{a,x}$ to derive the final line. This expression establishes the microscopic foundation for the time evolution of the distribution function of coarse-grained conserved quantities, forming the basis for subsequent projection-operator formulations of fluctuating hydrodynamics.

3.2.3 Projection

In what follows, we proceed to the second step, namely the *projection* procedure. To derive the hydrodynamic equations, we project the dynamics onto the subspace characterized by fixed values of the conserved quantities, thereby obtaining the time-evolution equation for the probability distribution function $f_t(u)$ of the coarse-grained hydrodynamic variables $\hat{u}_{a,x}$.

We introduce the projection operator $\mathcal{P}$ as

$$(\mathcal{P} \hat{A})[\Gamma] := \int d\Gamma' \hat{A}(\Gamma') \delta(\hat{u}(\Gamma') - \hat{u}(\Gamma)) / \Omega(\hat{u}(\Gamma)), \tag{3.2.16}$$

$$\Omega(\hat{u}(\Gamma)) := \int d\Gamma' \delta(\hat{u}(\Gamma') - \hat{u}(\Gamma)). \tag{3.2.17}$$

This operator $\mathcal{P}$ projects any microscopic observable $\hat{A}$ defined over the full phase space onto a subspace spanned by configurations sharing the same coarse-grained conserved quantities. Indeed, if $\hat{u}_{a,x}(\Gamma') = \hat{u}_{a,x}(\Gamma'')$ holds, then $(\mathcal{P} \hat{A})[\Gamma'] = (\mathcal{P} \hat{A})[\Gamma'']$, showing that $\mathcal{P}$ identifies all microstates corresponding to the same values of conserved quantities.

With this operator, we decompose the full distribution as

$$\hat{\rho}_t = \mathcal{P} \hat{\rho}_t + \mathcal{Q} \hat{\rho}_t, \tag{3.2.18}$$

where $\mathcal{Q} := 1 - \mathcal{P}$. Physically, the $\mathcal{P}$ subspace corresponds to the slow hydrodynamic modes, while the $\mathcal{Q}$ subspace captures the remaining fast microscopic degrees of freedom. The $\mathcal{Q}$ space thus encodes dissipative effects arising from fluctuations orthogonal to the hydrodynamic manifold. Fig. 3.3 demonstrates the schematic decomposition of $\mathcal{P}$ subspace and $\mathcal{Q}$ subspace.

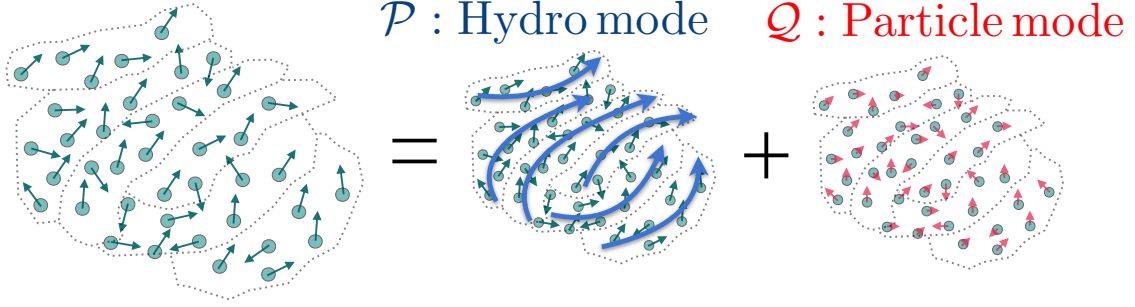


Figure 3.3: Schematic picture of the decomposition of $\mathcal{P}$ subspace and $\mathcal{Q}$ subspace. The $\mathcal{P}$ subspace corresponds to the slow hydrodynamic modes that describe the dynamics of coarse-grained fluid elements $u_{a,x}$. In contrast, the $\mathcal{Q}$ subspace represents the remaining microscopic degrees of freedom after the hydrodynamic modes have been projected out, and is therefore associated with particle modes. The $\mathcal{Q}$ sector encodes the fluctuation and dissipation originating from microscopic degrees of freedom.

Starting from the Liouville equation $\partial_t \hat{\rho}_t = \mathcal{L} \hat{\rho}_t$, substituting the above decomposition gives

$$\begin{cases} \partial_t(\mathcal{P} \hat{\rho}_t) = \mathcal{P} \mathcal{L}(\mathcal{P} \hat{\rho}_t + \mathcal{Q} \hat{\rho}_t), \\ \partial_t(\mathcal{Q} \hat{\rho}_t) = \mathcal{Q} \mathcal{L}(\mathcal{P} \hat{\rho}_t + \mathcal{Q} \hat{\rho}_t). \end{cases} \quad (3.2.19)$$

These two coupled equations can be solved formally for $\mathcal{Q} \hat{\rho}_t$ as

$$\mathcal{Q} \hat{\rho}_t = \int_{-\infty}^t ds e^{(t-s)\mathcal{Q}\mathcal{L}} \mathcal{Q} \mathcal{L} \mathcal{P} \hat{\rho}_s. \quad (3.2.20)$$

Explicit evaluation of this integral yields

$$\mathcal{Q} \hat{\rho}_t = \int_{-\infty}^t ds e^{(t-s)\mathcal{Q}\mathcal{L}} \mathcal{Q} \int \mathcal{D}u' \sum_{a',x'} \nabla_{x'} \hat{\mathcal{J}}_{a',x'} \delta(\hat{u}(\Gamma) - u') \frac{\delta}{\delta u'_{a',x'}} \left(\frac{f_s(u')}{\Omega(u')} \right), \quad (3.2.21)$$

where we write $\int \mathcal{D}u' := \prod_{a,x} \int du'_{a,x}$. This follows from

$$\begin{aligned} \mathcal{Q} \mathcal{L} \mathcal{P} \hat{\rho}_s &= \mathcal{L} \int d\Gamma' \hat{\rho}_s(\Gamma') \delta(\hat{u}(\Gamma') - \hat{u}(\Gamma)) / \Omega(\hat{u}(\Gamma)) \\ &= \int d\Gamma' \hat{\rho}_s(\Gamma') \mathcal{L} \prod_{a',x'} \delta(\hat{u}_{a',x'}(\Gamma') - \hat{u}_{a',x'}(\Gamma)) / \Omega(\hat{u}(\Gamma)) \\ &= \int d\Gamma' \hat{\rho}_s(\Gamma') \int \mathcal{D}u' \delta(\hat{u}_{a',x'}(\Gamma') - u'_{a',x'}) \mathcal{L} \delta(\hat{u}_{a',x'}(\Gamma) - u'_{a',x'}) / \Omega(u') \\ &= \int \mathcal{D}u' \int d\Gamma' \hat{\rho}_s(\Gamma') \sum_{a',x'} (\nabla_{x'} \hat{\mathcal{J}}_{a',x'}) \delta(\hat{u}_{a',x'}(\Gamma') - u'_{a',x'}) \cdot -\frac{\delta}{\delta u'_{a',x'}} \delta(\hat{u}(\Gamma) - u') / \Omega(u') \\ &= \int \mathcal{D}u' \sum_{a',x'} (\nabla_{x'} \hat{\mathcal{J}}_{a',x'}) \delta(\hat{u}(\Gamma) - u') \frac{\delta}{\delta u'_{a',x'}} \left\{ \int d\Gamma' \hat{\rho}_s(\Gamma') \delta(\hat{u}(\Gamma') - u') / \Omega(u') \right\}. \end{aligned}$$

Using these relations, we can now express the time evolution of $f_t(u)$ as

$$\partial_t f_t(u) = \int d\Gamma (\mathcal{P} + \mathcal{Q}) \hat{\rho}_t(\Gamma) \sum_{a,x} \nabla_x \hat{\mathcal{J}}_{a,x} \frac{\delta}{\delta u_{a,x}} \delta(\hat{u}(\Gamma) - u). \quad (3.2.22)$$

Substituting the above formal solution gives

$$\begin{aligned} \partial_t f_t(u) = & \sum_{a,x} \frac{\delta}{\delta u_{a,x}} \left(\langle \nabla_x \hat{\mathcal{J}}_{a,x} \rangle_{\text{LM}}^u f_t(u) \right) \\ & + \ell^{-1} \sum_{a,x,a',x'} \frac{\delta}{\delta u_{a,x}} \Omega(u) (\nabla_x \nabla_{x'} K_{ax,a'x'}(u, u'; t-s)) \frac{\delta}{\delta u_{a',x'}} \left(\frac{f_s(u)}{\Omega(u)} \right), \end{aligned} \quad (3.2.23)$$

where we defined

$$\langle \hat{A} \rangle_{\text{LM}}^u := \int d\Gamma \hat{A} \delta(\hat{u}(\Gamma) - u) / \Omega(\hat{u}(\Gamma)), \quad (3.2.24)$$

$$K_{ax,a'x'}(u, u'; t-s) := \ell \int_{-\infty}^t ds \left\langle \left(\mathcal{Q} \hat{\mathcal{J}}_{a,x} \right) \left(e^{(t-s)\mathcal{Q}\mathcal{L}} \delta(u - u') \mathcal{Q} \hat{\mathcal{J}}_{a',x'} \right) \right\rangle_{\text{LM}}^u, \quad (3.2.25)$$

$$\Omega(u) := \int d\Gamma' \delta(\hat{u}(\Gamma') - u). \quad (3.2.26)$$

The second term represents a non-Markovian convolution integral. Here, we approximate $K(u, u'; t-s)$ as in [254], assuming that the term decays rapidly in time, while the conserved quantities do not change rapidly.

$$K_{ax,a'x'}(u, u'; t-s) \simeq \ell \int_{-\infty}^t ds \left\langle \left(\mathcal{Q} \hat{\mathcal{J}}_{a,x} \right) \left(e^{(t-s)\mathcal{Q}\mathcal{L}} \mathcal{Q} \hat{\mathcal{J}}_{a',x'} \right) \right\rangle_{\text{LM}}^u \delta(u - u'). \quad (3.2.27)$$

This approximation is not rigorous, but physically justified by the coarse-graining and projection procedures: the coarse-grained coordinate x represents an average over ℓ microscopic particles, making $\mathcal{P}$ -space dynamics slow, while the $\mathcal{Q}$ -space dynamics—corresponding to fast microscopic fluctuations—decay rapidly in time. Therefore, the correlation of local currents $\hat{\mathcal{J}}_{a,x}$ in $\mathcal{Q}$ -space can be regarded as Markovian on the hydrodynamic time scale.

Under the Markovian approximation, the time-evolution equation reduces to a Fokker–Planck-type equation:

$$\begin{aligned} \partial_t f_t(u) = & \sum_{a,x} \frac{\delta}{\delta u_{a,x}} \left(\langle \nabla_x \hat{\mathcal{J}}_{a,x} \rangle_{\text{LM}}^u f_t(u) \right) \\ & + \ell^{-1} \sum_{a,x,a',x'} \frac{\delta}{\delta u_{a,x}} \Omega(u) (\nabla_x \nabla_{x'} K_{ax,a'x'}(u)) \frac{\delta}{\delta u_{a',x'}} \left(\frac{f_t(u)}{\Omega(u)} \right), \end{aligned} \quad (3.2.28)$$

with

$$K_{ax,a'x'}(u) := \ell \int_0^\infty ds \left\langle \left(\mathcal{Q} \hat{\mathcal{J}}_{a,x} \right) \left(e^{s\mathcal{Q}\mathcal{L}} \mathcal{Q} \hat{\mathcal{J}}_{a',x'} \right) \right\rangle_{\text{LM}}^u. \quad (3.2.29)$$

Finally, by decomposing $K_{ax,a'x'}$ into its symmetric and antisymmetric parts,

$$K_{ax,a'x'}^{(S,A)} := \frac{1}{2}(K_{ax,a'x'} \pm K_{a'x',ax}), \quad (3.2.30)$$

the Fokker–Planck equation can be recast as

$$\begin{aligned} \partial_t f_t(u) = & \sum_{a,x} \frac{\delta}{\delta u_{a,x}} \left[\left\langle \nabla_x \hat{\mathcal{J}}_{a,x} \right\rangle_{\text{LM}}^u f_t(u) + \ell^{-1} \sum_{a',x'} \left(\nabla_x K_{ax,a'x'}^{(A)}(u) \right) \nabla_x (\ln \Omega(u)) f_t(u) \right] \\ & + \ell^{-1} \sum_{a,x,a',x'} \frac{\delta}{\delta u_{a,x}} \Omega(u) \left(\nabla_x \nabla_{x'} K_{ax,a'x'}^{(S)}(u) \right) \frac{\delta}{\delta u_{a',x'}} \left(\frac{f_t(u)}{\Omega(u)} \right). \end{aligned} \quad (3.2.31)$$

This Fokker–Planck equation constitutes the mesoscopic stochastic dynamics of the fluctuating hydrodynamic fields derived from microscopic Hamiltonian dynamics via systematic coarse-graining and projection.

3.2.4 Fluctuating hydrodynamic equation

In correspondence with the Fokker–Planck equation

$$\begin{aligned} \partial_t f_t(u) = & \sum_{a,x} \frac{\delta}{\delta u_{a,x}} \left[\left\langle \nabla_x \hat{\mathcal{J}}_{a,x} \right\rangle_{\text{LM}}^u f_t(u) + \ell^{-1} \sum_{a',x'} \left(\nabla_x K_{ax,a'x'}^{(A)}(u) \right) \nabla_x (\ln \Omega(u)) f_t(u) \right] \\ & + \ell^{-1} \sum_{a,x,a',x'} \frac{\delta}{\delta u_{a,x}} \Omega(u) \left(\nabla_x \nabla_{x'} K_{ax,a'x'}^{(S)}(u) \right) \frac{\delta}{\delta u_{a',x'}} \left(\frac{f_t(u)}{\Omega(u)} \right), \end{aligned}$$

the corresponding Langevin equation can be expressed as

$$\partial_t u_{a,x} = -\nabla_x \left[\left\langle \hat{\mathcal{J}}_{a,x} \right\rangle_{\text{LM}}^u + \sum_{a',x'} K_{ax,a'x'} \nabla_{x'} \frac{\delta}{\delta (\ell u_{a',x'})} (\ln \Omega(u)) + \xi_{a,x}(t) \right], \quad (a = r, p, \epsilon) \quad (3.2.32)$$

where the stochastic term $\xi_{a,x}(t)$ satisfies the fluctuation–dissipation relation

$$\langle\langle \xi_{a,x}(t) \xi_{a',x'}(t') \rangle\rangle = 2\ell^{-1} K_{ax,a'x'}^{(S)} \delta(t - t'), \quad (3.2.33)$$

where $\langle\langle \dots \rangle\rangle$ stands for the noise average.

By interpreting the phase-space measure $\Omega(u)$ as the partition function, $\Omega(u) \sim e^{S/k_B}$ (and hereafter setting $k_B = 1$), we identify

$$\ln \Omega(u) = S, \quad (3.2.34)$$

where S is the thermodynamic entropy, and

$$\frac{\delta \ln \Omega(u)}{\delta (\ell u_{a',x'})} = \frac{\delta S}{\delta (\ell u_{a',x'})} = \lambda_{a',x'}, \quad (3.2.35)$$

where $\lambda_{a,x}$ is the local thermodynamic parameter (e.g., inverse temperature). It is important to note that S represents an extensive variable, whereas λ corresponds to its conjugate intensive variable. Using these relations, we may rewrite the Langevin equation as

$$\begin{aligned}\partial_t u_{a,x} &= -\nabla_x \left[\langle \hat{\mathcal{J}}_{a,x} \rangle_{\text{LM}}^u + \sum_{a',x'} K_{ax,a'x'} \nabla_{x'} \lambda_{a',x'} + \xi_{a,x}(t) \right] \\ &= -\nabla_x \left[\langle \hat{\mathcal{J}}_{a,x} \rangle_{\text{LM}}^u - \sum_{a',x'} K_{ax,a'x'}^{(A)} \Lambda_{a'}^{a''} \nabla_{x'} u_{a',x'} - \sum_{a',x'} K_{ax,a'x'}^{(S)} \Lambda_{a'}^{a''} \nabla_{x'} u_{a',x'} + \xi_{a,x}(t) \right].\end{aligned}\tag{3.2.36}$$

In the second line above, we have defined

$$\Lambda_{a'}^{a''} := - \left(\frac{\partial \lambda_{a'',x}}{\partial u_{a',x}} \right)_{\text{eq}}, \tag{3.2.37}$$

whose inverse matrix,

$$\Lambda^{-1} = - \left(\frac{\partial \lambda}{\partial \mathbf{u}} \right)_{\text{eq}}^{-1} = - \left(\frac{\partial \mathbf{u}}{\partial \lambda} \right)_{\text{eq}}, \tag{3.2.38}$$

corresponds to the susceptibility matrix. We further define

$$D_{ax,a'x'}^{(S,A)} := \sum_{a''} K_{ax,a'x'}^{(S,A)} \Lambda_{a'}^{a''}. \tag{3.2.39}$$

and finally arrive at the following fluctuating hydrodynamics equation for the long-range interacting systems:

$$\partial_t u_{a,x} = -\nabla_x \left[\langle \hat{\mathcal{J}}_{a,x} \rangle_{\text{LM}}^u - \sum_{a',x'} D_{ax,a'x'}^{(A)} \nabla_{x'} u_{a',x'} - \sum_{a',x'} D_{ax,a'x'}^{(S)} \nabla_{x'} u_{a',x} + \xi_{a,x}(t) \right]. \tag{3.2.40}$$

The first two terms, $\langle \hat{\mathcal{J}}_{a,x} \rangle_{\text{LM}}^u - \sum_{a',x'} D_{ax,a'x'}^{(A)} \nabla_{x'} u_{a',x'}$, are antisymmetric under time reversal and correspond to the reversible dynamics, while the third term, $-\sum_{a',x'} D_{ax,a'x'}^{(S)} \nabla_{x'} u_{a',x}$, and the stochastic term $\xi_{a,x}(t)$ represent irreversible dynamics — specifically, the dissipative and fluctuating contributions, respectively, which are related through the fluctuation–dissipation theorem.

For short-range interaction, $K_{ax,a'x'}$ decays sufficiently rapidly as $|x - x'|$ increases, i.e., $K_{ax,a'x'} \sim K_{ax,a'x} \delta(x - x')$, leading to coarse-grained forms

$$K_{a,a'} := \sum_{x'} K_{ax,a'x'}, \quad K_{a,a'}^{(S,A)} := \sum_{x'} K_{ax,a'x'}^{(S,A)} = \sum_{x'} \frac{1}{2} (K_{ax,a'x'} \pm K_{a'x',ax}), \tag{3.2.41}$$

$$\langle \langle \xi_{a,x}(t) \xi_{a',x'}(t') \rangle \rangle := 2\ell^{-1} K_{a,a'}^{(S)} \delta_{a,a'} \delta_{x,x'} \delta(t - t'). \tag{3.2.42}$$

This approximation is justified because each coarse-grained site at position x already contains ℓ microscopic particles, and the kernel $K_{ax,a'x'}$ exhibits Markovian behavior under this coarse-graining. Consequently, for the short-range case, we reduce the fluctuating hydrodynamic equation to the following form:

$$\partial_t u_{a,x}(t) = -\nabla_x \left[\langle \hat{\mathcal{J}}_{a,x} \rangle_{\text{LM}}^u - \sum_{a'} D_{a,a'}^{(A)} \nabla_x u_{a',x} - \sum_{a'} D_{a,a'}^{(S)} \nabla_x u_{a',x} + \xi_{a,x}(t) \right]. \quad (3.2.43)$$

3.2.5 Approximation of the local microcanonical ensemble by the local Gibbs ensemble via the large deviation principle

In general, it is difficult to directly evaluate expectation values with respect to the local microcanonical ensemble. To overcome this difficulty, we employ the large deviation principle to approximate the local microcanonical distribution by a local Gibbs distribution, which allows for practical computation of expectation values.

We first briefly summarize the concept of the large deviation principle [263]. The large deviation principle characterizes the rate at which probabilities decay away from the typical (law-of-large-numbers) value. Mathematically, it is expressed as follows.

Large Deviation Principle. Let A_N be a random variable taking values in a measurable set $\mathcal{S}$, with probability distribution $P(A_N)$. The system satisfies the large deviation principle if the following limit exists:

$$\lim_{N \rightarrow \infty} -\frac{1}{N} \ln P(A_N \in \mathcal{S}) = I, \quad (3.2.44)$$

where $I \geq 0$ is called the *rate function*, which generally takes the form $I(a)$ with $a = A_N/N$. The rate function quantifies how rapidly the probability $P(A_N)$ decays away from its typical value as N increases. Since $I(a)$ contains information about higher-order fluctuations, it captures deviations from Gaussian behavior.

As a simple example, consider tossing N independent fair coins, each showing heads with probability $1/2$. Denoting by $n = A_N$ the number of heads, the probability distribution $P(n)$ behaves asymptotically as

$$P(n) = \binom{N}{n} \left(\frac{1}{2}\right)^N \asymp e^{-NI(n/N)}, \quad (3.2.45)$$

$$I(a) = a \ln a + (1-a) \ln(1-a) + \ln 2, \quad a = n/N. \quad (3.2.46)$$

The rate function $I(a)$ captures how $P(n)$ decays as N increases, and its deviation from a quadratic form quantifies the non-Gaussianity of the fluctuations.

In general, when individual events are weakly correlated (as in the case of independent coin tosses), deviations from the typical value decay exponentially and obey the large deviation

principle. Conversely, if the decay is faster or slower than exponential, a finite rate function does not exist.

We now define the moment generating function associated with local fluctuations. Let $\hat{U}_{a,x} = \ell \hat{u}_{a,x}$, and define

$$Z(\xi) = \int d\Gamma e^{-\sum_{a,x} \xi_{a,x} \hat{U}_{a,x}(\Gamma)}. \quad (3.2.47)$$

The corresponding “partition function” $\Omega(u)$ can then be expressed as the inverse Laplace transform,

$$\Omega(u) = \prod_{a',x'} \int \frac{d\xi_{a',x'}}{2\pi i} Z(\xi) e^{-\sum_{a,x} \xi_{a,x} U_{a,x}}. \quad (3.2.48)$$

When the large deviation principle holds, both $Z(\xi)$ and $\Omega(u)$ obey the asymptotic forms

$$Z(\xi) \sim e^{\ell \mu(\xi)}, \quad (3.2.49)$$

$$\Omega(u) \sim e^{\ell s(u)}, \quad (3.2.50)$$

where $\mu(\xi) = \lim_{\ell \rightarrow \infty} \frac{1}{\ell} \ln Z(\xi)$ is the cumulant generating function, and $-s(u)$ corresponds to the rate function. The two functions are connected via the Legendre–Fenchel transform, known as the Gärtner–Ellis theorem:

$$s(u) = \inf_{\xi} \left[\mu(\xi) - \sum_{a,x} \xi_{a,x} u_{a,x} \right], \quad (3.2.51)$$

$$\mu(\xi) = \sup_u \left[s(u) + \sum_{a,x} \xi_{a,x} u_{a,x} \right]. \quad (3.2.52)$$

To obtain $s(u)$, one solves

$$\frac{\partial \mu(\xi)}{\partial \xi_{a,x}} - u_{a,x} = 0, \quad (3.2.53)$$

and denotes the solution by $\xi_{a,x}^*$. Then,

$$s(u) = \mu(\xi^*) - \sum_{a,x} \xi_{a,x}^* u_{a,x}. \quad (3.2.54)$$

Introducing the notations

$$S(u) := \ell s(u), \quad (3.2.55)$$

$$\lambda_{a,x} := -\xi_{a,x}^*, \quad (3.2.56)$$

we obtain

$$\frac{\partial S(u)}{\partial (\ell u_{a,x})} = \frac{\partial s(u)}{\partial u_{a,x}} = -\xi_{a,x}^* + \sum_{a',x'} \frac{\partial \xi_{a',x'}^*}{\partial u_{a,x}} \left(\frac{\partial \mu}{\partial \xi_{a',x'}^*} - u_{a',x'} \right) = \lambda_{a,x}. \quad (3.2.57)$$

This relation corresponds to the thermodynamic conjugacy between the entropy and the local intensive parameters.

Based on the above discussion, we can introduce an effective *local Gibbs ensemble* characterized by the entropy $S(u)$ obtained from the large deviation principle:

$$\hat{\rho}_{\text{LG}} = \frac{1}{Z_{\text{LG}}} \exp \left\{ \sum_{a,x} \xi_{a,x}^* \hat{U}_{a,x}(\Gamma) \right\} = \frac{1}{Z_{\text{LG}}} \exp \left\{ - \sum_{a,x} \lambda_{a,x} \hat{u}_{a,x}(\Gamma) \right\}. \quad (3.2.58)$$

The local thermodynamic parameters $\lambda_{a,x}$ are determined by

$$\langle \hat{u}_{a,x} \rangle_{\text{LG}} := \int d\Gamma \hat{u}_{a,x} \hat{\rho}_{\text{LG}} = \langle \hat{c}_{a,(x-1)\ell+n} \rangle_{\text{LG}} = u_{a,x}, \quad (3.2.59)$$

where $n = 1, \dots, \ell - 1$.

Thus, the local Gibbs ensemble is constructed on the basis of the large deviation principle. Using this ensemble, we can now derive asymptotic expressions for the local microcanonical averages of the coarse-grained currents, as well as for the corresponding bare transport coefficients in the limit of large ℓ .

For the local microcanonical average of the coarse-grained current, which are microscopic quantities defined at the site level, we have

$$\begin{aligned} \langle \hat{\mathcal{J}}_{a,x} \rangle_{\text{LM}}^u &= \langle \hat{j}_{a,(x-1)\ell+1} \rangle_{\text{LM}}^u = \int d\Gamma \hat{j}_{a,n}(\Gamma) \hat{\rho}_{\text{LM}}(\Gamma) \Big|_{n=(x-1)\ell+1} \\ &= \int d\Gamma \hat{j}_{a,n}(\Gamma) \prod_{a',x'} \delta(\hat{u}_{a',x'} - u_{a',x'}) / \Omega(u) \Big|_{n=(x-1)\ell+1}. \end{aligned} \quad (3.2.60)$$

Since this expression represents an average over ℓ particles, it is reasonable to approximate the local microcanonical ensemble by the local Gibbs ensemble in the large- ℓ limit:

$$\hat{\rho}_{\text{LM}} \sim \hat{\rho}_{\text{LG}}. \quad (3.2.61)$$

This substitution also applies to the coarse-grained stretch currents, allowing us to compute current expectation values explicitly within the local Gibbs framework.

In the preceding discussion, we introduced the replacement of the local microcanonical distribution by the local Gibbs distribution, justified by the large-deviation principle. We now turn to an explicit evaluation of physical observables under the local Gibbs ensemble. In particular, our goal is to compute the local Gibbs expectations of relevant microscopic quantities and to express them, when appropriate, as perturbations around equilibrium expectations.

In a local equilibrium state, the local Gibbs distribution admits a linearized representation around the global equilibrium ensemble. Denoting the local Gibbs distribution by $\hat{\hat{\rho}}_{\text{LG}}$, we expect that

$$\hat{\rho}_{\text{LG}} \sim \hat{\rho}_{\text{eq}} + \delta \hat{\rho}_{\text{LG}}, \quad (3.2.62)$$

where $\hat{\rho}_{\text{eq}}$ is the canonical equilibrium distribution. Accordingly, the local Gibbs expectation of an observable $\hat{A}$ may be expressed as a deviation from its equilibrium value:

$$\langle \hat{A} \rangle_{\text{LG}} = \langle \hat{A} \rangle + \delta \langle \hat{A} \rangle_{\text{LG}}, \quad (3.2.63)$$

where $\langle \dots \rangle$ denotes the equilibrium ensemble.

Although $\hat{\rho}_{\text{LG}}$ is defined in terms of coarse-grained fields $\hat{u}_{a,x}$, it may be rewritten as a distribution over the microscopic conserved quantities $\hat{c}_{a,n}$:

$$\begin{aligned} \hat{\rho}_{\text{LG}}(\Gamma; \lambda) &= \frac{1}{\tilde{Z}_{\text{LG}}} \exp \left[- \sum_{a,x} \lambda_{a,x} \hat{u}_{a,x}(\Gamma) \right] \\ &= \frac{1}{\tilde{Z}_{\text{LG}}} \prod_x \exp \left[- \sum_a \sum_{n=(x-1)\ell+1}^{x\ell} \lambda_{a,n} \hat{c}_{a,n}(\Gamma) \right]. \end{aligned} \quad (3.2.64)$$

Here the local thermodynamic parameters satisfy $\lambda_{a,x} = \ell \lambda_{a,n}$.

To develop a linearized description of $\hat{\rho}_{\text{LG}}(\lambda)$ around equilibrium, we recall that in equilibrium the local thermodynamic parameters reduce to their global values λ_a^{eq} , so that

$$\hat{\rho}_{\text{LG}}(\lambda^{\text{eq}}) = \hat{\rho}_{\text{eq}}. \quad (3.2.65)$$

For example, we consider the energy conservation law, i.e., $\hat{c}_{\epsilon,n} = \hat{h}_n$. For the energy case, the corresponding equilibrium thermodynamic parameter satisfies $\lambda_{\epsilon}^{\text{eq}} = \beta$.

Consequently, the local parameter admits the perturbative decomposition

$$\lambda_{a,n} \sim \lambda_a^{\text{eq}} + \delta \lambda_{a,n}. \quad (3.2.66)$$

The equilibrium expectation of conserved quantities takes the form

$$\langle \hat{c}_{a,n} \rangle_{\text{eq}} = \int d\Gamma \hat{c}_{a,n}(\Gamma) \hat{\rho}_{\text{LG}}(\Gamma; \lambda^{\text{eq}}) = u_a^{\text{eq}}. \quad (3.2.67)$$

Expanding the local Gibbs distribution

$$\hat{\rho}_{\text{LG}}(\lambda) = \hat{\rho}_{\text{LG}}(\lambda^{\text{eq}} + \delta \lambda) = \hat{\rho}_{\text{eq}} + \delta \hat{\rho}_{\text{LG}}, \quad (3.2.68)$$

we obtain the deviation of the local Gibbs expectation:

$$\langle \hat{c}_{a,n} \rangle_{\text{LG}} = u_a^{\text{eq}} + \delta u_{a,x}, \quad (3.2.69)$$

which identifies $\delta \lambda_{a,n}$ with

$$\delta u_{a,x} := u_{a,x} - u_a^{\text{eq}}. \quad (3.2.70)$$

Expanding $\delta \lambda_{a,n}$ as a power series in $\delta u_{a,x}$ yields

$$\delta \lambda_{a,n} = \sum_b \Lambda^a_b \delta u_{b,x} + \frac{1}{2} \sum_{b,b'} \Xi^a_{bb'} \delta u_{b,x} \delta u_{b',x} + \mathcal{O}(u^3), \quad (3.2.71)$$

$$\text{s.t. } (x-1)\ell + 1 \leq n \leq x\ell, \quad (3.2.72)$$

where

$$\begin{aligned}\Lambda^a_b &= \left(\frac{\partial \lambda}{\partial u} \right)_{ab} \Big|_{u=u^{\text{eq}}} = \left(\frac{\partial u}{\partial \lambda} \right)^{-1}_{ab} \Big|_{u=u^{\text{eq}}} = \left(\frac{\partial c^{\text{eq}}}{\partial \beta} \right)^{-1}_{ab} \\ &= \left(\langle \hat{c}_{a,n} \hat{c}_{b,n} \rangle_{\text{eq}} - \langle \hat{c}_{a,n} \rangle_{\text{eq}} \langle \hat{c}_{b,n} \rangle_{\text{eq}} \right)^{-1},\end{aligned}\tag{3.2.73}$$

and

$$\Xi^a_{bb'} = \frac{\partial^2 \lambda_{a,n}}{\partial u_{b,x} \partial u_{b',x}} \Big|_{u=u^{\text{eq}}} = \sum_c \Lambda^c_b \frac{\partial}{\partial \lambda_c} \left(\frac{\partial u}{\partial \lambda} \right)^{-1}_{ab'} \Big|_{u=u^{\text{eq}}} = \sum_c \Lambda^c_b \frac{\partial}{\partial \lambda_c} \left(\frac{\partial c^{\text{eq}}}{\partial \beta} \right)^{-1}_{ab'} \Big|_{u=u^{\text{eq}}}.\tag{3.2.74}$$

Thus, once the equilibrium expectations of the conserved quantities are known, all coefficients appearing in the linearized local Gibbs expansion may be determined explicitly.

In this section, we express the local Gibbs distribution $\hat{\rho}_{\text{LG}}(\lambda) = \hat{\rho}_{\text{LG}}(\lambda^{\text{eq}} + \delta\lambda) = \hat{\rho}_{\text{eq}} + \delta\hat{\rho}_{\text{LG}}(\lambda)$ in terms of the deviations $\delta\lambda_{a,n}$, and subsequently expand it up to second order in the coarse-grained variables $\delta u_{a,x}$.

Introducing the decomposition $\hat{c}_{a,n} = \langle \hat{c}_{a,n} \rangle_{\text{eq}} + \delta\hat{c}_{a,n}$, we obtain

$$\hat{\rho}_{\text{LG}}(\lambda)\tag{3.2.75}$$

$$\begin{aligned}&\approx \hat{\rho}_{\text{eq}} \left(1 - \sum_{a,n} \delta\lambda_{a,n} \delta\hat{c}_{a,n} + \frac{1}{2} \sum_{an,a'n'} \delta\lambda_{a,n} \delta\hat{c}_{a,n} \delta\lambda_{a',n'} \delta\hat{c}_{a',n'} + \mathcal{O}(\delta\lambda^3) \right) \\ &= \hat{\rho}_{\text{eq}} \left\{ 1 - \sum_{a,n} \delta\hat{c}_{a,n} \left(\sum_b \Lambda^a_b \delta u_{b,x} + \sum_{b,b'} \frac{1}{2} \Xi^a_{bb'} \delta u_{b,x} \delta u_{b',x} + \mathcal{O}(\delta u^3) \right) \right. \\ &\quad \left. + \frac{1}{2} \sum_{an,a'n'} \delta\hat{c}_{a,n} \delta\hat{c}_{a',n'} \left(\sum_b \Lambda^a_b \delta u_{b,x} + \mathcal{O}(\delta u^2) \right) \left(\sum_{b'} \Lambda^{a'}_{b'} \delta u_{b',x} + \mathcal{O}(\delta u^2) \right) + \mathcal{O}(\delta u^3) \right\} \\ &= \hat{\rho}_{\text{eq}} \left\{ 1 - \sum_n \sum_{a,b} \delta\hat{c}_{a,n} \Lambda^a_b \delta u_{b,x} \right. \\ &\quad \left. + \frac{1}{2} \left[- \sum_n \sum_{a,bb'} \delta\hat{c}_{a,n} \Xi^a_{bb'} \delta u_{b,x} \delta u_{b',x} + \sum_{nn'} \sum_{aa',bb'} \delta\hat{c}_{a,n} \delta\hat{c}_{a',n'} \Lambda^a_b \Lambda^{a'}_{b'} \delta u_{b,x} \delta u_{b',x} \right] + \mathcal{O}(\delta u^3) \right\}.\end{aligned}\tag{3.2.76}$$

Using this expansion, the expectation value of the coarse-grained current $\hat{\mathcal{J}}_{a,x}$ under the local Gibbs ensemble can be computed up to second order in δu as

$$\begin{aligned}\langle \hat{\mathcal{J}}_{a,x} \rangle_{\text{LG}} &= \text{Tr}(\hat{\mathcal{J}}_{a,x} \hat{\rho}_{\text{LG}}) \\ &\approx \text{Tr}(\hat{\mathcal{J}}_{a,x} \hat{\rho}_{\text{eq}}) - \sum_{a',n} \text{Tr}(\hat{\mathcal{J}}_{a,x} \delta\hat{c}_{a',n} \hat{\rho}_{\text{eq}}) \delta\lambda_{a',n} \\ &\quad + \frac{1}{2} \sum_{a'n,a''n'} \text{Tr}(\hat{\mathcal{J}}_{a,x} \delta\hat{c}_{a',n} \delta\hat{c}_{a'',n'} \hat{\rho}_{\text{eq}}) \delta\lambda_{a',n} \delta\lambda_{a'',n'}\end{aligned}$$

$$\begin{aligned}
& \approx \langle \hat{\mathcal{J}}_{a,x} \rangle_{\text{eq}} - \sum_b \sum_{a',n} \text{Tr}(\hat{\mathcal{J}}_{a,x} \delta \hat{c}_{a',n} \hat{\rho}_{\text{eq}}) \Lambda^{a'}_b \delta u_{b,x} \\
& + \frac{1}{2} \sum_{b,b'} \left[- \sum_{a',n} \text{Tr}(\hat{\mathcal{J}}_{a,x} \delta \hat{c}_{a',n} \hat{\rho}_{\text{eq}}) \Xi^{a'}_{bb'} \right. \\
& \quad \left. + \sum_{a'n,a''n'} \text{Tr}(\hat{\mathcal{J}}_{a,x} \delta \hat{c}_{a',n} \delta \hat{c}_{a'',n'} \hat{\rho}_{\text{eq}}) \Lambda^{a'}_b \Lambda^{a''}_{b'} \right] \delta u_{b,x} \delta u_{b',x} \\
& = \langle \hat{\mathcal{J}}_{a,x} \rangle_{\text{eq}} - \sum_b A^a_b \delta u_{b,x} + \frac{1}{2} \sum_{b,b'} H^a_{bb'} \delta u_{b,x} \delta u_{b',x}.
\end{aligned} \tag{3.2.77}$$

Here the coefficients are defined as

$$A^a_b := \sum_{a',n} \langle \hat{\mathcal{J}}_{a,x} \delta \hat{c}_{a',n} \rangle_{\text{eq}} \Lambda^{a'}_b, \tag{3.2.78}$$

$$H^a_{bb'} := - \sum_{a',n} \langle \hat{\mathcal{J}}_{a,x} \delta \hat{c}_{a',n} \rangle_{\text{eq}} \Xi^{a'}_{bb'} + \sum_{a'n,a''n'} \langle \hat{\mathcal{J}}_{a,x} \delta \hat{c}_{a',n} \delta \hat{c}_{a'',n'} \rangle_{\text{eq}} \Lambda^{a'}_b \Lambda^{a''}_{b'}. \tag{3.2.79}$$

Thus the local-Gibbs ensemble of the current is obtained explicitly as a perturbative expansion around equilibrium. We next prepare for the computation of the bare transport coefficients by evaluating the expectation of the projected current, $\langle \mathcal{P} \hat{\mathcal{J}}_{a,x} \rangle_{\text{LM}}^u$.

Replacing the local microcanonical ensemble with the corresponding local Gibbs ensemble, the projection operator may be written as

$$\begin{aligned}
(\mathcal{P} \hat{\mathcal{J}}_{a,x})(\Gamma) &:= \int d\Gamma' \hat{\mathcal{J}}_{a,x}(\Gamma') \prod_{a',x'} \delta(\hat{u}_{a',x'}(\Gamma') - \hat{u}_{a',x'}(\Gamma)) / \Omega(\hat{u}(\Gamma)) \\
&\approx \int d\Gamma' \hat{\mathcal{J}}_{a,x}(\Gamma') \prod_{x'} \exp \left[- \sum_{a'} \hat{\lambda}_{a',x'}(\Gamma) \hat{u}_{a',x'}(\Gamma') \right] / Z(\Gamma),
\end{aligned} \tag{3.2.80}$$

with the normalization factor

$$\hat{Z}(\Gamma) := \int d\Gamma' \prod_{x'} \exp \left[- \sum_{a'} \hat{\lambda}_{a',x'}(\Gamma) \hat{u}_{a',x'}(\Gamma') \right]. \tag{3.2.81}$$

The multipliers $\hat{\lambda}_{a',x'}$ are fixed by the condition

$$\int d\Gamma' \hat{c}_{a,n}(\Gamma') \prod_{x'} \exp \left[- \sum_{a'} \hat{\lambda}_{a',x'}(\Gamma) \hat{u}_{a',x'}(\Gamma') \right] / \tilde{Z} = \hat{u}_{a,x}(\Gamma) \approx \hat{u}_{a,x}(\Gamma). \tag{3.2.82}$$

This construction mirrors the structure of the local microcanonical distribution

$$\begin{aligned}
\hat{\rho}_{\text{LM}}(\Gamma') &= \prod_{a',x'} \delta(\hat{u}_{a',x'}(\Gamma') - u_{a',x'}) / \Omega(u) \\
&\approx \prod_{x'} \exp \left[- \sum_{a'} \lambda_{a',x'} \hat{u}_{a',x'}(\Gamma') \right] / Z = \hat{\rho}_{\text{LG}},
\end{aligned} \tag{3.2.83}$$

where the multipliers $\lambda_{a',x'}$ are determined from

$$\int d\Gamma' \hat{c}_{a,n}(\Gamma') \prod_{x'} \exp \left[- \sum_{a'} \lambda_{a',x'} \hat{u}_{a',x'}(\Gamma') \right] / Z = u_{a,x}. \quad (3.2.84)$$

Thus, we define

$$\begin{aligned} \hat{\rho}_{\text{LG}}(\Gamma'; \hat{\lambda}(\Gamma)) &:= \prod_{x'} \exp \left[- \sum_{a'} \hat{\lambda}_{a',x'}(\Gamma) \hat{u}_{a',x'}(\Gamma') \right] / \hat{Z}(\Gamma) \\ &= \prod_{x'} \exp \left[- \sum_{a'} \sum_{n'=(x'-1)\ell+1}^{x'\ell} \hat{\lambda}_{a',n'}(\Gamma) \hat{c}_{a',n'}(\Gamma') \right] / \hat{Z}(\Gamma), \end{aligned} \quad (3.2.85)$$

with $\hat{\lambda}_{a',n'} := \hat{\lambda}_{a',x'}/\ell$, and may expand $\hat{\rho}_{\text{LG}}(\Gamma'; \hat{\lambda}(\Gamma)) \sim \hat{\rho}_{\text{LG}}(\Gamma'; \lambda_{\text{eq}} + \delta\hat{\lambda}(\Gamma))$, and by the same reasoning as before obtain

$$(\mathcal{P}\hat{\mathcal{J}}_{a,x})(\Gamma) \approx \sum_b A_b^a \delta\hat{u}_{b,x} + \frac{1}{2} \sum_{b,b'} H_{bb'}^a \delta\hat{u}_{b,x} \delta\hat{u}_{b',x} + \mathcal{O}(\delta\hat{u}^3) \quad (3.2.86)$$

3.2.6 Bare transport coefficients

In general, the bare transport coefficients

$$K_{ax,a'x'}(u) := \ell \int_0^\infty ds \left\langle (\mathcal{Q}\hat{\mathcal{J}}_{a,x})(e^{s\mathcal{Q}\mathcal{L}}\mathcal{Q}\hat{\mathcal{J}}_{a',x'}) \right\rangle_{\text{LM}}^u \quad (3.2.87)$$

are difficult to evaluate analytically. In the following, I show that the expression

$$K_{ax,a'x'}(u) = \ell \int_0^\infty ds \left\langle (\mathcal{Q}\hat{\mathcal{J}}_{a,x})(e^{s\mathcal{Q}\mathcal{L}}\mathcal{Q}\hat{\mathcal{J}}_{a',x'}) \right\rangle_{\text{LM}}^u \quad (3.2.88)$$

$$\approx \ell \int_0^\infty ds \left\langle (\mathcal{Q}\hat{\mathcal{J}}_{a,x})(e^{s\mathcal{L}}\mathcal{Q}\hat{\mathcal{J}}_{a',x'}) \right\rangle_{\text{LM}}^u \quad (3.2.89)$$

can be justified, i.e., the evolution operator $e^{\mathcal{Q}\mathcal{L}s}$ may be replaced by $e^{\mathcal{L}s}$ within the present accuracy.

To this end, let me define

$$\hat{\mathcal{R}}_s := e^{s\mathcal{Q}\mathcal{L}} \left(\sum_{x'} \mathcal{Q}\hat{\mathcal{J}}_{a',x'} \right). \quad (3.2.90)$$

Its time evolution is then written as

$$\partial_s \hat{\mathcal{R}}_s = (\mathcal{L} - \mathcal{P}\mathcal{L}) \hat{\mathcal{R}}_s. \quad (3.2.91)$$

We next evaluate the projected contribution $\mathcal{P}\mathcal{L}\hat{\mathcal{R}}_s$. By definition,

$$\begin{aligned}
(\mathcal{P}\mathcal{L}\hat{\mathcal{R}}_s)(\Gamma) &= \int d\Gamma' \left\{ \hat{H}(\Gamma'), \hat{R}_s(\Gamma') \right\} \exp \left(- \sum_{a',x'} \hat{\lambda}_{a',x'}(\Gamma) \hat{u}_{a',x'}(\Gamma') \right) / \hat{Z}(\Gamma) \\
&= \int d\Gamma' \left\{ \exp \left(- \sum_{a',x'} \hat{\lambda}_{a',x'}(\Gamma) \hat{u}_{a',x'}(\Gamma') \right) / \hat{Z}(\Gamma), \hat{H}(\Gamma') \right\} \hat{R}_s(\Gamma') \\
&= \int d\Gamma' \hat{\rho}_{\text{eq}}(\Gamma') \left(\sum_{a,x} \left\{ \delta \hat{u}_{a,x}(\Gamma'), \hat{H}(\Gamma') \right\} \sum_{a'} \Lambda_{a'}^a \delta \hat{u}_{a',x}(\Gamma) + \dots \right) \hat{\mathcal{R}}_s(\Gamma') \\
&= \sum_{a,x} \left(\int d\Gamma' \hat{\rho}_{\text{eq}}(\Gamma') \hat{\mathcal{R}}_s(\Gamma') \nabla_x \hat{\mathcal{J}}_{a,x}(\Gamma') \right) \sum_{a'} \Lambda_{a'}^a \delta \hat{u}_{a',x}(\Gamma) + \mathcal{O}(\delta \hat{u}(\Gamma)^2). \quad (3.2.92)
\end{aligned}$$

In going from the second to the third line, the local Gibbs distribution was expanded around the equilibrium measure $\hat{\rho}_{\text{eq}}$, and the deviation

$$\delta \hat{u}_{a,x} := \hat{u}_{a,x} - u_a^{\text{eq}}$$

was introduced. Therefore, in a sufficiently small neighborhood of equilibrium, one finds

$$\partial_s \hat{\mathcal{R}}_s = (\mathcal{L} - \mathcal{P}\mathcal{L}) \hat{\mathcal{R}}_s \sim \mathcal{L} \hat{\mathcal{R}}_s, \quad (3.2.93)$$

because the projected part is at least of order $\mathcal{O}(\delta \hat{u}^2)$. Since $\hat{u}_{a,x}$ represents a coarse-grained variable obtained as an average of ℓ microscopic variables $\hat{c}_{a,n}$, its fluctuations scale as $\mathcal{O}(\ell^{-1/2})$. Thus, in the limit of large ℓ , the above approximation becomes asymptotically exact. Consequently, replacing $e^{\mathcal{Q}\mathcal{L}s}$ by $e^{\mathcal{L}s}$ in the definition of the bare transport coefficients is justified.

Although the direct computation of the bare coefficients is generally challenging, numerical studies have demonstrated that, for the Fermi–Pasta–Ulam–Tsingou (FPUT) model

$$H = \sum_{n=1}^N \left[\frac{1}{2} p_n^2 + \frac{1}{2} r_n^2 + \frac{k_3}{3} r_n^3 + \frac{k_4}{4} r_n^4 \right], \quad (3.2.94)$$

the bare transport coefficients converge to finite values [254].

3.2.7 Absence of equilibrium currents in the case of only energy conservation

In the case of only energy conservation, we show that the equilibrium energy current $\langle \hat{\mathcal{J}}_{\epsilon,x} \rangle_{\text{LM}}^u$ vanishes.

Let $\hat{\mathbf{T}}$ denote the time-reversal operator. Generally, local currents change sign under time reversal, we obtain

$$\hat{\mathbf{T}} \hat{\mathcal{J}}_{\epsilon,x} = -\hat{\mathcal{J}}_{\epsilon,x}, \quad (3.2.95)$$

On the other hand, the relevant conserved quantities, namely $u_{r,x}$ and $u_{\epsilon,x}$, are even under time reversal:

$$\hat{T}u_{\epsilon,x} = u_{\epsilon,x}, \quad (3.2.96)$$

The local microcanonical measure is defined by

$$\hat{\rho}_{\text{LM}}(\Gamma') = \prod_{a',x'} \delta(\hat{u}_{a',x'}(\Gamma') - u_{a',x'}) / \Omega(u). \quad (3.2.97)$$

As is standard in the theory of fluctuating hydrodynamics, we approximate this measure by a local Gibbs distribution,

$$\hat{\rho}_{\text{LM}}(\Gamma') \approx \prod_{x'} \exp\left(-\sum_{a'} \lambda_{a',x'} \hat{u}_{a',x'}(\Gamma')\right) / Z =: \hat{\rho}_{\text{LG}}, \quad (3.2.98)$$

and further expand the corresponding averages with respect to small deviations $\delta u_{a,x}$ from the equilibrium values of the conserved quantities.

Accordingly, the expectation of the current is expanded as

$$\langle \hat{\mathcal{J}}_{\epsilon,x} \rangle_{\text{LM}}^u \approx \langle \hat{\mathcal{J}}_{\epsilon,x} \rangle_{\text{LG}}^u \quad (3.2.99)$$

$$\approx \langle \hat{\mathcal{J}}_{\epsilon,x} \rangle_{\text{eq}} - A_{\epsilon}^{\epsilon} \delta u_{\epsilon,x} + \frac{1}{2} H_{\epsilon\epsilon}^{\epsilon} \delta u_{\epsilon,x} \delta u_{\epsilon,x}. \quad (3.2.100)$$

It is evident that the equilibrium current satisfies

$$\langle \hat{\mathcal{J}}_{\epsilon,x} \rangle_{\text{eq}} = 0, \quad (3.2.101)$$

because equilibrium is invariant under time reversal while $\hat{\mathcal{J}}_{\epsilon,x}$ is odd.

Moreover, the left-hand side $\langle \hat{\mathcal{J}}_{\epsilon,x} \rangle_{\text{LM}}^u$ is odd under time reversal, whereas the right-hand side is expressed entirely in terms of conserved quantities, all of which are even under $\hat{T}$. Thus, equality between both sides is possible only if

$$\langle \hat{\mathcal{J}}_{\epsilon,x} \rangle_{\text{LM}}^u = 0, \quad (3.2.102)$$

which completes the proof that the equilibrium coarse-grained currents vanish in the existence of only energy conservation.

3.2.8 Fluctuating hydrodynamic equation in the case of only energy conservation

In the case of only energy conservation, the contribution of $\mathcal{P}\hat{\rho}$ in (3.2.22) vanishes due to the time-reversal symmetry. As seen in the previous subsection, the equilibrium currents

$\langle \hat{\mathcal{J}}_{\epsilon,x} \rangle_{\text{LM}}^u$ also vanish. Consequently, in the case of only energy conservation, we finally arrive at the fluctuating hydrodynamic equation for long-range interacting systems:

$$\partial_t u_{\epsilon,x}(t) = \nabla_x \left[\sum_{x'} D_{x,x'} \nabla_{x'} u_{\epsilon,x'} + \xi_{\epsilon,x}(t) \right], \quad (3.2.103)$$

where $u_{\epsilon,x}$ is the coarse-grained local energy and $D_{x,x'}$ is the non-local diffusion constant obtained by the Green–Kubo-like relation:

$$D_{x,x'} = \frac{1}{c_V k_B T^2} \int_0^\infty ds \langle \hat{\mathcal{J}}_{\epsilon,x}(s) \hat{\mathcal{J}}_{\epsilon,x'} \rangle. \quad (3.2.104)$$

In the system without local equilibrium currents, the bare transport coefficients coincide with the macroscopic transport coefficients, because there is no contribution of $\mathcal{P}\hat{\rho}$ in (3.2.22). $\xi_{\epsilon,x}(t)$ is the thermal noise that satisfies

$$\langle\langle \xi_{\epsilon,x}(t) \xi_{\epsilon,x'}(t') \rangle\rangle = 2c_V k_B T^2 D_{x,x'} \delta(t - t'). \quad (3.2.105)$$

For short-range interactions, $D_{x,x'}$ generically shows rapid decay in space, i.e., $D_{x,x'} \sim \mathcal{D}_0 \delta(x - x')$, and hence, the dynamics reduces to the standard fluctuating hydrodynamics $\partial_t u_{\epsilon,x}(t) = \nabla_x [\mathcal{D}_0 \nabla_x u_{\epsilon,x}(t) + \xi_{\epsilon,x}(t)]$.

Chapter 4

Energy Diffusion in the Long-range Interacting Spin Systems

In this chapter, we investigate the nature of energy diffusion in long-range interacting spin systems. Our analysis focuses on two prototypical models—the transverse Ising model and the XYZ model—defined on a D -dimensional lattice with interaction exponent $\alpha > D$, a condition ensuring thermodynamic extensivity. In one dimension, we demonstrate that both normal and anomalous diffusion can arise. The anomalous regime is shown to originate from an anomalous enhancement of the amplitude of the equal-time energy current correlation.

By applying the cumulant power-law clustering theorem to equal-time current correlations, we establish several theorems that lead to a unified conclusion: the sufficient condition for energy diffusion to be normal in one dimension is $\alpha > 3/2$, independently of the choice of model. Furthermore, fluctuating hydrodynamics consistently accounts for the emergence of Lévy diffusion when $\alpha < 3/2$, thereby indicating that the above condition is optimal.

In higher dimensions $D \geq 2$, our results show that energy diffusion remains normal as long as $\alpha > D$, which is precisely the threshold required for thermodynamic extensivity. These findings provide a coherent and comprehensive understanding of energy diffusion in long-range interacting spin systems across spatial dimensions.

4.1 Models and setup

Let us first consider the one-dimensional uniform classical spin Hamiltonian

$$H = \sum_{n=1}^N \hat{h}_n, \tag{4.1.1}$$

where $\hat{h}_n$ is the n th local Hamiltonian and N is the number of sites. We consider the prototypical spin systems for $\hat{h}_n$:

$$\begin{aligned}\hat{h}_n^{(I)} &= -J \sum_{r=1}^{N/2} \frac{1}{2r^\alpha} (S_n^z S_{n+r}^z + S_{n-r}^z S_n^z) + h S_n^x, \\ \hat{h}_n^{(XYZ)} &= - \sum_{r=1}^{N/2} \frac{1}{2r^\alpha} \sum_{\sigma=x,y,z} J_\sigma (S_n^\sigma S_{n+r}^\sigma + S_{n-r}^\sigma S_n^\sigma),\end{aligned}\tag{4.1.2}$$

where $\hat{h}_n^{(I)}$ and $\hat{h}_n^{(XYZ)}$ represent the local Hamiltonians for the long-range transverse Ising and XYZ models, respectively. The continuous variable S_n^σ denotes the σ component of the spin at site n and satisfies $\sum_\sigma (S_n^\sigma)^2 = 1$. Periodic boundary condition is imposed such that $n \pm N \equiv n$. The parameter α represents the power-law exponent of the long-range interaction, with $\alpha > 1$ to guarantee the thermodynamic extensivity.

The time evolution of the classical spin system is governed by the equation

$$\frac{\partial}{\partial t} S_i^\sigma = \left[\mathbf{S}_i \times \left(-\frac{\partial H}{\partial \mathbf{S}_i} \right) \right]_\sigma =: \{S_i^\sigma, H\},\tag{4.1.3}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket in the spin dynamics, defined as (3.2.4). Any function dependent on the spin variables $f(\{S_i\})$ is then expressed with the Poisson bracket:

$$\frac{\partial}{\partial t} f(\{\mathbf{S}_i\}) = \sum_{i,\sigma} \frac{\partial f}{\partial S_i^\sigma} \{S_i^\sigma, H\} = \{f, H\}.\tag{4.1.4}$$

The transverse Ising and XYZ models with long-range interaction are believed to possess only energy as a local conserved quantity for general parameter values. Therefore, we investigate the energy diffusion $\langle \delta \hat{h}_n(t) \delta \hat{h}_0 \rangle$ for the long-range transverse Ising and XXZ model in (4.1.2).

4.2 Thermal conductivity and Green-Kubo formula

When the system exhibits normal energy diffusion, the energy diffusion constant $\mathfrak{D}$ is related to the thermal conductivity κ and the specific heat per unit length c_V through the relation $\mathfrak{D} = \kappa/c_V$. Since c_V remains finite for generic thermodynamic states, the problem of determining whether energy diffusion is normal is reduced to examining the finiteness of the thermal conductivity. In this context, the thermal conductivity is formulated via the Green-Kubo formula in terms of the equilibrium energy current correlation function [264]. The energy current itself is defined through the continuity equation for the local energy density,

$$\partial_t \hat{h}_n = -(\mathcal{J}_{n+1}^\epsilon - \mathcal{J}_n^\epsilon).\tag{4.2.1}$$

For systems with nearest-neighbor interactions, one may straightforwardly derive an explicit form of the current. However, in systems with long-range interactions, a more careful formulation is required. In such cases, the energy current can be expressed as

$$\mathcal{J}_n^\epsilon = \sum_{\substack{i \geq n, j \leq n-1 \\ |i-j| \leq N/2}} t_{i \leftarrow j}^\epsilon, \quad t_{i \leftarrow j}^\epsilon := \{\hat{h}_i, \hat{h}_j\}, \quad (4.2.2)$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket for the spin dynamics introduced in (3.2.4). This expression naturally emerges by considering the time evolution of the local energy,

$$\begin{aligned} \frac{\partial}{\partial t} \hat{h}_n &= \{\hat{h}_n, H\} = \sum_j \{\hat{h}_n, \hat{h}_j\} \\ &= \sum_{j \leq n-1} \{\hat{h}_n, \hat{h}_j\} - \sum_{i \geq n+1} \{\hat{h}_i, \hat{h}_n\} \\ &= \sum_{j \leq n-1} \{\hat{h}_n, \hat{h}_j\} + \sum_{\substack{i \geq n+1 \\ j \leq n-1}} \{\hat{h}_i, \hat{h}_j\} - \sum_{i \geq n+1} \{\hat{h}_i, \hat{h}_n\} - \sum_{\substack{i \geq n+1 \\ j \leq n-1}} \{\hat{h}_i, \hat{h}_j\} \\ &= \sum_{\substack{i \geq n \\ j \leq n-1}} \{\hat{h}_i, \hat{h}_j\} - \sum_{i \geq n+1, j \leq n} \{\hat{h}_i, \hat{h}_j\}, \end{aligned} \quad (4.2.3)$$

and by comparing the above expression with the continuity equation $\partial_t \hat{h}_n = -(\mathcal{J}_{n+1}^\epsilon - \mathcal{J}_n^\epsilon)$, one identifies the current as in (4.2.2). In this representation, $t_{i \leftarrow j}^\epsilon$ is naturally interpreted as the energy transmitted from site j to site i across the virtual boundary between $n-1$ and n , so that $\mathcal{J}_n^\epsilon$ is the total energy transmission through this boundary [55]. See the schematic in Fig. 4.1.

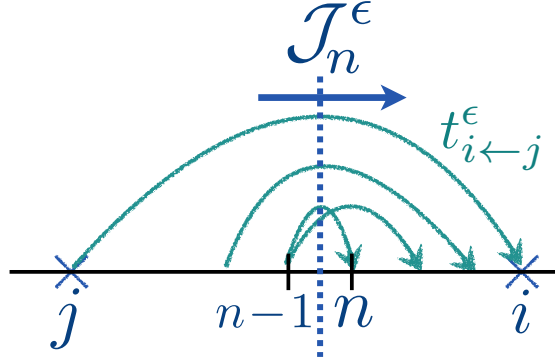


Figure 4.1: The schematic definition of the energy current $\mathcal{J}_n^\epsilon$ in one dimension.

The Green-Kubo formula for the finite-size thermal conductivity is then given by [264]:

$$\kappa_N = \frac{1}{k_B T^2} \int_0^\infty dt C_N(t), \quad C_N(t) = \sum_{n=1}^N \langle \mathcal{J}_n^\epsilon(t) \mathcal{J}_0^\epsilon \rangle, \quad (4.2.4)$$

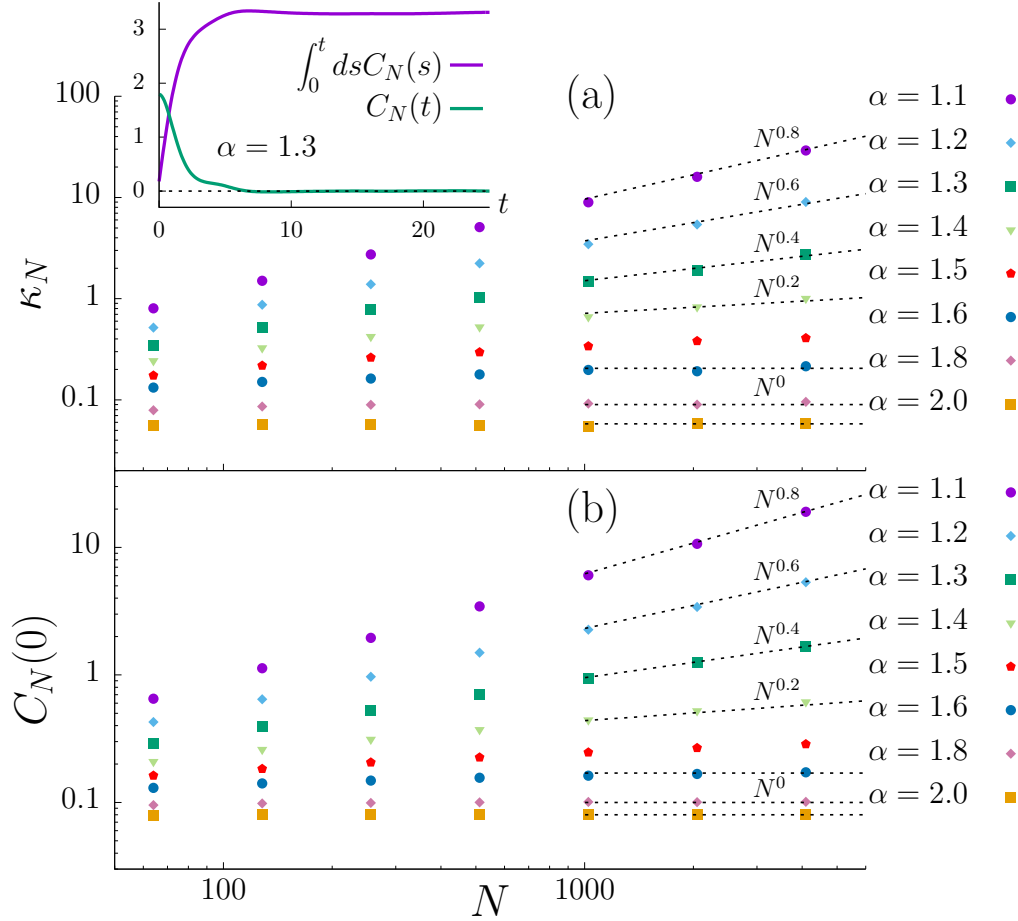


Figure 4.2: Thermal conductivity of the long-range transverse Ising model with parameters $J = h = 1$ and $T = 10$ ($k_B \equiv 1$). (a) Size dependence of κ_N . The inset shows the time-dependent current correlation as well as the Green-Kubo integral for $\alpha = 1.3$ and $N = 1024$. (b) Size dependence of the equal-time correlation $C_N(0)$. [This figure is reproduced with permission and modified from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

where k_B denotes the Boltzmann constant and $\langle \cdot \rangle$ represents the equilibrium expectation at temperature T . The explicit subscript N is used to emphasize the system-size dependence.

To examine the nature of energy diffusion, we performed numerical simulations for the long-range transverse Ising model. To integrate the equation (4.1.3), we employ the fourth-order symplectic integrator method [265] with a time step of $dt = 0.01$. To compute the correlation function, initial spin configurations are sampled using the Monte Carlo method at temperature T . Thermal averages are taken over 10^6 independent initial configurations.

The numerical results are displayed in Fig. 4.2. Figure 4.2(a) shows κ_N as a function of system size N , while the inset illustrates the rapid temporal decay of the current correlation for $\alpha = 1.3$, which leads to a convergent Green–Kubo integral. Such rapid decay is expected on physical grounds, as the system lacks continuous translational symmetry and is believed to be non-integrable (see, e.g., [161]). This behavior stands in sharp contrast to low-dimensional oscillator chains, which typically exhibit slow power-law decay due to momentum conservation associated with continuous translational symmetry [57–61]. As illustrated in the inset of Fig. 4.2(a), the autocorrelation function $C_N(t)$ exhibits rapid decay in time, and the time integral $\int_0^t ds C_N(s)$ converges by $t = 20$. Accordingly, we set $t = 20$ as the upper limit of integration in the calculation of κ_N . Additional numerical evidence supporting the rapid decay for various parameters and models is provided in Fig. 4.3 below.

As shown in the main panel of Fig. 4.2(a), the thermal conductivity displays anomalous system-size dependence for $\alpha < 1.5$, whereas for larger α , κ_N clearly saturates, indicating normal energy diffusion. These results therefore demonstrate the existence of a transition in the nature of energy diffusion.

Importantly, the observed anomaly is not attributable to a long-time tail in the current correlation, as explicitly shown in the inset of Fig. 4.2(a). Instead, it originates from a distinct mechanism: an anomalous enhancement of the equal-time current correlation amplitude. This mechanism becomes evident in Fig. 4.2(b), which shows the size dependence of the equal-time total current correlation $C_N(0)$. The qualitative agreement between the behaviors of $C_N(0)$ and κ_N indicates that the amplitude of the equal-time current correlation directly governs the anomalous energy transport.

These observations lead us to the conclusion that *the equal-time current correlation plays a decisive role in determining the qualitative nature of energy diffusion*. The remainder of this chapter demonstrates that this viewpoint provides a coherent and unifying framework for classifying energy diffusion in long-range interacting spin systems.

4.3 Upper bound for the equal-time energy current correlations

In this section, using Theorem 1, we obtain the upper bound for equal-time energy current correlation $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$.

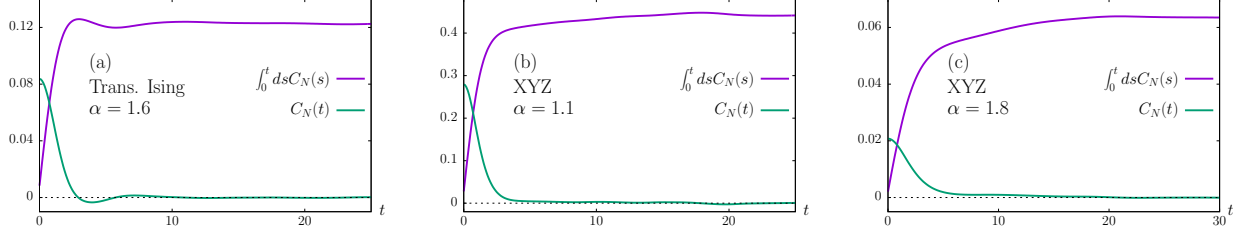


Figure 4.3: Rapid decay of total current correlation $C_N(t) := \sum_n \langle \mathcal{J}_n(t) \mathcal{J}_0(0) \rangle$ and the convergence of Green-Kubo integral $\int_0^\infty dt C_N(t)$. (a): Transverse Ising case for $\alpha = 1.6$ and $J = 1, h = -0.5, N = 1024$, (b): XYZ case for $\alpha = 1.1$ and $J_x = 1, J_y = 0.8, J_z = 0.5, N = 256$ and (c): XYZ case for $\alpha = 1.8$ and $J_x = 1, J_y = 0.8, J_z = 0.5, N = 256$. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

Theorem 2. *For the one-dimensional long-range interacting spin systems (4.1.2) in the infinite line, the equal-time equilibrium current correlation of $\beta < \beta_c$ in the regime $1 < \alpha \leq 2$ is upper-bounded as*

$$|\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle| < c' n^{-(2\alpha-2)}, \quad (4.3.1)$$

where c' is a constant that does not depend on n .

For the regime $\alpha > 2$, the transverse Ising and XY models admit an upper bound of the form $c' n^{2-2\alpha}$, whereas the XYZ model is instead bounded by $c' n^{-\alpha}$. This distinction originates from the structure of the energy current: in the transverse Ising (XYZ) case the current operator is expressed solely in terms of 2-body (3-body) spin variables, and therefore the corresponding current-current correlation necessarily involves 4-body (6-body) spin operators. The n th-order moment of such observables can be systematically decomposed into all admissible contributions built from joint cumulants up to order n . This hierarchical structure is completely analogous to the standard cumulant expansion for c-numbers, exemplified for $n = 2$ and $n = 3$ respectively by

$$\langle A_{X_1} A_{X_2} \rangle = \langle A_{X_1} A_{X_2} \rangle_c + \langle A_{X_1} \rangle_c \langle A_{X_2} \rangle_c, \quad (4.3.2)$$

$$\begin{aligned} \langle A_{X_1} A_{X_2} A_{X_3} \rangle &= \langle A_{X_1} A_{X_2} A_{X_3} \rangle_c + \langle A_{X_1} \rangle_c \langle A_{X_2} A_{X_3} \rangle_c + \langle A_{X_2} \rangle_c \langle A_{X_3} A_{X_1} \rangle_c \\ &\quad + \langle A_{X_3} \rangle_c \langle A_{X_1} A_{X_2} \rangle_c + \langle A_{X_1} \rangle_c \langle A_{X_2} \rangle_c \langle A_{X_3} \rangle_c. \end{aligned} \quad (4.3.3)$$

Invoking Theorem 1, we analyze the equal-time current correlation by constructing a general diagrammatic representation that governs the contributions of multi-body operators at arbitrary order. Through this diagrammatic framework, we ultimately derive the upper bound stated in Theorem 2. Although Theorem 2 is obtained under the assumption of a high-temperature regime, its physical implications are expected to extend far beyond this

limitation, as the bound of Theorem 1 is robust from a physical standpoint. It is worth emphasizing that Theorems 1, 2, and 3 remain valid even in quantum systems.

Theorem 2 constitutes a central result in our characterization of universal energy diffusion. From the theorem, we obtain the upper bound for the equal-time correlation amplitude

$$C_N(0) < c' N^{3-2\alpha} + \text{const.}$$

This estimate exhibits excellent agreement with the numerical evidence presented in Fig. 4.2(b). *Consequently, we conclude that the condition $\alpha > 3/2$ is sufficient to guarantee normal diffusion, independently of the choice of microscopic models or parameters.* Moreover, as discussed in detail below, Theorem 2 also furnishes a theoretical foundation for anomalous diffusion within the framework of fluctuating hydrodynamics.

4.4 Proof of Theorem 2

In this section, we employ Theorem 1 to derive rigorous upper bounds on the equal-time energy current correlations for one-dimensional prototypical long-range interacting spin systems. Separate proofs are provided for the long-range transverse Ising and XYZ models, as the operator structure of the current and the symmetry properties of the dynamics differ substantially among them.

4.4.1 Transverse Ising model

We begin with the one-dimensional long-range transverse Ising model, whose Hamiltonian is defined by

$$H = \sum_{n=1}^N \left[-J \sum_{r=1}^{N/2} \frac{S_n^z S_{n+r}^z}{r^\alpha} + h S_n^x \right]. \quad (4.4.1)$$

The local energy operator at site n is given by

$$\hat{h}_n := -\frac{J}{2} \sum_{r=1}^{N/2} \left[\frac{S_n^z S_{n+r}^z}{r^\alpha} + \frac{S_n^z S_{n-r}^z}{r^\alpha} \right] + h S_n^x, \quad (4.4.2)$$

which satisfies the continuity equation

$$\partial_t \hat{h}_n = -(\mathcal{J}_{n+1}^\epsilon - \mathcal{J}_n^\epsilon), \quad (4.4.3)$$

$$\mathcal{J}_n^\epsilon = - \sum_{\substack{i \geq n, j < n \\ |i-j| < N/2}} t_{i \leftarrow j}^\epsilon, \quad (4.4.4)$$

where the quantity $t_{i \leftarrow j}^\epsilon$ represents the microscopic energy transfer from site j to site i and is defined via the spin Poisson bracket as

$$t_{i \leftarrow j}^\epsilon := \{\hat{h}_i, \hat{h}_j\} = \frac{Jh}{2} \frac{S_i^z S_j^y - S_i^y S_j^z}{d_{i,j}^\alpha}. \quad (4.4.5)$$

Here the bracket $\{\cdot, \cdot\}$ denotes the classical spin Poisson bracket introduced in Eq. (3.2.4). The current $\mathcal{J}_n^\epsilon$ is therefore interpreted as the net energy flux across the virtual boundary separating sites $n-1$ and n .

The symmetries of the Hamiltonian impose several useful constraints. First, owing to the invariance under the global transformation

$$(S_i^x, S_i^y, S_i^z) \longrightarrow (S_i^x, -S_i^y, -S_i^z), \quad i = 1, \dots, N,$$

we obtain

$$\langle S_i^y \rangle = \langle S_i^z \rangle = 0. \quad (4.4.6)$$

Moreover, since the Hamiltonian contains no S^y components, any correlator involving two S^y operators factorizes as

$$\langle S_i^y S_j^y \mathcal{O} \rangle = \delta_{ij} \langle (S_i^y)^2 \mathcal{O} \rangle, \quad (4.4.7)$$

for any observable $\mathcal{O}$. These structural identities play a crucial role in simplifying the cumulant expansion of current correlations.

Using Eqs. (4.4.6) and (4.4.7), together with the cumulant bounds established in Theorem 1, we now proceed to prove the transverse Ising case of Theorem 2.

Theorem 2.1. *In the one-dimensional long-range interacting transverse Ising model, under the condition $\beta < \beta_c$, the equal-time current correlation is upper-bounded as*

$$|\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle| \leq C_0 n^{2-2\alpha}, \quad (4.4.8)$$

for $\alpha > 1$.

Proof. The equal-time current correlation $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ admits the following explicit representation:

$$\begin{aligned} \langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle &= \left(\frac{Jh}{2} \right)^2 \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \frac{\langle (S_k^z S_\ell^y - S_k^y S_\ell^z)(S_{k'}^z S_{\ell'}^y - S_{k'}^y S_{\ell'}^z) \rangle}{d_{k,\ell}^\alpha d_{k',\ell'}^\alpha} \\ &= \left(\frac{Jh}{2} \right)^2 \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{\substack{\{a,b\}=\{k,\ell\} \\ \{a',b'\}=\{k',\ell'\}}} \varepsilon_{a,b} \varepsilon_{a',b'} \frac{\langle S_a^z S_b^y S_{a'}^z S_{b'}^y \rangle}{d_{a,b}^\alpha d_{a',b'}^\alpha} \\ &= \left(\frac{Jh}{2} \right)^2 \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{\substack{\{a,b\}=\{k,\ell\} \\ \{a',b'\}=\{k',\ell'\}}} \varepsilon_{a,b} \varepsilon_{a',b'} \delta_{b,b'} \frac{\langle S_a^z S_{a'}^z (S_b^y)^2 \rangle}{d_{a,b}^\alpha d_{a',b}^\alpha}, \end{aligned} \quad (4.4.9)$$

where the simplification in the final line follows directly from the identity (4.4.7). The coefficients $\varepsilon_{a,b}$ and $\varepsilon_{a',b'}$ denote the second-order Levi-Civita symbols, defined by

$$\varepsilon_{a,b} = \begin{cases} +1, & (a,b) = (k,\ell), \\ -1, & (a,b) = (\ell,k), \end{cases} \quad (4.4.10)$$

with $\varepsilon_{a',b'}$ defined analogously.

To provide intuitive insight, we now present a diagrammatic interpretation of the term

$$d_{a,b}^{-\alpha} d_{a',b}^{-\alpha}$$

appearing in (4.4.9). Figure 4.4 illustrates the diagram in which vertices correspond to localized regions of the lattice (representing local spin operators), and edges represent power-law decays associated with the distances between the corresponding sites.

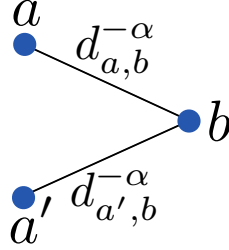


Figure 4.4: Diagrammatic representation of the factor $d_{a,b}^{-\alpha} d_{a',b}^{-\alpha}$ appearing in Eq. (4.4.9). [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

The three-body correlation function $\langle S_a^z S_{a'}^z (S_b^y)^2 \rangle$ can be decomposed into cumulants as

$$\langle S_a^z S_{a'}^z (S_b^y)^2 \rangle = \langle S_a^z S_{a'}^z (S_b^y)^2 \rangle_c + \langle S_a^z S_{a'}^z \rangle_c \langle (S_b^y)^2 \rangle_c, \quad (4.4.11)$$

where we have used the symmetry property (4.4.6). Applying Theorem 1, we obtain the bounds

$$\langle S_a^z S_{a'}^z (S_b^y)^2 \rangle_c \leq \|S_a^z\| \|S_{a'}^z\| \|(S_b^y)^2\| \left(\frac{c_1}{d_{a,a'}^\alpha d_{a',b}^\alpha} + \frac{c'_1}{d_{a,b}^\alpha d_{b,a'}^\alpha} + \frac{c''_1}{d_{a',a}^\alpha d_{a,b}^\alpha} \right), \quad (4.4.12)$$

$$\langle S_a^z S_{a'}^z \rangle_c \leq \|S_a^z\| \|S_{a'}^z\| \frac{c_2}{d_{a,a'}^\alpha}. \quad (4.4.13)$$

We shall employ the following elementary inequalities repeatedly throughout the subsequent analysis:

$$\frac{1}{d_{i,j}^{2\alpha}} < \frac{1}{d_{i,j}^\alpha}, \quad (4.4.14)$$

where the lattice spacing is set to 1, and

$$\sum_i \frac{1}{d_{k,i}^\alpha} \frac{1}{d_{i,j}^\alpha} \frac{1}{d_{j,i}^\alpha} \frac{1}{d_{i,\ell}^\alpha} \leq \sum_i \frac{1}{d_{k,i}^\alpha} \frac{1}{d_{i,j}^\alpha} \sum_{i'} \frac{1}{d_{j,i'}^\alpha} \frac{1}{d_{i',\ell}^\alpha}, \quad (k, \ell = 1, \dots, N). \quad (4.4.15)$$

For completeness, we also provide in Fig. 4.5 the diagrammatic illustrations corresponding to Lemma 4, inequality (4.4.14), and inequality (4.4.15). In these diagrams, constant

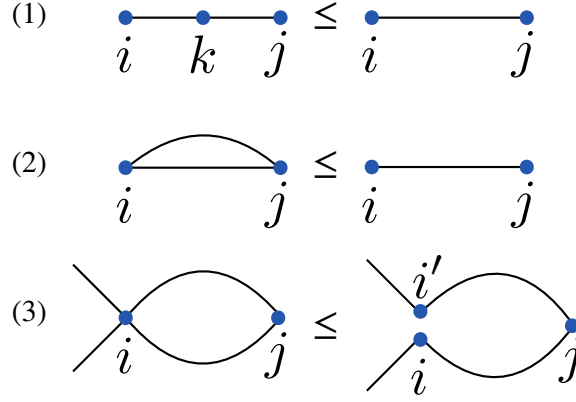


Figure 4.5: Diagrammatic representations of Lemma 4, inequality (4.4.14), and inequality (4.4.15). Only the distance-dependent structure is shown, and multiplicative constants are omitted for clarity. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

prefactors of order $\mathcal{O}(1)$ are omitted so as to highlight exclusively the contributions arising from spatial dependence.

By using Theorem 1, Lemma 4, and (4.4.13)-(4.4.15), we obtain the following upper bound of $|\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle|$:

$$\begin{aligned}
 |\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle| &\leq \left(\frac{Jh}{2}\right)^2 \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{\substack{\{a,b\}=\{k,\ell\} \\ \{a',b'\}=\{k',\ell'\}}} \delta_{b,b'} \frac{|\langle S_a^z S_{a'}^z (S_b^y)^2 \rangle_c| + |\langle S_a^z S_{a'}^z \rangle_c \langle (S_b^y)^2 \rangle_c|}{d_{a,b}^\alpha d_{a',b}^\alpha} \\
 &\leq \left(\frac{Jh}{2}\right)^2 \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{\substack{\{a,b\}=\{k,\ell\} \\ \{a',b'\}=\{k',\ell'\}}} \delta_{b,b'} \frac{1}{d_{a,b}^\alpha d_{a',b}^\alpha} \left(\frac{c_1}{d_{a,a'}^\alpha d_{a',b}^\alpha} + \frac{c'_1}{d_{a,b}^\alpha d_{b,a'}^\alpha} + \frac{c''_1}{d_{a',a}^\alpha d_{a,b}^\alpha} + \frac{c_2}{d_{a,a'}^\alpha} \right) \\
 &\leq \left(\frac{Jh}{2}\right)^2 \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{\substack{\{a,b\}=\{k,\ell\} \\ \{a',b'\}=\{k',\ell'\}}} \delta_{b,b'} \frac{1}{d_{a,b}^\alpha d_{a',b}^\alpha} \left(\frac{c_1}{d_{a,a'}^\alpha} + \sum_{b''} \frac{c'_1}{d_{a,b''}^\alpha d_{b'',a'}^\alpha} + \frac{c''_1}{d_{a',a}^\alpha} + \frac{c_2}{d_{a,a'}^\alpha} \right) \\
 &\leq \left(\frac{Jh}{2}\right)^2 \{(c_1 + c''_1 + c_2)u + c'_1 u^2\} \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{1}{d_{k,\ell'}^{2\alpha}} \leq C_0 n^{2-2\alpha}, \tag{4.4.16}
 \end{aligned}$$

where $C_0 = (Jh/2)^2 \{(c_1 + (c''_1 + c_2)u + c'_1 u^2) / \{(1 - 2\alpha)(2 - 2\alpha)\}\}$.

We describe the diagrammatic derivation of the upper bound of $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$, as illustrated in Fig. 4.6. This figure corresponds to Eq. (4.4.16) and visually represents how each contribution is connected and how the leading terms that determine the upper bound are extracted.

First, by utilizing the cumulant power-law clustering properties in Eqs. (4.4.12) and (4.4.13), edges connecting the vertices are systematically added. By classifying the contributions based on how the cumulant clusters are connected, four distinct terms are identified,

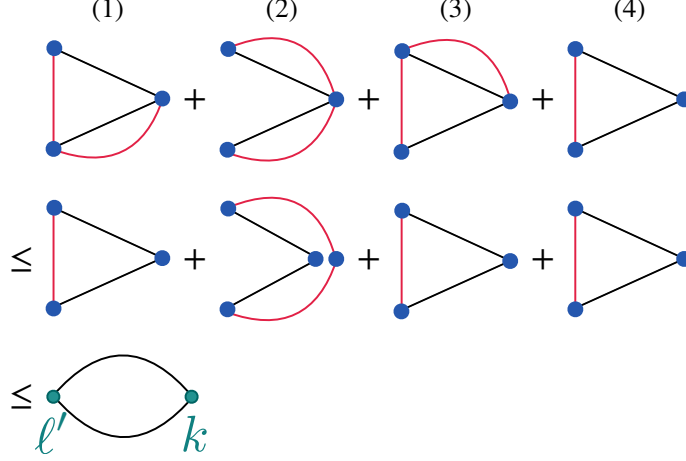


Figure 4.6: The diagrammatic derivation for the upper bound of $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ in (4.4.16). [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

as shown in the first line of Fig. 4.6. These correspond to the second line of Eq. (4.4.16) and reflect the general structure of the cumulant decomposition. Next, by applying the inequalities in Eqs. (4.4.14) and (4.4.15) and performing simplifications based on spin symmetries, we obtain the second line in Fig. 4.6. This corresponds to the third line in Eq. (4.4.16) and demonstrates that the vanishing of S^y components in the Hamiltonian contributes efficiently to the diagrammatic simplification for the transverse Ising model. Finally, by employing Lemma 4, the chain connecting k and ℓ' is reorganized into a closed loop, as depicted in the third line of Fig. 4.6. This corresponds to the fourth line in Eq. (4.4.16) and summarizes the effect of long-range interactions into the final upper bound expression.

By following these steps systematically in accordance with Eq. (4.4.16) and Fig. 4.6, we establish the claims of the theorem. That is, through the combination of cumulant clustering, the symmetries of Hamiltonian, and the chain lemma 4, the upper bound on the equal-time energy current correlation in the long-range transverse Ising system is rigorously derived. $\square$

4.4.2 XYZ model

In this subsection, we consider the one-dimensional long-range interacting XYZ model, which is described by the following Hamiltonian

$$H = - \sum_{n=1}^N \sum_{r=1}^{N/2} \left[J_x \frac{S_n^x S_{n+r}^x}{r^\alpha} + J_y \frac{S_n^y S_{n+r}^y}{r^\alpha} + J_z \frac{S_n^z S_{n+r}^z}{r^\alpha} \right]. \quad (4.4.17)$$

The local energy at site n is defined as

$$\hat{h}_n := - \sum_{r=1}^{N/2} \sum_{\sigma=x,y,z} \frac{J_\sigma}{2} \left[\frac{S_n^\sigma S_{n+r}^\sigma}{r^\alpha} + \frac{S_n^\sigma S_{n-r}^\sigma}{r^\alpha} \right], \quad (4.4.18)$$

and satisfies the following continuity equations

$$\partial_t \hat{h}_n = -(\mathcal{J}_{n+1}^\epsilon - \mathcal{J}_n^\epsilon), \quad (4.4.19)$$

$$\mathcal{J}_n^\epsilon = - \sum_{\substack{i \geq n, j < n \\ |i-j| < N/2}} t_{i \leftarrow j}^\epsilon. \quad (4.4.20)$$

Energy transmission operator $t_{i \leftarrow j}^\epsilon$ is defined as

$$t_{i \leftarrow j}^\epsilon = \{\hat{h}_i, \hat{h}_j\} = \sum_{m=1}^N \sum_{\sigma \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2}}{4} \sum_{\{a,b,c\}=\{i,j,m\}} \tilde{\epsilon}_{abc} \frac{S_a^{\sigma_1} S_b^{\sigma_2} S_c^{\sigma_3}}{d_{a,c}^\alpha d_{b,c}^\alpha}, \quad (4.4.21)$$

where $\sum_{\sigma \in P_{\text{cyc}}^{xyz}}$ denotes sums over all circular permutation of the variables (x, y, z) . In other words, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ represents one of (x, y, z) , (y, z, x) or (z, x, y) , and $\sum_{\sigma \in P_{\text{cyc}}^{xyz}}$ is defined as the summation over these permutations. $\tilde{\epsilon}_{abc}$ is defined as follows:

$$\tilde{\epsilon}_{abc} = \begin{cases} +1, & \text{for } (a, b, c) = (i, j, m), (m, j, i), (i, m, j), \\ -1, & \text{for } (a, b, c) = (j, i, m), (j, m, i), (m, i, j), \end{cases} \quad (4.4.22)$$

Note that $\tilde{\epsilon}_{abc}$ is different from Levi-Civita symbols.

From the symmetry of the Hamiltonian, we note that the following properties hold

$$\langle S_i^x \rangle = \langle S_i^y \rangle = \langle S_i^z \rangle = 0, \quad (4.4.23)$$

$$\langle S_i^\sigma S_j^\tau \rangle = \delta_{\sigma,\tau} \langle S_i^\sigma S_j^\sigma \rangle, \quad (4.4.24)$$

for $i, j = 1, \dots, n$ and $\sigma, \tau \in \{x, y, z\}$. The relation (4.4.23), (4.4.24) is shown that the Hamiltonian is invariant to the canonical transformation $(S_i^\sigma, S_i^\tau, S_i^v) \rightarrow (S_i^\sigma, -S_i^\tau, -S_i^v)$ for $\{\sigma, \tau, v\} = \{x, y, z\}$ and $i = 1, \dots, N$.

Using these properties and Theorem 1, we establish the XYZ case of Theorem 2:

Theorem 2.2. *In the one-dimensional long-range interacting XYZ model, under the condition $\beta < \beta_c$, the equal-time current correlation is upper-bounded as*

$$|\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle| \leq C_2 n^{2-2\alpha}, \quad (4.4.25)$$

for $1 < \alpha \leq 2$, and

$$|\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle| \leq C'_2 n^{-\alpha}, \quad (4.4.26)$$

for $\alpha > 2$.

Proof. The equal-time current correlation $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ is given as follows:

$$\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle = \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{cyc}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \tilde{\varepsilon}_{abc} \tilde{\varepsilon}_{a'b'c'} \frac{\langle S_a^{\sigma_1} S_b^{\sigma_2} S_c^{\sigma_3} S_{a'}^{\tau_1} S_{b'}^{\tau_2} S_{c'}^{\tau_3} \rangle}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \quad (4.4.27)$$

The term $1/(d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha)$ in (4.4.27) is illustrated diagrammatically in Fig.4.7.

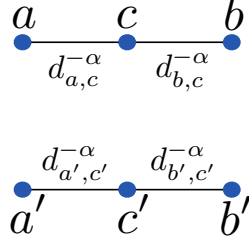


Figure 4.7: The diagrammatic representation of the term $d_{a,c}^{-\alpha} d_{b,c}^{-\alpha} d_{a',c'}^{-\alpha} d_{b',c'}^{-\alpha}$ in (4.4.27) $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

The 6-body correlation $\langle S_a^{\sigma_1} S_b^{\sigma_2} S_c^{\sigma_3} S_{a'}^{\tau_1} S_{b'}^{\tau_2} S_{c'}^{\tau_3} \rangle$ can be decomposed into cumulant products as follows:

$$\begin{aligned} & \langle S_a^{\sigma_1} S_b^{\sigma_2} S_c^{\sigma_3} S_{a'}^{\tau_1} S_{b'}^{\tau_2} S_{c'}^{\tau_3} \rangle \\ &= \langle S_a^{\sigma_1} S_b^{\sigma_2} S_c^{\sigma_3} S_{a'}^{\tau_1} S_{b'}^{\tau_2} S_{c'}^{\tau_3} \rangle_c + \sum_{\substack{\{i_1, \dots, i_6\} \\ = \{a, b, c, a', b', c'\} \\ \mu \in \{x, y, z\}, \nu_k \neq \mu}} \langle S_{i_1}^\mu S_{i_2}^\mu \rangle_c \langle S_{i_3}^{\nu_1} S_{i_4}^{\nu_2} S_{i_5}^{\nu_3} S_{i_6}^{\nu_4} \rangle_c + \langle S_{i_1}^x S_{i_2}^x \rangle_c \langle S_{i_3}^y S_{i_4}^y \rangle_c \langle S_{i_5}^z S_{i_6}^z \rangle_c \end{aligned} \quad (4.4.28)$$

where we use the relation (4.4.23), (4.4.24). Next, we use Theorem 1 as follows:

$$\langle S_a^{\sigma_1} S_b^{\sigma_2} S_c^{\sigma_3} S_{a'}^{\tau_1} S_{b'}^{\tau_2} S_{c'}^{\tau_3} \rangle_c \leq c_1 \|S_a^{\sigma_1}\| \dots \|S_{c'}^{\tau_3}\| \sum_{\substack{\{j_1, \dots, j_6\} = \\ \{a, b, c, a', b', c'\} \\ \text{:connected}}} \frac{1}{d_{j_1, j_2}^\alpha \dots d_{j_5, j_6}^\alpha} \quad (4.4.29)$$

$$\langle S_{i_3}^{\nu_1} S_{i_4}^{\nu_2} S_{i_5}^{\nu_3} S_{i_6}^{\nu_4} \rangle_c \leq c_2 \|S_{i_3}^{\nu_1}\| \dots \|S_{i_6}^{\nu_4}\| \sum_{\substack{\{j_1, \dots, j_4\} \\ = \{i_1, \dots, i_4\} \\ \text{:connected}}} \frac{1}{d_{j_1, j_2}^\alpha d_{j_2, j_3}^\alpha d_{j_3, j_4}^\alpha} \quad (4.4.30)$$

$$\langle S_{i_p}^\mu S_{i_q}^\mu \rangle_c \leq c_3 \|S_{i_p}^\mu\| \|S_{i_q}^\mu\| \frac{1}{d_{i_p, i_q}^\alpha} \quad (4.4.31)$$

By using Theorem 1 and (4.4.29)-(4.4.31), we obtain the following inequality for $|\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle|$:

$$\begin{aligned}
|\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle| &\leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{cyc}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \\
&\times \left\{ |\langle S_a^{\sigma_1} S_b^{\sigma_2} S_c^{\sigma_3} S_{a'}^{\tau_1} S_{b'}^{\tau_2} S_{c'}^{\tau_3} \rangle_c| \right. \\
&+ \sum_{\substack{\{i_1, \dots, i_6\} = \\ \{a, b, c, a', b', c'\} \\ \mu \in \{x, y, z\}, \nu_k \neq \mu}} \left| \langle S_{i_1}^\mu S_{i_2}^\mu \rangle_c \langle S_{i_3}^{\nu_1} S_{i_4}^{\nu_2} S_{i_5}^{\nu_3} S_{i_6}^{\nu_4} \rangle_c \right| + \left| \langle S_{i_1}^x S_{i_2}^x \rangle_c \langle S_{i_3}^y S_{i_4}^y \rangle_c \langle S_{i_5}^z S_{i_6}^z \rangle_c \right| \Big\} \\
&\leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{cyc}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \sum_{\substack{\{i_1, \dots, i_6\} = \\ \{a, b, c, a', b', c'\}}} \\
&\times \left\{ \sum_{\substack{i_1, \dots, i_6 \\ : \text{connected}}} \frac{c_1}{d_{i_1, i_2}^\alpha \dots d_{i_5, i_6}^\alpha} + \sum_{\substack{i_1 \in \{a, b, c\} \\ i_2 \in \{a', b', c'\}}} \frac{c_3}{d_{i_1, i_2}^\alpha} \sum_{\substack{i_3, \dots, i_6 \\ : \text{connected}}} \frac{c_2}{d_{i_3, i_4}^\alpha d_{i_4, i_5}^\alpha d_{i_5, i_6}^\alpha} \right. \\
&+ \left. \sum_{\substack{\{i_1, i_3, i_5\} = \{a, b, c\} \\ \{i_2, i_4, i_6\} = \{a', b', c'\}}} \frac{c_3}{d_{i_1, i_2}^\alpha} \frac{c_3}{d_{i_3, i_4}^\alpha} \frac{c_3}{d_{i_5, i_6}^\alpha} \right\} \tag{4.4.32}
\end{aligned}$$

Let us first consider the contribution of the first term in (4.4.32):

$$\sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{cyc}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \sum_{\substack{\{i_1, \dots, i_6\} = \\ \{a, b, c, a', b', c'\} \\ : \text{connected}}} \frac{c_1}{d_{i_1, i_2}^\alpha \dots d_{i_5, i_6}^\alpha} \tag{4.4.33}$$

Fig. 4.8 comprehensively covers all the cases of clustering linkage in (4.4.33). Here, diagrams with identical topology are regarded as equivalent. The existence of 51 topologically distinct graphs can be verified using Burnside's lemma (also known as Cauchy–Frobenius lemma).

The methods of clustering linkage can be classified into two cases: (1) when there are two or more connected components linking the upper and lower graphs in Fig. 4.7, and (2) when there is only one connected component.

In the case (1) in Fig. 4.8, let us first compute the upper bound of the contribution to $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ in the following typical examples: (1)-(i) the first row, fifth column from the left in Fig. 4.8 (1), and (1)-(ii) the fourth row, ninth column from the left in Fig. 4.8 (1).

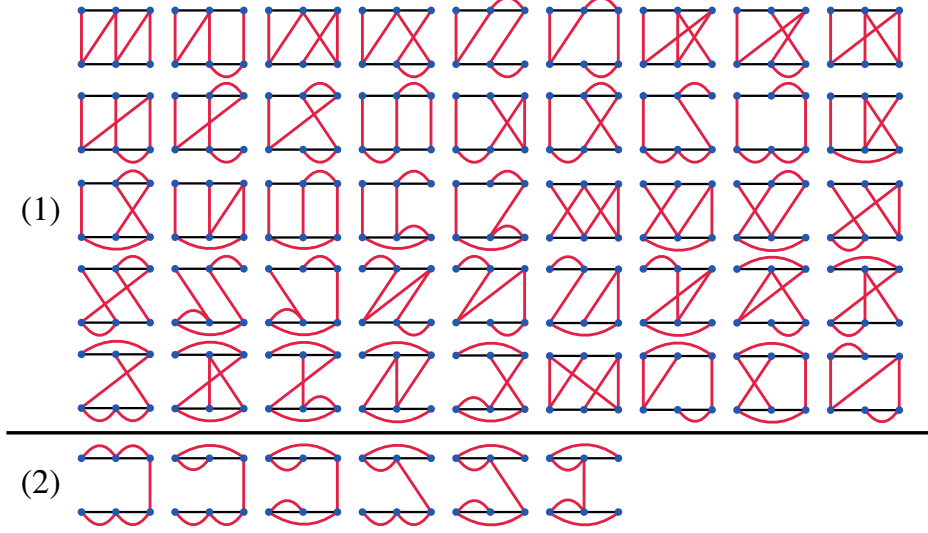


Figure 4.8: The diagrammatic representation of the contribution of the first term in (4.4.32) to the upper bound of $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$. It can be classified into two cases: (1) when there are two or more connected components linking the upper and lower graphs in Fig. 4.7, and (2) when there is only one connected component. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

In the case (1)-(i), we obtain the following upper bound of the contribution to $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$:

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \frac{c_1}{d_{a,a'}^\alpha d_{a',c}^\alpha d_{c,b}^\alpha d_{b,c'}^\alpha d_{c',b}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \frac{c_1}{d_{a,a'}^\alpha d_{b',c'}^\alpha d_{c',b}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \sum_{c'=1}^N \frac{c_1}{d_{a,a'}^\alpha d_{a',c'}^\alpha d_{c',b'}^\alpha d_{b',c''}^\alpha d_{c'',b}^\alpha d_{b,c}^\alpha d_{c,a}^\alpha} \\
& \leq \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} u^5 c_1 \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{1}{d_{k,\ell'}^{2\alpha}} \leq c n^{2-2\alpha}, \tag{4.4.34}
\end{aligned}$$

where $c = u^5 c_1 / \{(1 - 2\alpha)(2 - 2\alpha)\} \cdot \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2} / 16$. The corresponding diagrammatic derivation is presented in the upper half of Fig. 4.9.

In the case (1)-(ii), we similarly obtain the following upper bound of the contribution to

$\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$:

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \frac{c_1}{d_{a,b}^\alpha d_{b,a'}^\alpha d_{a',c'}^\alpha d_{c',c}^\alpha d_{c,b'}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \frac{c_1}{d_{a,b}^\alpha d_{a',c'}^\alpha d_{c',c}^\alpha d_{c,b'}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \sum_{c''=1}^N \frac{c_1}{d_{a,c}^\alpha d_{c,c'}^\alpha d_{c',a'}^\alpha d_{a',c''}^\alpha d_{c'',b'}^\alpha d_{b',c}^\alpha d_{c,b}^\alpha d_{b,a}} \\
& \leq \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} u^6 c_1 \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{1}{d_{k,\ell'}^{2\alpha}} \leq c' n^{2-2\alpha}, \tag{4.4.35}
\end{aligned}$$

where $c' = u^6 c_1 / \{(1-2\alpha)(2-2\alpha)\} \cdot \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2} / 16$. The corresponding diagrammatic derivation is presented in the lower half of Fig. 4.9.

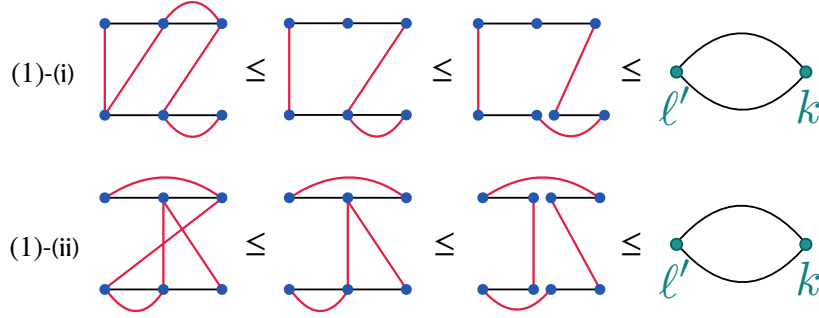


Figure 4.9: Typical examples of the diagrammatic derivation for the upper bound in the case Fig. 4.8 (1). [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

The same upper bound can be obtained for all remaining cases in Fig. 4.8 (1) as well:

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \sum_{\substack{\{i_1, \dots, i_6\} = \\ \{a, b, c, a', b', c'\} \\ \text{:the case (1) in Fig. 4.8}}} \frac{c_1}{d_{i_1, i_2}^\alpha \dots d_{i_5, i_6}^\alpha} \\
& \leq \mathcal{O}(1) n^{2-2\alpha}. \tag{4.4.36}
\end{aligned}$$

Next, we consider the case (2) in Fig. 4.8. In this case, (k, ℓ, m) and (k', ℓ', m') are connected via a single edge. We construct closed loops for both (k, ℓ, m) and (k', ℓ', m') . Furthermore, considering the dominant contribution in power-law clustering, the ordering relation between $k, \ell, m, k', \ell', m'$ is as follows:

$$\forall u \in \{k, \ell, m\}, \forall v \in \{k', \ell', m'\}, u > v. \tag{4.4.37}$$

We define $p, p' \in \{1, \dots, N\}$ as $p = \min\{k, \ell, m\}$, $p' = \max\{k', \ell', m'\}$. Here, $p \in \{\ell, m\}$ and $p' \in \{k', m'\}$, which implies that k, ℓ, p, p' are arranged as $k \geq n > p > p' \geq 0 > \ell'$.

Since we obtain the following inequality for the contribution of the case (2) in Fig.4.8:

$$\begin{aligned} & \sum_{\substack{k, i, i', \ell' \\ :k \geq n > p > p' \geq 0 > \ell'}} \sum_{q, q'} \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, p, q\} \\ \{a', b', c'\} = \{k', p', q'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \sum_{\substack{\{i_1, \dots, i_6\} = \\ \{a, b, c, a', b', c'\} \\ \text{:the case (2) in Fig.4.8}}} \frac{c_1}{d_{i_1, i_2}^\alpha \dots d_{i_5, i_6}^\alpha} \\ & \leq \mathcal{O}(1) \sum_{\substack{k, p, p', \ell' \\ :k \geq n > p > p' \geq 0 > \ell'}} \frac{1}{d_{k,p}^{2\alpha}} \frac{1}{d_{p,p'}^\alpha} \frac{1}{d_{p',\ell'}^{2\alpha}}, \end{aligned} \quad (4.4.38)$$

where we denote q, q' as $q = \{\ell, m\} \setminus \{p\}$ and $q' = \{k', m'\} \setminus \{p'\}$. The corresponding schematic diagram is presented in Fig. 4.10.

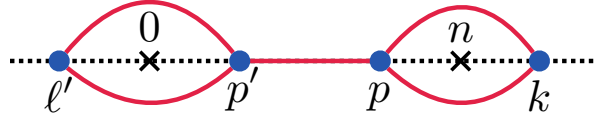


Figure 4.10: The schematic diagram of the case (2) in Fig. 4.8. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

Let us demonstrate (4.4.38) in a typical example: the second column from the left in Fig. 4.8 (2). For this example, we obtain the following inequality:

$$\begin{aligned} & \sum_{\substack{k, p, p', \ell' \\ :k \geq n > p > p' \geq 0 > \ell'}} \sum_{q, q'} \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, p, q\} \\ \{a', b', c'\} = \{\ell', p', q'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \frac{c_1}{d_{c,a}^\alpha d_{a,b}^\alpha d_{b,b'}^\alpha d_{b',c'}^\alpha d_{c',a'}^\alpha} \\ & \leq \sum_{\substack{k, p, p', \ell' \\ :k \geq n > p > p' \geq 0 > \ell'}} \sum_{q, q'} \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, p, q\} \\ \{a', b', c'\} = \{\ell', p', q'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \sum_{c'=1}^N \frac{\bar{a}c_1}{d_{a,b}^\alpha d_{b,b'}^\alpha d_{b',c'}^\alpha d_{c',a'}^\alpha} \\ & \leq \sum_{\substack{k, p, p', \ell' \\ :k \geq n > p > p' \geq 0 > \ell'}} \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} u^3 c_1 \frac{1}{d_{k,p}^{2\alpha}} \frac{1}{d_{p,p'}^\alpha} \frac{1}{d_{p',\ell'}^{2\alpha}} = c' \sum_{\substack{k, p, p', \ell' \\ :k \geq n > p > p' \geq 0 > \ell'}} \frac{1}{d_{k,p}^{2\alpha}} \frac{1}{d_{p,p'}^\alpha} \frac{1}{d_{p',\ell'}^{2\alpha}}, \end{aligned} \quad (4.4.39)$$

where $c' = u^3 c_1 \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2} / 16$. The corresponding diagrammatic derivation is presented in Fig. 4.11.

The same inequality can also be obtained for all remaining cases in Fig. 4.8 (2). Thus, (4.4.38) has been demonstrated.

In the case (2), the following upper bound for the contribution to $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ is derived from

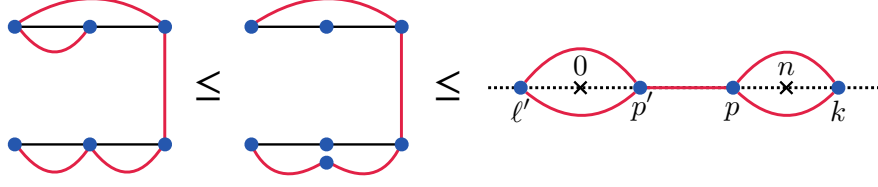


Figure 4.11: A typical example of the diagrammatic derivation for the upper bound in the case Fig. 4.8 (2). [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

(4.4.38):

$$\begin{aligned}
& \sum_{\substack{k,p,p',\ell' \\ :k \geq n > p > p' \geq 0 > \ell'}} \frac{1}{d_{k,p}^{2\alpha}} \frac{1}{d_{p,p'}^\alpha} \frac{1}{d_{p',\ell'}^{2\alpha}} = \sum_{\substack{k,p,p',\ell' \\ :k \geq n > p > p' \geq 0 > \ell'}} \frac{1}{(d_{k,n} + d_{n,p})^{2\alpha}} \frac{1}{d_{p,p'}^\alpha} \frac{1}{(d_{p',0} + d_{0,\ell'})^{2\alpha}} \\
& \leq \sum_{\substack{p,p' \\ :n > p > p' \geq 0}} \sum_{x,y=1}^{N/2} \frac{1}{(d_{n,p} + x)^{2\alpha}} \frac{1}{d_{p,p'}^\alpha} \frac{1}{(d_{p',0} + y)^{2\alpha}} \leq \frac{1}{(1-2\alpha)^2} \sum_{\substack{p,p' \\ :n > p > p' \geq 0}} \frac{1}{d_{n,p}^{2\alpha-1}} \frac{1}{d_{p,p'}^\alpha} \frac{1}{d_{p,0}^{2\alpha-1}} \\
& \leq \frac{1}{(1-2\alpha)^2} \sum_{\substack{p,p' \\ :n > p > p' \geq 0}} \frac{1}{d_{n,p}^\alpha} \frac{1}{d_{p,p'}^\alpha} \frac{1}{d_{p,0}^\alpha} \leq \frac{u^2}{(1-2\alpha)^2} \frac{1}{d_{n,0}^\alpha} = \frac{u^2}{(1-2\alpha)^2} n^{-\alpha}. \tag{4.4.40}
\end{aligned}$$

This bound decays faster than $n^{2-2\alpha}$ when $\alpha \leq 2$. Hence, the term shown in Fig.4.8 (2) does not constitute the leading term for $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ for $1 < \alpha \leq 2$. However, this term should be dominant for $\alpha > 2$.

Next, let us consider the contribution of the second term in (4.4.32):

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m,m'=1}^N \sum_{\sigma,\tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a,b,c\}=\{k,\ell,m\} \\ \{a',b',c'\}=\{k',\ell',m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \sum_{\substack{\{i_1,\dots,i_6\}= \\ \{a,b,c,a',b',c'\} \\ i_1 \in \{a,b,c\} \\ i_2 \in \{a',b',c'\} \\ i_3,\dots,i_6:\text{connected}}} \frac{c_3}{d_{i_1,i_2}^\alpha} \frac{c_2}{d_{i_3,i_4}^\alpha d_{i_4,i_5}^\alpha d_{i_5,i_6}^\alpha}
\end{aligned} \tag{4.4.41}$$

Fig. 4.12 comprehensively covers all the cases of clustering linkage in (4.4.41). Here, diagrams with identical topology are regarded as equivalent. The existence of 17 topologically distinct graphs can be verified using Burnside's lemma (also known as Cauchy–Frobenius lemma).

Let us compute the upper bound of the contribution to $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ in the following typical examples: (iii) the first row, fifth column from the left in Fig. 4.12, and (iv) the first row, sixth column from the left in Fig. 4.12.

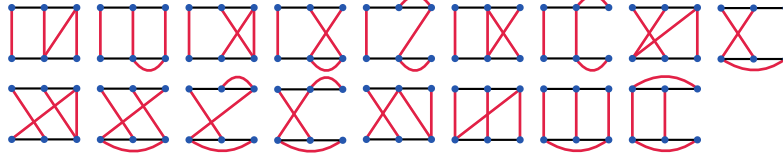


Figure 4.12: The diagrammatic representation of the contribution of the second term in (4.4.32) to the upper bound of $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

In the case (iii), we obtain the following upper bound of the contribution to $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$:

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \frac{c_3}{d_{a,a'}^\alpha} \frac{c_2}{d_{b,c'}^\alpha d_{c',c}^\alpha d_{c,b'}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \frac{c_3}{d_{a,a'}^\alpha} \sum_{e, f=1}^N \frac{c_2}{d_{b,e}^\alpha d_{e,f}^\alpha d_{f,b'}^\alpha} \\
& \leq \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} u^6 c_2 c_3 \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{1}{d_{k,\ell'}^{2\alpha}} \leq c'' n^{2-2\alpha}, \tag{4.4.42}
\end{aligned}$$

where $c'' = u^6 c_2 c_3 / \{(1 - 2\alpha)(2 - 2\alpha)\} \cdot \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2} / 16$. The corresponding diagrammatic derivation is presented in the upper half of Fig. 4.13.

In the case (iv), we similarly obtain the following upper bound of the contribution to $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$:

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \frac{c_3}{d_{a,b'}^\alpha} \frac{c_2}{d_{c,a'}^\alpha d_{a',b}^\alpha d_{b,b'}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b',c'}^\alpha} \frac{c_3}{d_{a,b'}^\alpha} \frac{c_2}{d_{c,a'}^\alpha d_{a',b}^\alpha d_{b,b'}^\alpha} \\
& \leq \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} u^4 c_2 c_3 \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{1}{d_{k,\ell'}^{2\alpha}} \leq c''' n^{2-2\alpha}, \tag{4.4.43}
\end{aligned}$$

where $c''' = u^4 c_2 c_3 / \{(1 - 2\alpha)(2 - 2\alpha)\} \cdot \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2} / 16$. The corresponding diagrammatic derivation is presented in the lower half of Fig. 4.13.

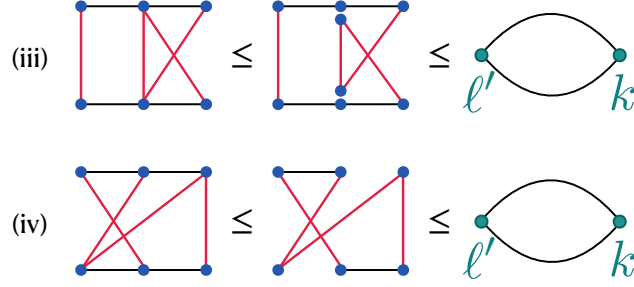


Figure 4.13: Typical examples of the diagrammatic derivation for the upper bound in the case Fig. 4.12. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

The same upper bound can be obtained for all remaining cases in Fig. 4.12 as well:

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \sum_{\substack{\{i_1, \dots, i_6\} = \\ \{a, b, c, a', b', c'\} \\ i_1 \in \{a, b, c\} \\ i_2 \in \{a', b', c'\} \\ i_3, \dots, i_6: \text{connected}}} \frac{c_3}{d_{i_1, i_2}^\alpha} \frac{c_2}{d_{i_3, i_4}^\alpha d_{i_4, i_5}^\alpha d_{i_5, i_6}^\alpha} \\
& \leq \mathcal{O}(1) n^{2-2\alpha}
\end{aligned} \tag{4.4.44}$$

Lastly, let us consider the contribution of the third term in (4.4.32).

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \sum_{\substack{\{i_1, i_3, i_5\} = \{a, b, c\} \\ \{i_2, i_4, i_6\} = \{a', b', c'\}}} \frac{c_3}{d_{i_1, i_2}^\alpha} \frac{c_3}{d_{i_3, i_4}^\alpha} \frac{c_3}{d_{i_5, i_6}^\alpha} \\
& \tag{4.4.45}
\end{aligned}$$

Fig. 4.14 encompasses all possible ways of clustering linkage, which are limited to two cases. Here, graphs with the same topology are considered equivalent.

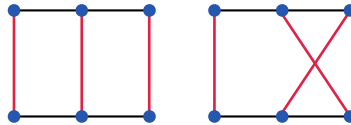


Figure 4.14: The diagrammatic representation of the contribution of the third term in (4.4.32) to the upper bound of $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

For the diagram on the left in Fig. 4.14, as in the case of Fig. 4.8(1) and Fig. 4.12, we can demonstrate that it is bounded by $\mathcal{O}(1) n^{2-2\alpha}$, using (4.4.14), (4.4.15), and Lemma 4.

Next, let us consider the diagram on the right. We classify the cases as follows: (a) when neither b nor b' (the two points at the right end of the graph) are equal to k or ℓ' , and (b) when $b = k$ and $b' = k'$.

First, for the case (a), let $p \in \{b, b'\}$ be the point such that $p \neq k, \ell'$. Without loss of generality, we can assume $p = b$. Thus, we obtain the following upper bound for (4.4.45):

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, p, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\} \\ p \neq k, \ell'}} \frac{1}{d_{a,c}^\alpha d_{p,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \frac{c_3}{d_{a,a'}^\alpha} \frac{c_3}{d_{p,c'}^\alpha} \frac{c_3}{d_{b,c'}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{a, c, a', b', c'} \frac{(c_3)^3}{d_{a,a'}^\alpha d_{a',c'}^\alpha d_{c',b'}^\alpha d_{b',c}^\alpha d_{c,a}^\alpha} \frac{u}{d_{c,c'}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{a, c, a', b', c'} \frac{u(c_3)^3}{d_{a,a'}^\alpha d_{a',c'}^\alpha d_{c',b'}^\alpha d_{b',c}^\alpha d_{c,a}^\alpha} \\
& \leq \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} u^4 (c_3)^3 \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{1}{d_{k,\ell'}^{2\alpha}} \leq c_a n^{2-2\alpha}, \tag{4.4.46}
\end{aligned}$$

where $c_a = u^4 (c_3)^3 / \{(1 - 2\alpha)(2 - 2\alpha)\} \cdot \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2} / 16$. The corresponding diagrammatic derivation is illustrated in the upper half of the Fig. 4.15.

Next, for the case (b): $b = k, b' = \ell'$, the following upper bound is derived:

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, c\} = \{\ell, m\} \\ \{a', c'\} = \{k', m'\}}} \frac{1}{d_{a,c}^\alpha d_{k,c}^\alpha d_{a',c'}^\alpha d_{\ell',c'}^\alpha} \frac{c_3}{d_{a,a'}^\alpha} \frac{c_3}{d_{k,c'}^\alpha} \frac{c_3}{d_{b,\ell'}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{c, c'} \frac{(c_3)^3}{d_{k,c'}^\alpha d_{c',\ell'}^\alpha d_{\ell',c}^\alpha d_{c,k}^\alpha} \frac{u^2}{d_{c,c'}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{c, c'} \frac{u^2 \bar{a}(c_3)^3}{d_{k,c'}^\alpha d_{c',\ell'}^\alpha d_{\ell',c}^\alpha d_{c,k}^\alpha} \\
& \leq \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} u^4 \bar{a}(c_3)^3 \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{1}{d_{k,\ell'}^{2\alpha}} \leq c_a n^{2-2\alpha}, \tag{4.4.47}
\end{aligned}$$

with the corresponding diagrammatic derivation shown in the lower half of the Fig. 4.15.

From these results, we can also derive the following upper bound for the contribution of the third term in (4.4.32):

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \sum_{\sigma, \tau \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2} J_{\tau_1} J_{\tau_2}}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{1}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha} \sum_{\substack{\{i_1, i_3, i_5\} = \{a, b, c\} \\ \{i_2, i_4, i_6\} = \{a', b', c'\}}} \frac{c_3}{d_{i_1, i_2}^\alpha} \frac{c_3}{d_{i_3, i_4}^\alpha} \frac{c_3}{d_{i_5, i_6}^\alpha} \\
& \leq \mathcal{O}(1) n^{2-2\alpha} \tag{4.4.48}
\end{aligned}$$

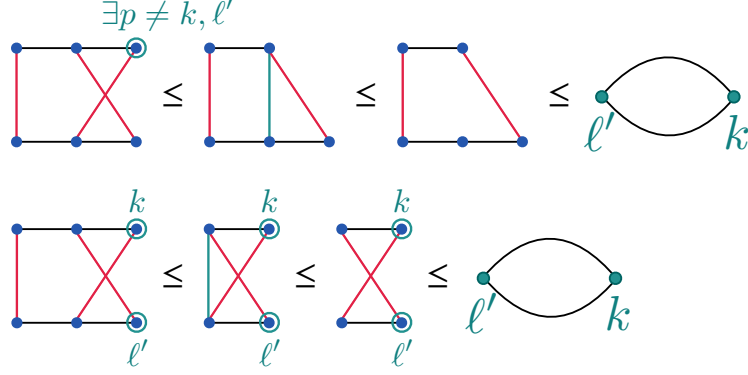


Figure 4.15: A typical example of the diagrammatic derivation of the upper bound of the third term in (4.4.32). [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

Based on the above discussion, it has been demonstrated that all terms in (4.4.32) can be bounded as follows:

$$|\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle| < \mathcal{O}(1)n^{2-2\alpha} + \mathcal{O}(1)n^{-\alpha} \quad (4.4.49)$$

Hence, for $1 < \alpha \leq 2$, the equal-time current correlation $|\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle|$ is bounded by $\mathcal{O}(1)n^{2-2\alpha}$, while for $\alpha > 2$, $|\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle|$ is bounded by $\mathcal{O}(1)n^{-\alpha}$. $\square$

4.5 Fluctuating hydrodynamics and Lévy diffusion

We investigate the anomalous transport regime by employing Theorem 2. To this end, we introduce a fluctuating hydrodynamic description appropriate for long-range interacting systems, formulated in terms of a continuous field representation on an infinite line. The hydrodynamic equation for the energy density field $\epsilon(x, t)$ is given by

$$\frac{\partial}{\partial t} \epsilon(x, t) = \frac{\partial}{\partial x} \left[\int dx' D(x - x') \frac{\partial}{\partial x'} \epsilon(x', t) + \xi(x, t) \right], \quad (4.5.1)$$

where $\epsilon(x, t)$ denotes the energy per unit length at position x , and $\xi(x, t)$ represents the thermal noise. The noise correlation is prescribed by the fluctuation–dissipation relation

$$\langle\langle \xi(x, t) \xi(x', t') \rangle\rangle = 2c_V k_B T^2 D(x - x') \delta(t - t'). \quad (4.5.2)$$

Fluctuating hydrodynamics of this form may be derived phenomenologically [61, 213], or from first principles using Zwanzig’s projection operator method [254, 256, 257], as discussed in detail in Section 3.2.

A key ingredient of this description is the nonlocal diffusion kernel $D(x - x')$. In systems with only short-range (e.g. nearest-neighbor) interactions, $D(x - x')$ typically decays rapidly,

allowing one to approximate it by a local kernel of the form $D(x - x') \simeq \mathfrak{D} \delta(x - x')$. In contrast, for long-range interacting systems, $D(x - x')$ exhibits algebraic decay and retains its nonlocal character. Using this hydrodynamic framework, we consider the equilibrium space-time correlation function of energy fluctuations:

$$C(x, t) := \langle\langle \delta\epsilon(x, t) \delta\epsilon(0, 0) \rangle\rangle, \quad (4.5.3)$$

where $\delta\epsilon(x, t) := \epsilon(x, t) - \bar{\epsilon}_x$, and $\bar{\epsilon}_x$ denotes the long-time average energy density at position x .

A direct computation yields the evolution equation for $C(x, t)$,

$$\partial_t C(x, t) = \partial_x \int dx' D(x - x') \partial_{x'} C(x', t). \quad (4.5.4)$$

Introducing the Fourier transform $a(x, t) = (2\pi)^{-1} \int dk a(k, t) e^{ikx}$, we obtain

$$\partial_t C(k, t) = -k^2 D(k) C(k, t). \quad (4.5.5)$$

Suppose that the small-wavenumber behavior of the diffusion kernel is characterized by a power law:

$$D(k) \sim k^{-\nu}. \quad (4.5.6)$$

Then the asymptotic behavior of the correlation function $C(x, t)$ is expected to exhibit the scaling form

$$C(x, t) \sim \begin{cases} t^{-1/(2-\nu)} f_{\text{Lévy}}\left(\frac{x}{t^{1/(2-\nu)}}\right), & \nu > 0, \\ t^{-1/2} f_{\text{Gauss}}\left(\frac{x}{t^{1/2}}\right), & \nu < 0, \end{cases} \quad (4.5.7)$$

where $f_{\text{Lévy}}$ and f_{Gauss} denote, respectively, the Lévy-stable and Gaussian scaling functions.

If the nonlocal diffusion kernel decays algebraically in real space,

$$D(x - x') \sim |x - x'|^{-\eta} \quad (4.5.8)$$

for large separations, then the associated exponent satisfies $\nu = 1 - \eta$. Consequently, normal diffusion arises for $\eta > 1$, whereas $\eta < 1$ leads to anomalous (Lévy) diffusion. This analysis provides a hydrodynamic interpretation of the diffusion regimes inferred from Theorem 2.

We now proceed to a more microscopic characterization of the nonlocal diffusion coefficient. Within the discrete lattice formulation, the nonlocal diffusion coefficient can be expressed in terms of a Green-Kubo-like formula [254] as

$$D(n - n') = (k_B T^2 c_V)^{-1} \int_0^\infty dt \langle \mathcal{J}_n^\epsilon(t) \mathcal{J}_{n'}^\epsilon \rangle, \quad (4.5.9)$$

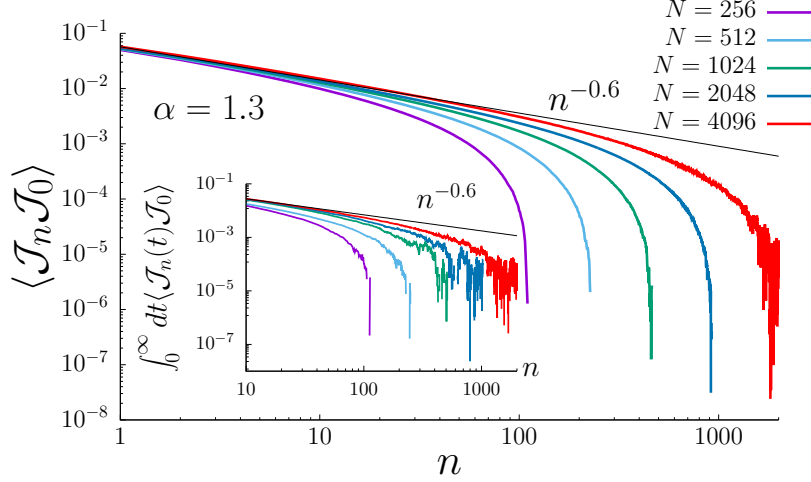


Figure 4.16: The correlation function $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ for the transverse Ising model with the parameter $\alpha = 1.3$ and $T = 10$ ($k_B \equiv 1$) in the main panel. The inset shows the Green-Kubo-like formula. The solid line is the theoretical line (4.3.1), i.e., $n^{-(2\alpha-2)}$. The equal-time correlation $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ explains the power law exponent, $\eta = 2\alpha - 2$. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

for which the amplitude of the equal-time contribution can be rigorously bounded by Theorem 2. In the main panel of Fig. 4.16, we present the equal-time current correlation $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ for the transverse Ising model with exponent $\alpha = 1.3$. The inset displays the corresponding Green-Kubo-type integrated correlations for the same parameter. We have verified that $\langle \mathcal{J}_n^\epsilon(t) \mathcal{J}_0^\epsilon \rangle$ decays rapidly in time such that the integral converges (data not shown). The results, plotted on a log-log scale, exhibit clear power-law behavior that becomes increasingly pronounced with system size. A finite-size effect is visible at the largest distances. Strikingly, the extracted power-law exponent η coincides with the theoretical upper bound obtained in (4.3.1), namely $\eta = 2\alpha - 2 = 0.6$. The inset reveals the same exponent, demonstrating that the equal-time correlations determine the qualitative scaling.

As supplementary material for Fig. 4.16, we provide Figs. 4.17 and 4.18. Fig. 4.17 shows the equal-time current correlations $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ together with the Green-Kubo-type integrals $\int_0^\infty dt \langle \mathcal{J}_n^\epsilon(t) \mathcal{J}_0^\epsilon(0) \rangle$ for the transverse Ising model. Panel (a) corresponds to $\alpha = 1.1$, while panel (b) treats $\alpha = 1.6$. In each case, the main panel displays the equal-time correlations, and the inset shows the Green-Kubo-type integral. For every parameter set examined, the time-dependent correlations decay sufficiently fast to ensure finiteness of the integral. The log-log plots exhibit power laws whose exponents η agree with the theoretical bounds (A.10) and (4.4.25), namely $\eta = 2\alpha - 2$. These observations confirm that the upper bounds in Theorem 2.1 are optimal across the considered parameter regimes.

Fig. 4.18 presents analogous results for the XY and XYZ models. Panel (a) corresponds to

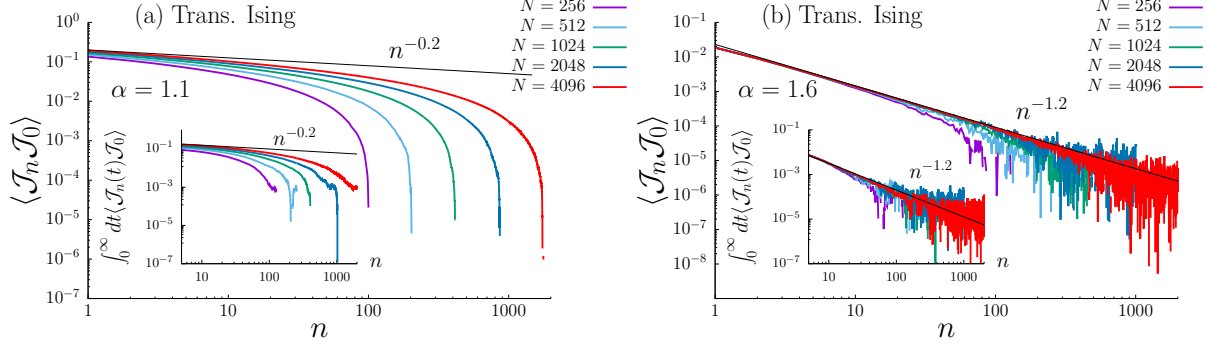


Figure 4.17: The correlation function $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ for the transverse Ising model in the main panel, with the parameter $\alpha = 1.1$ in (a) and $\alpha = 1.6$ in (b). Other parameters are $J = h = 1$. The inset shows the Green-Kubo like formula. The solid line is the theoretical line, i.e., $n^{2-2\alpha}$. The power-law behavior can be explained with the equal-time correlation $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ and the power-law exponent is equal to $\eta = 2\alpha - 2$. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

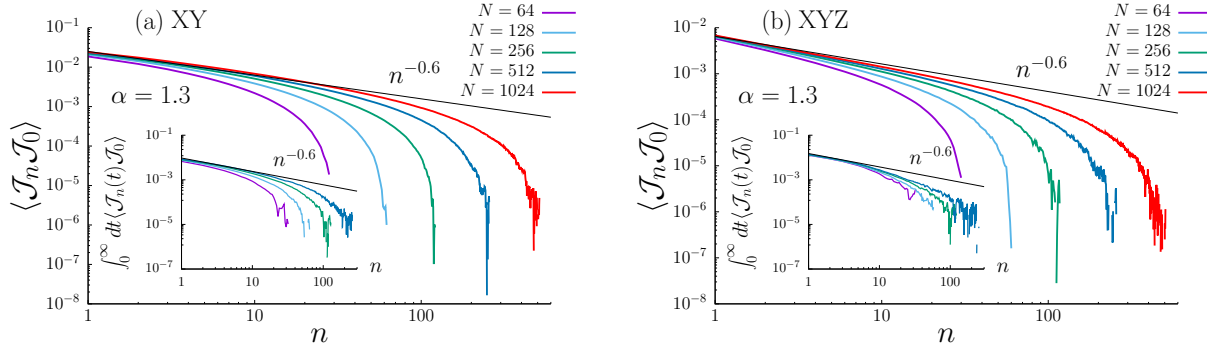


Figure 4.18: The correlation function $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ for the XY model in (a) and the XYZ model in (b), with the parameter $\alpha = 1.3$ in the main panel. Other parameters in the XY model are $J_x = 1, J_y = -0.8$, and in the XYZ model, $J_x = 1, J_y = 0.8, J_z = 0.5$. The inset shows the Green-Kubo like formula. The solid line is the theoretical line, i.e., $n^{2-2\alpha}$. The power-law behavior can be explained with the equal-time correlation $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ and the power law exponent is equal to $\eta = 2\alpha - 2$. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

the XY model with $\alpha = 1.3$, and panel (b) to the XYZ model with the same exponent. Again, the main panels show the equal-time correlations $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$, while the insets show the Green-Kubo-type integrals. The extracted exponents η agree with the bounds (A.10) and (4.4.25), reinforcing the optimality of the theoretical estimate.

These findings strongly support the scaling form (4.5.7), irrespective of the specific model, with

$$\nu = 3 - 2\alpha. \quad (4.5.10)$$

To further validate this prediction, we numerically computed the space–time correlations for the transverse Ising and XYZ models for several exponents α . In the Hamiltonian description, the space–time correlation function is defined as

$$C(n, t) := \langle \delta \hat{h}_n(t) \delta \hat{h}_0(0) \rangle, \quad (4.5.11)$$

where $\delta \hat{h}_n(t) := \hat{h}_n(t) - \langle \hat{h}_n \rangle$. The results are shown in Fig. 4.19. Figs. 4.19(a) and 4.19(b) correspond to the transverse Ising model, while panels (c) and (d) correspond to the XYZ model. Figures 4.19(a) and (c) treat $\alpha = 1.3$, whereas panels (b) and (d) treat $\alpha = 1.6$. The results demonstrate that both models exhibit Lévy-type scaling for $\alpha = 1.3$, while diffusive scaling emerges for $\alpha = 1.6$, fully consistent with the scaling form (4.5.7) together with (5.5.13).

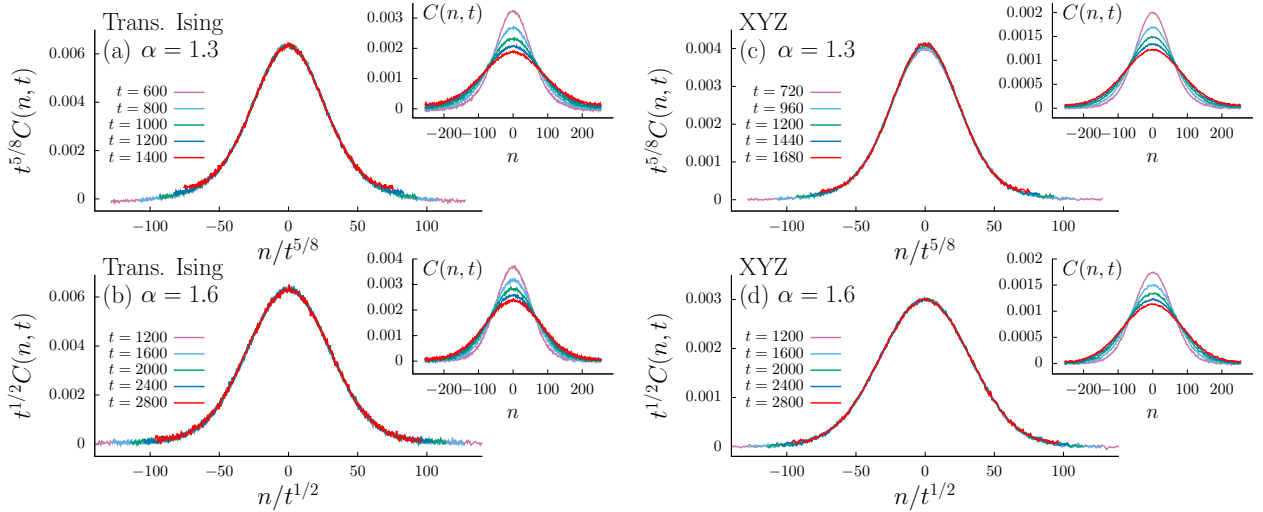


Figure 4.19: Space-time energy correlation function $C(n, t) := \langle \delta \hat{h}_n(t) \delta \hat{h}_0(0) \rangle$ for the transverse Ising model with the parameters $J = 1$ and $h = 1$ ((a) and (b)) and the XYZ model with the parameters $J_x = 1, J_y = 0.8, J_z = 0.5$ ((c) and (d)). $T = 10$ ($k_B \equiv 1$). System size is $N = 512$. In (a) and (c), we use the parameter $\alpha = 1.3$, while (b) and (d) are for $\alpha = 1.6$. It is clear that in both models, $\alpha = 1.3$ case obeys the Lévy scaling, while $\alpha = 1.6$ shows the diffusive scaling. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

As supplementary material for Fig. 4.19, we provide Figs. 4.20 and 4.21. Fig. 4.20 presents the space–time energy correlations $C(n, t) = \langle \delta \hat{h}_n(t) \delta \hat{h}_0(0) \rangle$ for the transverse Ising and XYZ

models with $\alpha = 1.1$. Panel (a) corresponds to the transverse Ising model, and panel (b) to the XYZ model. Both datasets clearly display Lévy-type scaling, confirming the fluctuating-hydrodynamic prediction $\nu = 3 - 2\alpha$.

Fig. 4.21 shows the XY model for $\alpha = 1.1, 1.3$, and 1.6 . The results again indicate that the cases $\alpha = 1.1$ and $\alpha = 1.3$ obey Lévy scaling, whereas the case $\alpha = 1.6$ exhibits normal diffusive scaling. These observations for the XY model provide further confirmation of the scaling theory based on fluctuating hydrodynamics.

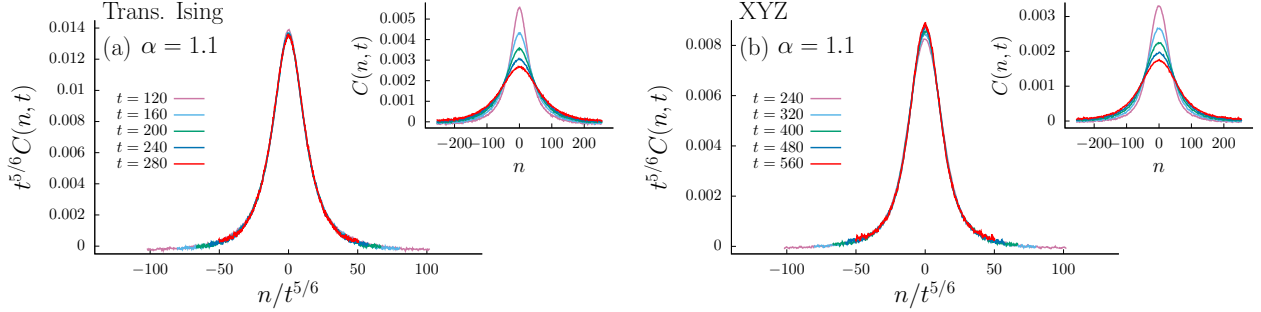


Figure 4.20: Space-time energy correlation function $C(n, t) := \langle \delta \hat{h}_n(t) \delta \hat{h}_0(0) \rangle$ for the transverse Ising model with the parameters $\alpha = 1.1$ and $J = 1$, $h = 1$ in (a), and the XYZ model with the parameters $\alpha = 1.1$ and $J_x = 1$, $J_y = 0.8$, $J_z = 0.5$ in (b). The system size is $N = 512$. It is clear that both models obey the Lévy scaling. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

4.5.1 Validity of fluctuating hydrodynamics

To assess the validity of the fluctuating-hydrodynamics description, we perform a direct comparison between the space-time correlation functions obtained from the fluctuating hydrodynamic equation and those computed from microscopic numerical simulations.

In the inset of Fig. 4.16, we demonstrate the convergence of the Green-Kubo-like expression (4.5.9) for the nonlocal diffusion coefficient. Using these data, we numerically evaluate the space-time equilibrium correlation function

$$C(x, t) = \langle \langle \delta \epsilon(x, t) \delta \epsilon(0, 0) \rangle \rangle, \quad (4.5.12)$$

appearing in (4.5.3), by solving the integro-differential equation (4.5.4), which follows from the fluctuating-hydrodynamics framework (4.5.1). This procedure enables a direct comparison between the hydrodynamic prediction $C(x, t)$ and the correlation function

$$C(n, t) = \langle \delta \hat{h}_n(t) \delta \hat{h}_0(0) \rangle, \quad (4.5.13)$$

computed from the microscopic dynamics governed by the Hamiltonian (4.1.2).

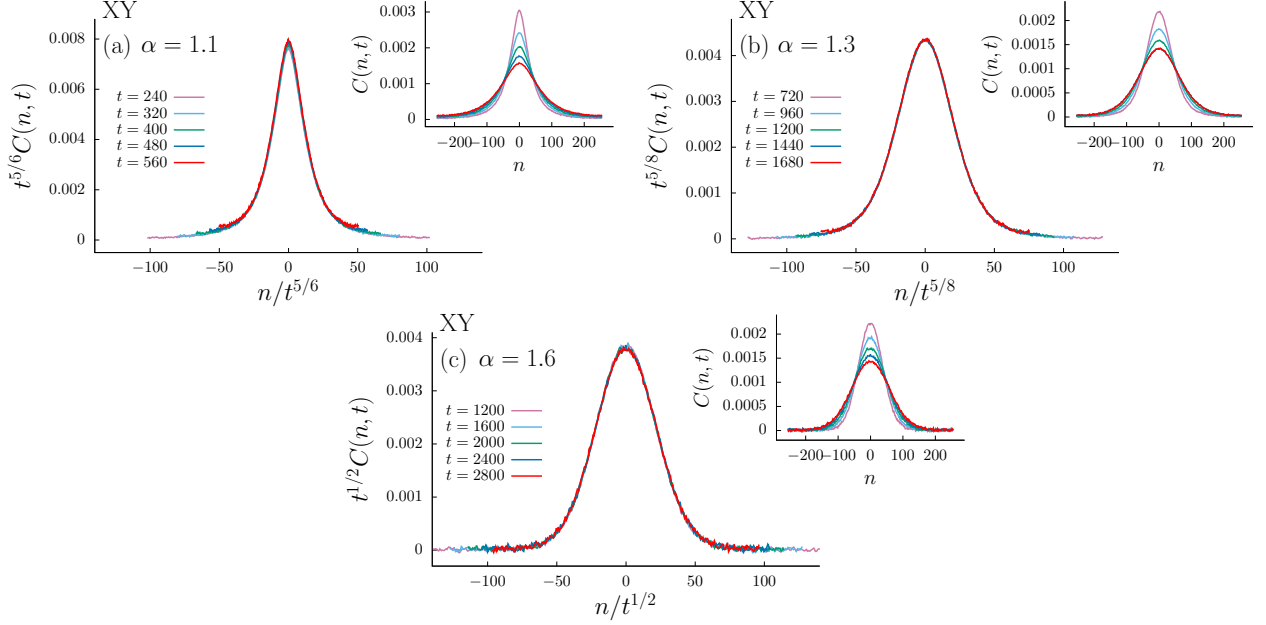


Figure 4.21: Space-time energy correlation function $C(n, t) := \langle \delta \hat{h}_n(t) \delta \hat{h}_0(0) \rangle$ for the XY model with the parameters $J_x = 1, J_y = -0.8$. We use the exponents $\alpha = 1.1$ in (a), $\alpha = 1.3$ in (b), and $\alpha = 1.6$ in (c). The system size is $N = 512$. It is clear that $\alpha = 1.1$ and $\alpha = 1.3$ cases obey the Lévy scaling, while $\alpha = 1.6$ case shows the diffusive scaling. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

Fig. 4.22 presents this comparison for the case $\alpha = 1.3$ in both the transverse Ising and XYZ models. As is evident from the figure, the agreement between the fluctuating-hydrodynamics prediction and the fully microscopic computation is remarkably close. This quantitative consistency provides compelling evidence that fluctuating hydrodynamics with a nonlocal diffusion kernel furnishes an accurate and predictive effective description of the dynamical properties of long-range interacting spin systems.

4.6 Higher dimensional case

In this section, we provide the sufficient condition of normal energy diffusion for prototypical long-range spin systems in D dimensions ($D \geq 2$). The formulation of the Green-Kubo formula in D -dimensional long-range systems is more complicated than in the one-dimensional case. Hence, first, we derive the Green-Kubo formula and the current expression in D dimensions. Next, using Theorem 1, we prove the upper bound for the equal-time total current correlations in D -dimensional case (Theorem 3). We provide separate proofs for the long-range transverse Ising and XYZ model.

Theorem 1 remains valid in arbitrary spatial dimensions. We therefore extend our analysis

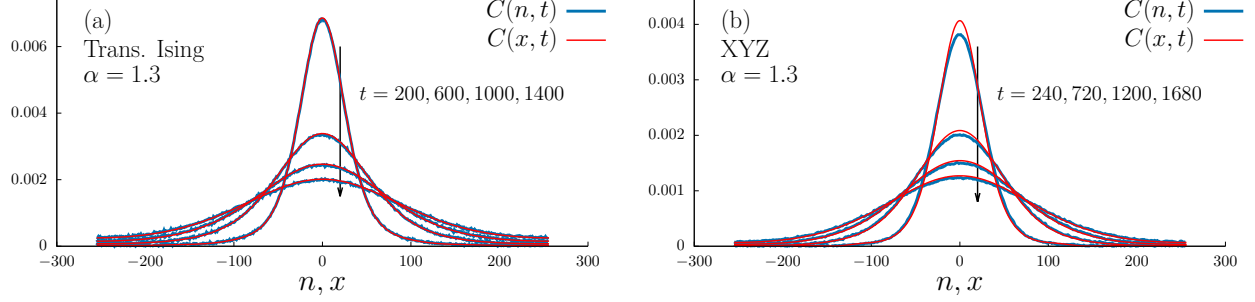


Figure 4.22: Comparison of space-time correlations between $C(n, t)$, obtained directly from the microscopic Hamiltonian, and $C(x, t)$, computed via fluctuating hydrodynamics using the input data $D(n - n')$ computed from a Green-Kubo-like formula. (a) Transverse Ising case with $\alpha = 1.3$, $J = 1$, $h = -0.5$, and $N = 512$; (b) XYZ case with $\alpha = 1.3$, $J_x = 1$, $J_y = 0.8$, $J_z = 0.5$, and $N = 512$. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

to a generic D -dimensional lattice consisting of N^D sites. To characterize the spreading of energy fluctuations, which propagate uniformly in concentric patterns, we consider the mean-square displacement

$$S(t) := \mathcal{N}^{-1} \sum_{i=1}^D \sum_{\mathbf{n}} n_i^2 \langle \delta\epsilon(\mathbf{n}, t) \delta\epsilon(\mathbf{0}, 0) \rangle, \quad (4.6.1)$$

where $\epsilon(\mathbf{n}, t)$ denotes the local energy at site vector $\mathbf{n}$ at time t , and $\mathcal{N} = \sum_{\mathbf{n}} \langle \delta\epsilon(\mathbf{n}) \delta\epsilon(\mathbf{0}) \rangle$ is the normalization constant following Ref. [203]. If the energy transport is normal, the diffusion constant is obtained from the long-time behavior of $S(t)$ as

$$\begin{aligned} \mathfrak{D} &= \lim_{t \rightarrow \infty} \frac{S(t)}{2t} = \frac{1}{k_B T^2 c_V} \int_0^\infty dt C_N^{(\epsilon, D)}(t), \\ C_N^{(\epsilon, D)}(t) &= \frac{1}{N^{D-1}} \sum_{i=1}^D \sum_{n_i=1}^N \langle \mathcal{J}_{n_i|n_i-1}^{(\epsilon, i)}(t) \mathcal{J}_{0|-1}^{(\epsilon, i)}(0) \rangle, \end{aligned} \quad (4.6.2)$$

where c_V is the specific heat per unit volume, and $\mathcal{J}_{n_i|n_i-1}^{(\epsilon, i)}$ denotes the energy current in the i th spatial direction, defined as the total energy transferred across the virtual interface between $\mathbf{n} - \mathbf{e}_i$ and $\mathbf{n}$, with $\mathbf{e}_i$ the unit vector in the i th direction.

Based on this formulation, we establish the following theorem.

Theorem 3. *For D -dimensional long-range interacting spin systems with interaction exponent $\alpha > D$ and spatial dimension $D \geq 2$, whose one-dimensional analogue is given by Eq. (4.1.2), the equal-time current correlation $C_N^{(\epsilon, D)}(0)$ is finite for $\beta < \beta_c$.*

This theorem, together with the general premise that the amplitude of equal-time current correlations determines the qualitative class of energy diffusion, leads to an important conclusion: in all higher dimensions $D \geq 2$, energy diffusion is necessarily normal provided that $\alpha > D$. This condition coincides with the requirement of thermodynamic extensivity, which ensures the finiteness of the specific heat c_V , and thus precludes the emergence of anomalous (Lévy-type) transport in these dimensions.

4.6.1 Formula of the diffusion constant in higher dimensions

In this subsection, we provide the detailed formulation of the energy diffusion constant (4.6.2) through the Green–Kubo expression for thermal conductivity. We begin by considering the continuity equation for the local energy. The energy current is defined as the collection of all energy transfers across a specified hyperplane. To introduce this notion in a D -dimensional lattice, it is necessary to construct subspaces that partition the lattice into two point-symmetric halves. We therefore provide an explicit constructive definition of such subspaces.

The continuity equation for the local energy operator $\hat{h}_{\mathbf{n}}$ at site $\mathbf{n}$ takes the form

$$\begin{aligned} \partial_t \hat{h}_{\mathbf{n}} &= \{\hat{h}_{\mathbf{n}}, H\} = \sum_{\mathbf{m}} \{\hat{h}_{\mathbf{n}}, \hat{h}_{\mathbf{m}}\} = \sum_{\mathbf{m}} t_{\mathbf{n} \leftarrow \mathbf{m}}^\epsilon = \sum_{\mathbf{r} \neq \mathbf{0}} t_{\mathbf{n} \leftarrow \mathbf{n} + \mathbf{r}}^\epsilon \\ &= \sum_{\mathbf{r} \in \mathcal{D}^+} t_{\mathbf{n} \leftarrow \mathbf{n} + \mathbf{r}}^\epsilon - t_{\mathbf{n} - \mathbf{r} \leftarrow \mathbf{n}}^\epsilon = - \sum_{\mathbf{r} \in \mathcal{D}^+} \nabla_{\mathbf{n}; \mathbf{r}} \mathcal{J}_{\mathbf{n}; \mathbf{r}}^\epsilon, \end{aligned} \quad (4.6.3)$$

where the energy transmission operator is defined by

$$t_{\mathbf{i} \leftarrow \mathbf{j}}^\epsilon := \{\hat{h}_{\mathbf{i}}, \hat{h}_{\mathbf{j}}\}, \quad (4.6.4)$$

and the discrete divergence and current are respectively given by

$$\nabla_{\mathbf{n}; \mathbf{r}} A_{\mathbf{n}; \mathbf{r}} := A_{\mathbf{n} + \mathbf{r}; \mathbf{r}} - A_{\mathbf{n}; \mathbf{r}}, \quad (4.6.5)$$

$$\mathcal{J}_{\mathbf{n}; \mathbf{r}}^\epsilon := -t_{\mathbf{n} - \mathbf{r} \leftarrow \mathbf{n}}^\epsilon. \quad (4.6.6)$$

Here, $\mathcal{D}^+$ denotes one of the two point-symmetric partitions of the full lattice excluding the origin. We now present a constructive definition of this set.

Let the underlying space be the D -dimensional torus $\mathbb{T}^D := (\mathbb{Z}/N\mathbb{Z})^D$. We fix an ordering of coordinates $\{i_1, i_2, \dots, i_D\} = \{1, \dots, D\}$ and consider a site $\mathbf{n}$ as the origin. For any vector $\mathbf{x}$, we first decompose the space according to the sign of the i_1 -th coordinate:

$$\mathcal{D}_{i_1}^0 := \{\mathbf{x} \in \mathbb{T}^D \mid x_{i_1} = 0\}, \quad (4.6.7)$$

$$\mathcal{D}_{i_1}^+ := \{\mathbf{x} \in \mathbb{T}^D \mid x_{i_1} > 0\}, \quad (4.6.8)$$

$$\mathcal{D}_{i_1}^- := \{\mathbf{x} \in \mathbb{T}^D \mid x_{i_1} < 0\}. \quad (4.6.9)$$

We then recursively subdivide the zero sets. Specifically, $\mathcal{D}_{i_1}^0$ is decomposed as

$$\mathcal{D}_{i_1; i_2}^0 := \{\mathbf{x} \in \mathcal{D}_{i_1}^0 \mid x_{i_2} = 0\}, \quad (4.6.10)$$

$$\mathcal{D}_{i_1; i_2}^+ := \{\mathbf{x} \in \mathcal{D}_{i_1}^0 \mid x_{i_2} > 0\}, \quad (4.6.11)$$

$$\mathcal{D}_{i_1; i_2}^- := \{\mathbf{x} \in \mathcal{D}_{i_1}^0 \mid x_{i_2} < 0\}. \quad (4.6.12)$$

Continuing in this manner, we define for each $e = 1, \dots, D$ the sets

$$\mathcal{D}_{i_1; \dots; i_e}^0 := \{\mathbf{x} \in \mathcal{D}_{i_1; \dots; i_{e-1}}^0 \mid x_{i_e} = 0\}, \quad (4.6.13)$$

$$\mathcal{D}_{i_1; \dots; i_e}^+ := \{\mathbf{x} \in \mathcal{D}_{i_1; \dots; i_{e-1}}^0 \mid x_{i_e} > 0\}, \quad (4.6.14)$$

$$\mathcal{D}_{i_1; \dots; i_e}^- := \{\mathbf{x} \in \mathcal{D}_{i_1; \dots; i_{e-1}}^0 \mid x_{i_e} < 0\}. \quad (4.6.15)$$

These sets satisfy

$$\mathcal{D}_{i_1; \dots; i_D}^0 = \{\mathbf{0}\}, \quad (4.6.16)$$

$$\bigcup_{\substack{a=\pm \\ e=1, \dots, D}} \mathcal{D}_{i_1; \dots; i_e}^a = \mathbb{T}^D \setminus \{\mathbf{0}\}, \quad (4.6.17)$$

$$\mathcal{D}_{i_1; \dots; i_e}^a \cap \mathcal{D}_{i_1; \dots; i_f}^b = \emptyset \quad (a, b \in \{+, -\}, e, f \in \{1, \dots, D\}). \quad (4.6.18)$$

Since $\mathcal{D}_{i_1; \dots; i_e}^+$ and $\mathcal{D}_{i_1; \dots; i_e}^-$ are point-symmetric with respect to the origin, we define

$$\mathcal{D}^+ := \bigcup_{e=1}^D \mathcal{D}_{i_1; \dots; i_e}^+, \quad \mathcal{D}^- := \bigcup_{e=1}^D \mathcal{D}_{i_1; \dots; i_e}^-. \quad (4.6.19)$$

These satisfy

$$\mathcal{D}^+ \cup \mathcal{D}^- = \mathbb{T}^D \setminus \{\mathbf{0}\}, \quad (4.6.20)$$

$$\mathcal{D}^+ \cap \mathcal{D}^- = \emptyset, \quad (4.6.21)$$

and $\mathcal{D}^+$ and $\mathcal{D}^-$ form a point-symmetric partition of the space. Thus, we obtain a concrete and systematic construction of $\mathcal{D}^+$, which depends on the chosen ordering $(i_1, \dots, i_D)$.

For the two-dimensional case, an explicit illustration of the sets $\mathcal{D}^+$ and $\mathcal{D}^-$ is shown in Fig. 4.23. Defining $\mathcal{D}_{i_1}^+ = \{\mathbf{x} \mid x_{i_1} > 0\}$, $\mathcal{D}_{i_1}^- = \{\mathbf{x} \mid x_{i_1} < 0\}$, and similarly $\mathcal{D}_{i_1; i_2}^\pm = \{\mathbf{x} \mid x_{i_1} = 0, x_{i_2} \gtrless 0\}$, we obtain $\mathcal{D}^a = \mathcal{D}_{i_1}^a \cup \mathcal{D}_{i_1; i_2}^a$ for $a = \pm$. As indicated in the figure, these sets partition the lattice into two complementary halves, excluding the origin.

Next, on the basis of the continuity equation (4.6.3), we derive the Green–Kubo formula in D dimensions under the assumption that Fourier’s law is valid. In the diffusive regime, the mean-squared displacement $S(t)$ is related to the diffusion coefficient $\mathfrak{D}$ through

$$S(t) = \left\langle \sum_{i=1}^D n_i^2(t) \right\rangle = \sum_{\mathbf{n}} \sum_{i=1}^D n_i^2 P(\mathbf{n}, t) = \frac{1}{k_B T^2 c_V} \sigma(t) \sim 2\mathfrak{D}t, \quad (4.6.22)$$

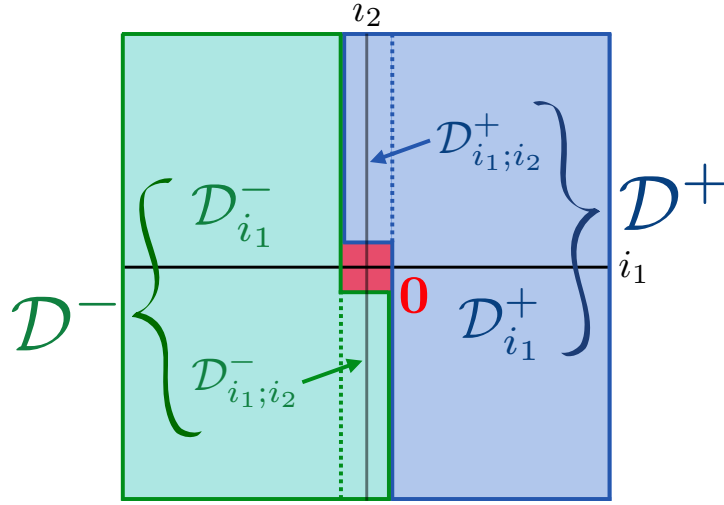


Figure 4.23: The schematic definition of $\mathcal{D}^+ = \mathcal{D}_{i_1}^+ \cup \mathcal{D}_{i_1;i_2}^+$ and $\mathcal{D}^- = \mathcal{D}_{i_1}^- \cup \mathcal{D}_{i_1;i_2}^-$ in 2-dimensional case. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

where $P(\mathbf{n}, t)$ denotes the probability density of displacement $\mathbf{n}$, defined by

$$P(\mathbf{n}, t) = \mathcal{N}^{-1} \langle \delta \hat{h}_{\mathbf{n}}(t) \delta \hat{h}_{\mathbf{0}} \rangle, \quad \mathcal{N} := \sum_{\mathbf{n}} \langle \delta \hat{h}_{\mathbf{n}} \delta \hat{h}_{\mathbf{0}} \rangle, \quad (4.6.23)$$

and the spatial second moment of the energy distribution is given by

$$\sigma(t) := \sum_{i=1}^D \sum_{\mathbf{n}} n_i^2 \langle \delta \hat{h}_{\mathbf{n}}(t) \delta \hat{h}_{\mathbf{0}} \rangle. \quad (4.6.24)$$

We now establish the relation between this spatial energy profile and the correlation function of the energy current. To this end, we divide the space according to the domain $\mathcal{D}^+ := \mathcal{D}^+(i, i_2, \dots, i_D)$ introduced previously. Employing the continuity equation (4.6.3), one obtains the following expression for $\sigma(t)$:

$$\begin{aligned} \sigma(t) &= \sum_{i=1}^D \sum_{\mathbf{n}} n_i^2 \langle \delta \hat{h}_{\mathbf{n}}(t) \delta \hat{h}_{\mathbf{0}}(0) \rangle \\ &= \sum_{i=1}^D \sum_{\mathbf{n}} n_i^2 \left\langle - \left(\int_0^t ds \sum_{\mathbf{r} \in \mathcal{D}^+} \nabla_{\mathbf{r}} \mathcal{J}_{\mathbf{n}:\mathbf{r}}^{\epsilon}(s) \right) \delta \hat{h}_{\mathbf{0}}(0) \right\rangle \\ &= \sum_{i=1}^D \sum_{\mathbf{n}} \sum_{\mathbf{r} \in \mathcal{D}^+} (2r_i n_i + r_i^2) \int_0^t ds \langle \mathcal{J}_{\mathbf{n}:\mathbf{r}}^{\epsilon}(s) \delta \hat{h}_{\mathbf{0}}(0) \rangle \\ &= \sum_{i=1}^D \sum_{\mathbf{n}} \sum_{\mathbf{r} \in \mathcal{D}^+} (2r_i n_i + r_i^2) \int_0^t ds \langle \mathcal{J}_{\mathbf{n}:\mathbf{r}}^{\epsilon}(0) \delta \hat{h}_{\mathbf{0}}(-s) \rangle \\ &= \sum_{i=1}^D \sum_{\mathbf{n}} \sum_{\mathbf{r} \in \mathcal{D}^+} (2r_i n_i + r_i^2) \int_0^t ds \left\langle \mathcal{J}_{\mathbf{n}:\mathbf{r}}^{\epsilon}(0) \left(- \sum_{\mathbf{r}' \in \mathcal{D}^+} \int_0^s ds' \nabla_{\mathbf{r}'} \mathcal{J}_{\mathbf{0}:\mathbf{r}'}^{\epsilon}(-s') \right) \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^D \sum_{\mathbf{n}} \sum_{\mathbf{r}, \mathbf{r}' \in \mathcal{D}^+} (2r_i n_i + r_i^2) \int_0^t ds \left\langle -(\mathcal{J}_{\mathbf{n};\mathbf{r}}^\epsilon(0) - \mathcal{J}_{\mathbf{n}-\mathbf{r}';\mathbf{r}}^\epsilon(0)) \left(\int_0^s ds' \mathcal{J}_{\mathbf{0};\mathbf{r}'}^\epsilon(-s') \right) \right\rangle \\
&= \sum_{i=1}^D \sum_{\mathbf{n}} \sum_{\mathbf{r}, \mathbf{r}' \in \mathcal{D}^+} 2r_i r'_i \int_0^t ds \int_0^s ds' \langle \mathcal{J}_{\mathbf{n};\mathbf{r}}^\epsilon(0) \mathcal{J}_{\mathbf{0};\mathbf{r}'}^\epsilon(-s') \rangle \\
&= \sum_{i=1}^D \sum_{\mathbf{n}} \sum_{\mathbf{r}, \mathbf{r}' \in \mathcal{D}^+} 2r_i r'_i \int_0^t ds \int_0^s ds' \langle \mathcal{J}_{\mathbf{n};\mathbf{r}}^\epsilon(s') \mathcal{J}_{\mathbf{0};\mathbf{r}'}^\epsilon(0) \rangle \\
&= \sum_{i=1}^D \frac{1}{N^D} \left\langle \left(\int_0^t ds \sum_{\mathbf{n}} \sum_{\mathbf{r} \in \mathcal{D}_i^+} r_i \mathcal{J}_{\mathbf{n};\mathbf{r}}^\epsilon(s) \right) \left(\int_0^t ds' \sum_{\mathbf{n}'} \sum_{\mathbf{r}' \in \mathcal{D}^+} r'_i \mathcal{J}_{\mathbf{n}';\mathbf{r}'}^\epsilon(s') \right) \right\rangle \\
&= \sum_{i=1}^D \frac{1}{N^D} \left\langle \left(\int_0^t ds \sum_{n_i} \mathcal{J}_{n_i|n_i-1}^{(\epsilon,i)}(s) \right) \left(\int_0^t ds' \sum_{n'_i} \mathcal{J}_{n'_i|n'_i-1}^{(\epsilon,i)}(s') \right) \right\rangle, \tag{4.6.25}
\end{aligned}$$

where we define the energy current in the i th direction as

$$\mathcal{J}_{n_i|n_i-1}^{(\epsilon,i)} := \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell: \ell_i < n_i}} t_{\mathbf{k} \leftarrow \ell}^\epsilon, \tag{4.6.26}$$

which represents the total energy transmission $t_{\mathbf{k} \leftarrow \ell}^\epsilon$ across the hyperplane separating $\mathbf{n} - \mathbf{e}_i$ and $\mathbf{n}$. As illustrated schematically in Fig. 4.24, the quantity $\mathcal{J}_{n_i|n_i-1}^{(\epsilon,i)}$ captures all contributions to the energy transmission crossing this boundary, thereby enabling a consistent formulation of the Green–Kubo expression for energy diffusion in D dimensions.

Thus, when Fourier’s law is valid, we obtain the Green–Kubo formula for the diffusion coefficient in D dimensions:

$$\mathfrak{D} = \lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{k_B T^2 c_V} \int_0^t ds C_N^{(\epsilon,D)}(s), \tag{4.6.27}$$

$$C_N^{(\epsilon,D)}(s) = N^{1-D} \sum_{i=1}^D \sum_{n_i} \langle \mathcal{J}_{n_i|n_i-1}^{(\epsilon,i)}(s) \mathcal{J}_{0|-1}^{(\epsilon,i)}(0) \rangle. \tag{4.6.28}$$

In D dimensions, prototypical long-range interacting spin systems are generally expected to be non-integrable. It is therefore natural to assume that the current autocorrelation function $C_N^{(\epsilon,D)}(t)$ decays rapidly in time. Under this assumption, the magnitude of the equal-time current correlations plays a central role in determining the energy diffusion coefficient. Motivated by this observation, we now turn to an analysis of the amplitude of $C_N^{(\epsilon,D)}(0)$, which may be interpreted as the equal-time total energy current correlation.

4.7 Proof of Theorem 3

In this section, we prove Theorem 3 to derive the upper bound for the equal-time total current correlation $C_N^{(D)}(0)$ in D dimensions. Under the assumptions that c_V is finite and that $C_N^{(D)}(t)$ decays rapidly in time, the amplitude of $C_N^{(D)}(0)$ determines the sufficient condition for normal energy diffusion. We provide the separated proofs for the transverse Ising and XYZ models.

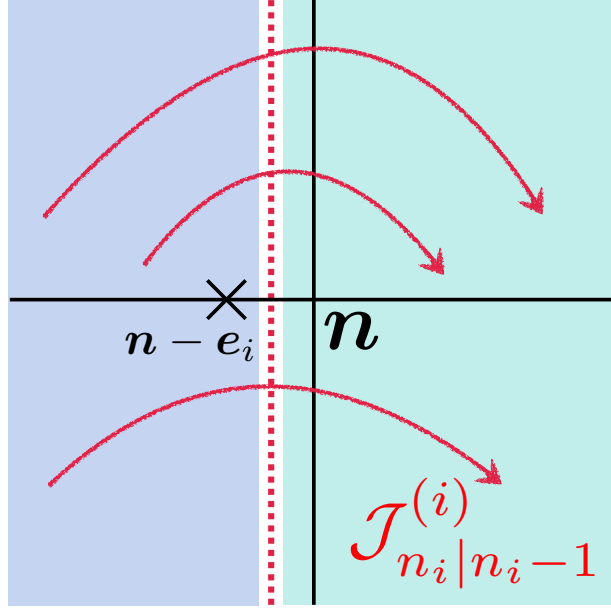


Figure 4.24: The schematic definition of the current $\mathcal{J}_{n_i | n_i - 1}^{(\epsilon, i)}$ in i th direction. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

4.7.1 Transverse Ising model

We first study the D -dimensional long-range transverse Ising model, whose Hamiltonian is given by

$$H = - \sum_{i \neq j} J \frac{S_i^z S_j^z}{d_{i,j}^\alpha} + \sum_i h S_i^x. \quad (4.7.1)$$

The local energy at site $\mathbf{n}$ is defined as

$$\hat{h}_{\mathbf{n}} := - \sum_{j \neq \mathbf{n}} \frac{J}{2} \frac{S_{\mathbf{n}}^z S_j^z}{d_{\mathbf{n},j}^\alpha} + h S_{\mathbf{n}}^x, \quad (4.7.2)$$

and the current $\mathcal{J}_{n_i | n_i - 1}^{(\epsilon, i)}$ is expressed as

$$\mathcal{J}_{n_i | n_i - 1}^{(\epsilon, i)} = \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell: \ell_i < n_i}} t_{\mathbf{k} \leftarrow \ell}^\epsilon, \quad (4.7.3)$$

$$t_{\mathbf{k} \leftarrow \ell}^\epsilon = \{\hat{h}_{\mathbf{k}}, \hat{h}_{\ell}\} = \frac{Jh}{2} \frac{S_{\mathbf{k}}^z S_{\ell}^y - S_{\mathbf{k}}^y S_{\ell}^z}{d_{\mathbf{k},\ell}^\alpha}. \quad (4.7.4)$$

Here, we establish the transverse Ising part of Theorem 3 as follows:

Theorem 3.1. *For the D -dimensional long-range transverse Ising model ($D \geq 2$), under the conditions $\alpha > D/2 + 1$, $\alpha > D$, and $\beta < \beta_c$, the equal-time total current correlation $C_N^{(D)}(0)$*

converges to a finite value independent of the system size N . Consequently, $C_N^{(D)}(0)$ remains finite for all $\alpha > D$.

Proof. The equal-time total current correlation $C_N^{(D)}(0)$ can be written as

$$C_N^{(D)}(0) = N^{1-D} \sum_{i=1}^D \sum_{n_i} \langle \mathcal{J}_{n_i|n_i-1}^{(\epsilon,i)} \mathcal{J}_{0|-1}^{(\epsilon,i)} \rangle. \quad (4.7.5)$$

Following exactly the same line of reasoning as in the one-dimensional case, and invoking Theorem 1 together with the relevant symmetries implied by the Hamiltonian, we obtain the following upper bound for the equal-time current correlation:

$$\left| \langle \mathcal{J}_{n_i|n_i-1}^{(\epsilon,i)} \mathcal{J}_{0|-1}^{(\epsilon,i)} \rangle \right| \leq \mathcal{O}(1) \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell': \ell'_i < 0}} \frac{1}{d_{k,\ell'}^{2\alpha}} \quad (4.7.6)$$

For $D \geq 2$, the right-hand side admits the bound

$$\begin{aligned} \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell': \ell'_i < 0}} \frac{1}{d_{k,\ell'}^{2\alpha}} &\leq \int_{\substack{x_i \geq n_i \\ y_i < 0}} dx_i dy_i \prod_{k \neq i} \int_{-N/2}^{N/2} dx_k dy_k \left\{ (x_i - y_i)^2 + \sum_{j \neq i} (x_j - y_j)^2 \right\}^{-2\alpha/2} \\ &\leq \Gamma_{D-1} \int_{-N/2}^{N/2} d^{D-1} \mathbf{u} \int_{\substack{x_i \geq n_i \\ y_i < 0}} dx_i dy_i \int_0^\infty d\xi \xi^{D-2} \{ (x_i - y_i)^2 + \xi^2 \}^{-2\alpha/2} \\ &\leq \frac{\Gamma_{D-1} B(\frac{D-1}{2}, \alpha - \frac{D-1}{2})}{2} N^{D-1} \int_0^{N/2} dx dy (n_i + x + y)^{D-1-2\alpha} \\ &\leq \frac{\Gamma_{D-1} B(\frac{D-1}{2}, \alpha - \frac{D-1}{2})}{2(D-2\alpha)(D+1-2\alpha)} N^{D-1} n_i^{D+1-2\alpha}, \end{aligned} \quad (4.7.7)$$

where $u_j = (x_j + y_j)/2$ for $j \neq i$, $\xi = \left(\sum_{j \neq i} (x_j - y_j)^2 \right)^{1/2}$, Γ_{D-1} denotes the surface-area coefficient of the $(D-2)$ -dimensional sphere, and $B(x, y)$ is the Beta function.

Consequently, the dependence of $C_N^{(D)}(0)$ on the system size N is obtained in a straightforward manner:

$$\left| C_N^{(D)}(0) \right| \leq \mathcal{O}(1) N^{D+2-2\alpha} + \text{const.} \quad (4.7.8)$$

Thus, we conclude that $C_N^{(D)}(0)$ converges to a finite value in the thermodynamic limit whenever $\alpha > D/2 + 1$. $\square$

4.7.2 XYZ model

We consider the D dimensional long-range interacting XYZ model, which is described by the following Hamiltonian

$$H = - \sum_{\mathbf{i} \neq \mathbf{j}} \sum_{\sigma=x,y,z} J_\sigma \frac{S_i^\sigma S_j^\sigma}{d_{i,j}^\alpha}. \quad (4.7.9)$$

The local energy at site $\mathbf{n}$ is defined as

$$\hat{h}_{\mathbf{n}} := - \sum_{j \neq \mathbf{n}} \sum_{\sigma=x,y,z} \frac{J_{\sigma}}{2} \frac{S_{\mathbf{n}}^{\sigma} S_j^{\sigma}}{d_{\mathbf{n},j}^{\alpha}}, \quad (4.7.10)$$

and the current $\mathcal{J}_{n_i|n_i-1}^{(\epsilon,i)}$ is described as follows

$$\mathcal{J}_{n_i|n_i-1}^{(\epsilon,i)} = - \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell: \ell_i < n_i}} t_{\mathbf{k} \leftarrow \ell}^{\epsilon} \quad (4.7.11)$$

$$t_{\mathbf{k} \leftarrow \ell}^{\epsilon} = \{\hat{h}_{\mathbf{k}}, \hat{h}_{\ell}\} = \sum_{\mathbf{m}} \sum_{\sigma \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2}}{4} \sum_{\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} = \{\mathbf{k}, \ell, \mathbf{m}\}} \tilde{\epsilon}_{abc} \frac{S_{\mathbf{a}}^{\sigma_1} S_{\mathbf{b}}^{\sigma_2} S_{\mathbf{c}}^{\sigma_3}}{d_{\mathbf{a},\mathbf{c}}^{\alpha} d_{\mathbf{b},\mathbf{c}}^{\alpha}}, \quad (4.7.12)$$

where $\tilde{\epsilon}_{abc}$ is defined as follows:

$$\tilde{\epsilon}_{abc} = \begin{cases} +1, & \text{for } (\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{k}, \ell, \mathbf{m}), (\mathbf{m}, \ell, \mathbf{k}), (\mathbf{k}, \mathbf{m}, \ell), \\ -1, & \text{for } (\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\ell, \mathbf{k}, \mathbf{m}), (\ell, \mathbf{m}, \mathbf{k}), (\mathbf{m}, \mathbf{k}, \ell). \end{cases} \quad (4.7.13)$$

Note that $\tilde{\epsilon}_{abc}$ is different from Levi-Civita symbols.

Here, we state the XYZ part of Theorem 3 as follows:

Theorem 3.2. *In the D -dimensional long-range interacting XYZ model, under the condition $\alpha > D/2 + 1$, $\alpha > D$ and $\beta < \beta_c$, $C_N^{(D)}(0)$ converges to the finite value independent of the system size N . It immediately follows that $C_N^{(D)}(0)$ is finite as long as $\alpha > D$.*

Proof. In complete analogy with the one-dimensional analysis, and by invoking Theorem 1 together with the symmetry properties implied by the Hamiltonian, all contributions to the current-current correlation, except for case (2) in Fig. 4.8, admit the following bound:

$$\left| \left\langle \mathcal{J}_{n_i|n_i-1}^{(\epsilon,i)} \mathcal{J}_{0|-1}^{(\epsilon,i)} \right\rangle \right|_{\text{except Fig. 4.8(2)}} \leq \mathcal{O}(1) \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell': \ell'_i < 0}} \frac{1}{d_{\mathbf{k}, \ell'}^{2\alpha}} \quad (4.7.14)$$

We now turn to the remaining contribution, namely case (2) in Fig. 4.8. In this configuration, the triplets $(\mathbf{k}, \ell, \mathbf{m})$ and $(\mathbf{k}', \ell', \mathbf{m}')$ each form a closed loop, and these two loops are connected by a single edge. A schematic representation of this structure is provided in Fig. 4.25.

Let $\mathbf{p} \in \{\mathbf{k}, \ell, \mathbf{m}\}$ and $\mathbf{p}' \in \{\mathbf{k}', \ell', \mathbf{m}'\}$ denote the endpoints of the unique edge connecting the two loops. In the example shown in Fig. 4.25, these correspond to $\mathbf{p} = \ell$ and $\mathbf{p}' = \mathbf{m}'$.

Next, we introduce the sites

$$\mathbf{q} \in \{\mathbf{k}, \ell, \mathbf{m}\} \setminus \{\mathbf{p}\}, \quad \mathbf{q}' \in \{\mathbf{k}', \ell', \mathbf{m}'\} \setminus \{\mathbf{p}'\},$$

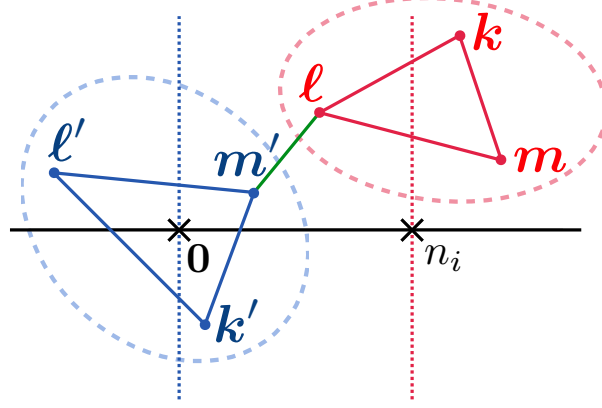


Figure 4.25: The schematic diagram of the case (2) in Fig.4.8 in D dimensions. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

which are defined according to their positions relative to the hyperplanes $x_i = n_i$ and $x_i = 0$:

$$\mathbf{q} := \begin{cases} \mathbf{k}, & \text{if } \mathbf{p} \in \{\mathbf{x} \mid x_i \geq n_i\}, \\ \mathbf{\ell}, & \text{if } \mathbf{p} \in \{\mathbf{x} \mid x_i < n_i\}, \end{cases} \quad (4.7.15)$$

$$\mathbf{q}' := \begin{cases} \mathbf{\ell}', & \text{if } \mathbf{p}' \in \{\mathbf{x} \mid x_i \geq 0\}, \\ \mathbf{k}', & \text{if } \mathbf{p}' \in \{\mathbf{x} \mid x_i < 0\}. \end{cases} \quad (4.7.16)$$

By construction, the pairs $(\mathbf{p}, \mathbf{q})$ and $(\mathbf{p}', \mathbf{q}')$ lie on opposite sides of their respective hyperplanes, and therefore satisfy

$$(p_i - n_i)(q_i - n_i) \leq 0, \quad p'_i q'_i \leq 0.$$

In the explicit configuration displayed in Fig. 4.25, we indeed have $\mathbf{q} = \mathbf{k}$ and $\mathbf{q}' = \mathbf{\ell}'$.

Proceeding in parallel with the one-dimensional argument, we obtain the following bound for the contributions of case (2) in Fig. 4.8:

$$\left| \left\langle \mathcal{J}_{n_i|n_i-1}^{(\epsilon, i)} \mathcal{J}_{0|-1}^{(\epsilon, i)} \right\rangle \right|_{\text{Fig. 4.8(2)}} \leq \mathcal{O}(1) \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p}', \mathbf{q}'} \frac{1}{d_{\mathbf{q}, \mathbf{p}}^{2\alpha}} \frac{1}{d_{\mathbf{p}, \mathbf{p}'}^{\alpha}} \frac{1}{d_{\mathbf{p}', \mathbf{q}'}^{2\alpha}} \quad (4.7.17)$$

Firstly, we estimate the summations over $\mathbf{q}$ and $\mathbf{q}'$. For the sum over $\mathbf{q}$, we obtain the

following bound:

$$\begin{aligned}
\sum_{\mathbf{q}: (p_i - n_i)(q_i - n_i) \leq 0} \frac{1}{d_{\mathbf{q}, \mathbf{p}}^{2\alpha}} &\leq \int_{(p_i - n_i)(q_i - n_i) \leq 0} d^d \mathbf{q} \frac{1}{\{(q_i - p_i)^2 + \sum_{j \neq i} (q_j - p_j)^2\}^{2\alpha/2}} \\
&\leq \int_{(p_i - n_i)(q_i - n_i) \leq 0} dq_i \Gamma_{D-1} \int_0^\infty d\xi \xi^{D-2} \frac{1}{\{(q_i - p_i)^2 + \xi^2\}^{2\alpha/2}} \\
&\leq \Gamma' \int_{(p_i - n_i)(q_i - n_i) \leq 0} dq_i \frac{1}{(q_i - p_i)^{2\alpha - D + 1}} \\
&\leq \Gamma' \int_0^{N/2} dx \frac{1}{(x + n_i - p_i)^{2\alpha - D + 1}} \\
&\leq \Gamma'' \frac{1}{(n_i - p_i)^{2\alpha - D}} = \Gamma'' \frac{1}{d_{\bar{\mathbf{n}}_{\mathbf{p}}, \mathbf{p}}^{2\alpha - D}}, \tag{4.7.18}
\end{aligned}$$

where $\Gamma' = \Gamma_{D-1} B(\frac{D-1}{2}, \alpha - \frac{D-1}{2})/2$, $\Gamma'' = \Gamma'/(2\alpha - D)$ and we denote $\bar{\mathbf{n}}_{\mathbf{p}}$ as the foot of the perpendicular dropped from $\mathbf{p}$ to the hyperplane $x_i = n_i$, as illustrated in the leftmost part in Fig.4.26.

Similarly, the summation over $\mathbf{q}'$ is bounded as

$$\sum_{\mathbf{q}': p'_i q'_i \leq 0} \frac{1}{d_{\mathbf{p}', \mathbf{q}'}^{2\alpha}} \leq \Gamma'' \frac{1}{d_{\mathbf{p}', \bar{\mathbf{0}}_{\mathbf{p}'}}^{2\alpha - D}} \tag{4.7.19}$$

where we denote $\bar{\mathbf{0}}_{\mathbf{p}'}$ as the foot of the perpendicular dropped from $\mathbf{p}'$ to the hyperplane $x_i = 0$, as depicted in the leftmost part in Fig.4.26.

Combining these bounds, we obtain

$$\left| \left\langle \mathcal{J}_{n_i | n_i - 1}^{(\epsilon, i)} \mathcal{J}_{0 | -1}^{(\epsilon, i)} \right\rangle \right|_{\text{Fig. 4.8(2)}} \leq \sum_{\mathbf{p}, \mathbf{p}'} \frac{1}{d_{\mathbf{p}, \mathbf{p}'}^{2\alpha - D}} \frac{1}{d_{\mathbf{p}, \mathbf{p}'}^\alpha} \frac{1}{d_{\mathbf{p}', \bar{\mathbf{0}}_{\mathbf{p}'}}^{2\alpha - D}} \leq \sum_{\mathbf{p}, \mathbf{p}'} \frac{1}{d_{\bar{\mathbf{n}}_{\mathbf{p}}, \mathbf{p}}^\alpha} \frac{1}{d_{\mathbf{p}, \mathbf{p}'}^\alpha} \frac{1}{d_{\mathbf{p}', \bar{\mathbf{0}}_{\mathbf{p}'}}^\alpha}, \tag{4.7.20}$$

where in advance, we assume $\alpha > D$.

To proceed further with the evaluation of this upper bound, we now present the following lemma.

Lemma 5. *If $\mathbf{i}, \mathbf{j}$ depend on $\mathbf{k}$ and $\tilde{u}_i := \sum_{\mathbf{k}} 1/d_{\mathbf{k}, \mathbf{i}(\mathbf{k})}^\alpha < \infty$, $\tilde{u}_j := \sum_{\mathbf{k}} 1/d_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha < \infty$ hold, under the condition $\alpha > D$, the following inequality holds:*

$$\sum_{\mathbf{k}} \frac{1}{d_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha} \frac{1}{d_{\mathbf{j}(\mathbf{k}), \mathbf{k}}^\alpha} \leq \frac{\tilde{u}}{d_{\bar{\mathbf{i}}, \bar{\mathbf{j}}}^\alpha}, \tag{4.7.21}$$

where $d_{\bar{\mathbf{i}}, \bar{\mathbf{j}}}^{-\alpha} := \sup_{\mathbf{k}} d_{\mathbf{i}(\mathbf{k}), \mathbf{j}(\mathbf{k})}^{-\alpha}$, and $\tilde{u} := 2^{\alpha-1}(\tilde{u}_i + \tilde{u}_j)$.

Proof of Lemma 5.

$$\begin{aligned}
\sum_{\mathbf{k}} \frac{1}{d_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha} \frac{1}{d_{\mathbf{j}(\mathbf{k}), \mathbf{k}}^\alpha} &= \sum_{\mathbf{k}} \frac{1}{d_{\mathbf{i}(\mathbf{k}), \mathbf{j}(\mathbf{k})}^\alpha} \frac{d_{\mathbf{i}(\mathbf{k}), \mathbf{j}(\mathbf{k})}^\alpha}{d_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha d_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha} \leq \frac{1}{d_{\bar{\mathbf{i}}, \bar{\mathbf{j}}}^\alpha} \sum_{\mathbf{k}} \frac{(d_{\mathbf{i}(\mathbf{k}), \mathbf{k}} + d_{\mathbf{k}, \mathbf{j}(\mathbf{k})})^\alpha}{d_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha d_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha} \\
&\leq 2^{\alpha-1} \frac{1}{d_{\bar{\mathbf{i}}, \bar{\mathbf{j}}}^\alpha} \sum_{\mathbf{k}} \frac{d_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha + d_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha}{d_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha d_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha} = 2^{\alpha-1} \frac{1}{d_{\bar{\mathbf{i}}, \bar{\mathbf{j}}}^\alpha} \sum_{\mathbf{k}} \left(\frac{1}{d_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha} + \frac{1}{d_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha} \right) = \frac{\tilde{u}}{d_{\bar{\mathbf{i}}, \bar{\mathbf{j}}}^\alpha} \tag{4.7.22}
\end{aligned}$$

Here, we use $\{(x+y)/2\}^\alpha \leq (x^\alpha + y^\alpha)/2$, following from the convexity of $f(x) = x^\alpha$ and the convexity inequality. $\square$

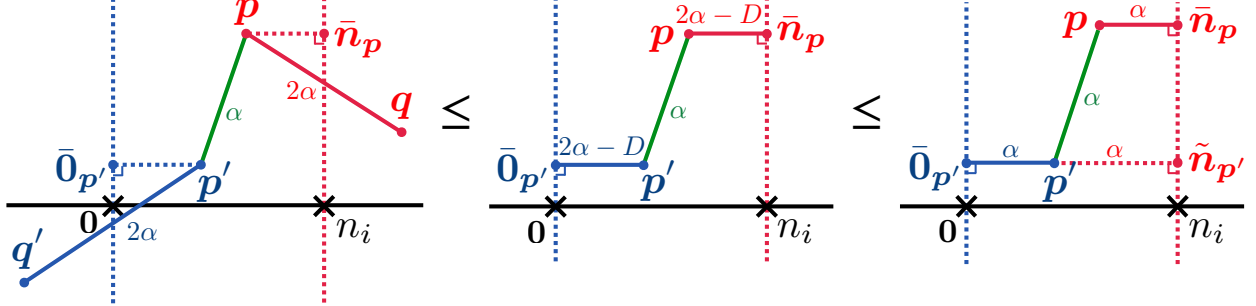


Figure 4.26: The diagrammatic derivation for the upper bound of the case (2) in Fig.4.8 in D dimensions. [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

Finally, utilizing Lemma.5, we obtain the following upper bound:

$$\begin{aligned}
\left| \left\langle \mathcal{J}_{n_i|n_i-1}^{(\epsilon,i)} \mathcal{J}_{0|-1}^{(\epsilon,i)} \right\rangle \right|_{\text{Fig.4.8(2)}} &\leq \sum_{\mathbf{p}, \mathbf{p}'} \frac{1}{d_{\bar{\mathbf{n}}_{\mathbf{p}}, \mathbf{p}}^\alpha} \frac{1}{d_{\mathbf{p}, \mathbf{p}'}^\alpha} \frac{1}{d_{\mathbf{p}', \bar{\mathbf{0}}_{\mathbf{p}'}}^\alpha} \\
&\leq \sum_{\mathbf{p}'} \frac{1}{d_{\mathbf{p}', \bar{\mathbf{0}}_{\mathbf{p}'}}^\alpha} 2^{\alpha-1} (u + \tilde{u}_{\bar{\mathbf{n}}_{\mathbf{p}}}) \sup_{\mathbf{p}} \frac{1}{d_{\bar{\mathbf{n}}_{\mathbf{p}}, \mathbf{p}}^\alpha} \\
&= 2^{\alpha-1} (u + \tilde{u}_{\bar{\mathbf{n}}_{\mathbf{p}}}) \sum_{\mathbf{p}'} \frac{1}{d_{\bar{\mathbf{n}}_{\mathbf{p}'}, \mathbf{p}'}^\alpha} \frac{1}{d_{\mathbf{p}', \bar{\mathbf{0}}_{\mathbf{p}'}}^\alpha} \\
&\leq 2^{\alpha-1} (u + \tilde{u}_{\bar{\mathbf{n}}_{\mathbf{p}}}) 2^{\alpha-1} (\tilde{u}_{\bar{\mathbf{0}}_{\mathbf{p}'}} + \tilde{u}_{\bar{\mathbf{n}}_{\mathbf{p}'}}) \sup_{\mathbf{p}'} \frac{1}{d_{\bar{\mathbf{n}}_{\mathbf{p}'}, \bar{\mathbf{0}}_{\mathbf{p}'}}^\alpha} = \mathcal{O}(1) \frac{1}{n_i^\alpha}, \quad (4.7.23)
\end{aligned}$$

where $\tilde{u}_{\bar{\mathbf{n}}_{\mathbf{p}}} = \sum_{\mathbf{p}} d_{\bar{\mathbf{n}}_{\mathbf{p}}, \mathbf{p}}^{-\alpha} \leq \int_1^{N/2} d^d \mathbf{x} x_i^{-\alpha} \leq \mathcal{O}(1) N^{D-\alpha} < \infty$ for $\alpha > D$. Furthermore, let $\bar{\mathbf{n}}_{\mathbf{p}}$ be the $\bar{\mathbf{n}}_{\mathbf{p}}$ that satisfies $\sup_{\mathbf{p}} d_{\bar{\mathbf{n}}_{\mathbf{p}}, \mathbf{p}}^{-\alpha} = d_{\bar{\mathbf{n}}_{\mathbf{p}}, \mathbf{p}'}^{-\alpha}$. As shown on the far right of Fig.4.26, $\bar{\mathbf{n}}_{\mathbf{p}}$ coincides with the foot of the perpendicular dropped from $\mathbf{p}'$ to the hyperplane $x_i = n_i$. The condition of Lemma 5 is satisfied, as $\tilde{u}_{\bar{\mathbf{0}}_{\mathbf{p}'}} = \sum_{\mathbf{p}'} d_{\bar{\mathbf{0}}_{\mathbf{p}'}, \mathbf{p}'}^\alpha \leq \int_1^{N/2} d^d \mathbf{x} x_i^{-\alpha} < \infty$ and $\tilde{u}_{\bar{\mathbf{n}}_{\mathbf{p}'}} = \sum_{\mathbf{p}'} d_{\bar{\mathbf{n}}_{\mathbf{p}'}, \mathbf{p}'}^\alpha \leq \int_1^{N/2} d^d \mathbf{x} x_i^{-\alpha} < \infty$.

From (4.7.14) and (4.7.23), we obtain the following upper bound for $C_N^{(D)}(0)$:

$$\begin{aligned}
\left| C_N^{(D)}(0) \right| &\leq \mathcal{O}(1) N^{D+2-2\alpha} + \mathcal{O}(1) N^{2-D-\alpha} + \text{const.} \\
&\leq \mathcal{O}(1) N^{D+2-2\alpha} + \text{const.}, \quad (4.7.24)
\end{aligned}$$

for all spatial dimensions $D \geq 2$. The second inequality follows from the fact that the term $\mathcal{O}(1) N^{2-D-\alpha}$ is subleading compared to $\mathcal{O}(1) N^{D+2-2\alpha}$ in higher dimensions.

Finally, we demonstrate that $C_N^{(\epsilon, D)}(0)$ converges to a finite value in the thermodynamic limit provided that $\alpha > D/2 + 1$. Under this condition, the dominant contribution $\mathcal{O}(1) N^{D+2-2\alpha}$ vanishes as $N \rightarrow \infty$, and hence $C_N^{(\epsilon, D)}(0)$ admits a finite, well-defined limit. $\square$

Chapter 5

Spin Diffusion in the Long-range Interacting Spin Systems

We investigate spin diffusion in classical long-range interacting spin systems, focusing on the long-range XXZ model as a prototypical spin-conserving system. In the extensive regime $\alpha > D$, we rigorously establish that the amplitude of equal-time spin current correlations determines the class of spin diffusion. Using the cumulant power-law clustering theorem, we prove that $\alpha > \alpha_c = 3/2$ is a sufficient condition for normal diffusion in one dimension, while normal diffusion persists for $\alpha > D$ in higher dimensions. Numerical results indicate that this bound is tight, with anomalous (Lévy) diffusion emerging for $\alpha < 3/2$, consistent with fluctuating hydrodynamics.

The scaling behavior observed for spin diffusion is completely analogous to that for energy diffusion, suggesting that it may be universal and independent of the specific type of local conserved quantities.

5.1 Model and setup

We consider a one-dimensional, translationally invariant classical spin system characterized by the Hamiltonian

$$H = \sum_{n=1}^N \hat{h}_n, \quad (5.1.1)$$

where $\hat{h}_n$ denotes the local Hamiltonian associated with site n , and N is the total number of lattice sites. As a representative class of spin-conserving models, we focus on the long-range interacting XXZ Hamiltonians, for which the local energy $\hat{h}_n$ is defined as

$$\hat{h}_n^{(\text{XXZ})} = - \sum_{r=1}^{N/2} \frac{1}{2r^\alpha} \left[\sum_{\sigma=x,y} J \left(S_n^\sigma S_{n+r}^\sigma + S_{n-r}^\sigma S_n^\sigma \right) + J_z \left(S_n^z S_{n+r}^z + S_{n-r}^z S_n^z \right) \right]. \quad (5.1.2)$$

The continuous variable S_n^σ denotes the σ component of the spin at site n and satisfies $\sum_\sigma (S_n^\sigma)^2 = 1$. Periodic boundary condition is imposed such that $n \pm N \equiv n$. The exponent α controls the spatial decay of the long-range interactions, and the condition $\alpha > 1$ ensures the thermodynamic extensivity of the Hamiltonian.

The temporal evolution of the spin variables is governed by the classical spin equation of motion (4.1.3).

For generic parameter values, the XXZ Hamiltonian is known to conserve both the total energy and the total spin. In particular, the z -component of the spin serves as a locally conserved density. We investigate the spin transport properties of the system, focusing on the space-time correlation function of spin $\langle S_n^z(t) S_0^z \rangle$ for the long-range XXZ model defined in (5.1.2). This quantity provides a direct probe of spin diffusion and its possible anomalous behavior induced by long-range interactions.

5.2 Spin conductivity and Green-Kubo formula

When the system exhibits normal spin diffusion, the spin diffusion constant $\mathcal{D}^s$ is expressed in terms of the spin conductivity κ^s and the inverse spin susceptibility Λ as $\mathcal{D}^s = \kappa^s \Lambda$, where Λ generally remains finite. Consequently, verifying whether spin diffusion is normal reduces to establishing the finiteness of the spin conductivity. To this end, we focus on the spin conductivity, which is formulated through the Green-Kubo relation involving the equilibrium spin current correlation [264].

In the XXZ model, the local z -component of spin, S_n^z , constitutes a locally conserved quantity, and the associated spin current is defined via the corresponding continuity equation:

$$\partial_t S_n^z = -\mathcal{J}_{n+1}^s + \mathcal{J}_n^s, \quad (5.2.1)$$

where the explicit form of the local spin current for systems with long-range interactions is given by

$$\mathcal{J}_n^s = \sum_{\substack{i \geq n, j \leq n-1 \\ |i-j| \leq N/2}} t_{i \leftarrow j}^s, \quad t_{i \leftarrow j}^s := 2\{S_i^z, \hat{h}_j\}. \quad (5.2.2)$$

Here, $\{\cdot, \cdot\}$ denotes the Poisson bracket for spin dynamics (3.2.4). The quantity $t_{i \leftarrow j}^s$ is interpreted as the infinitesimal spin transfer from site j to site i mediated across the virtual boundary between sites $n-1$ and n . The local spin current $\mathcal{J}_n^s$ thus represents the net spin transfer across this boundary.

The spin conductivity for a finite system of size N is then obtained via the Green-Kubo formula:

$$\kappa_N^s = \Lambda \int_0^\infty dt C_N^s(t), \quad C_N^s(t) = \sum_{n=1}^N \langle \mathcal{J}_n^s(t) \mathcal{J}_0^s \rangle, \quad (5.2.3)$$

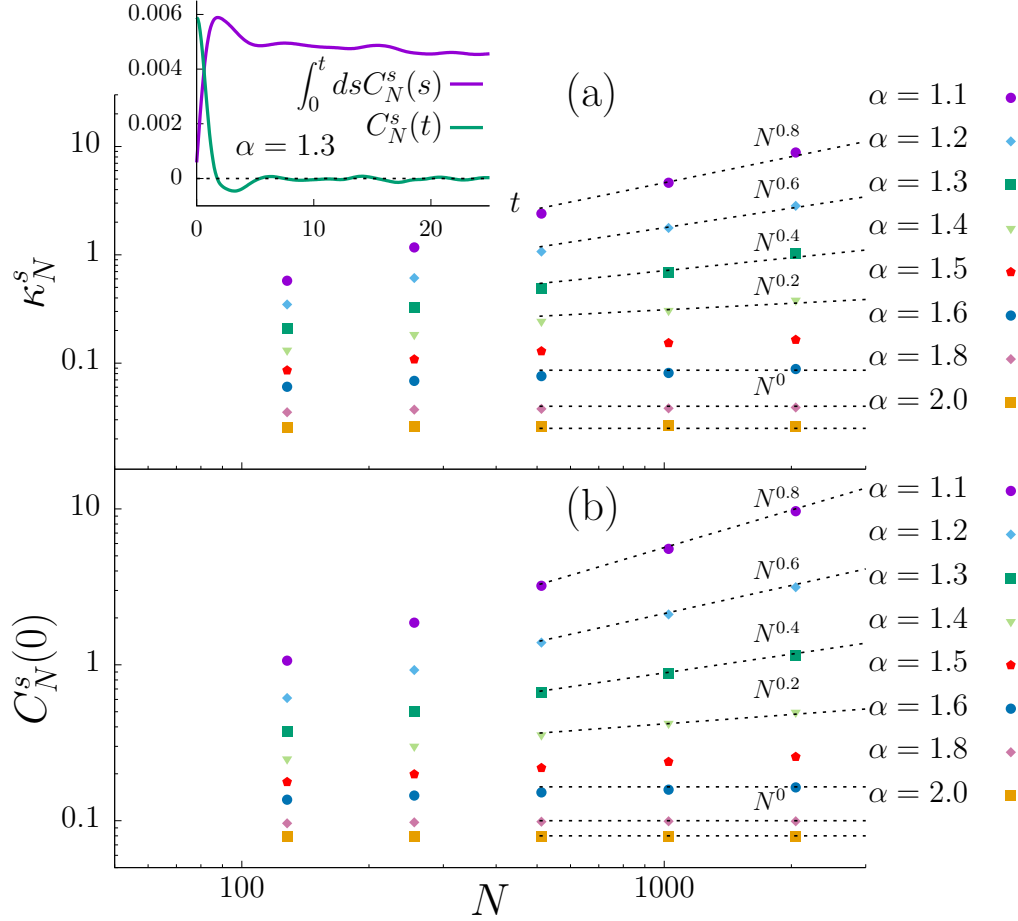


Figure 5.1: The spin conductivity for the long-range XX model with the parameters $J = 1$, and $T = 10$ ($k_B \equiv 1$). (a): κ_N as a function of size N . The inset is the time dependence as well as the integration of the total spin current correlation for $\alpha = 1.3$ and $N = 1024$. (b): $C_N(0)$ as a function of N .

where Λ is the inverse spin susceptibility and $\langle \cdot \rangle$ denotes the equilibrium average at inverse temperature β . The subscript N indicates the explicit dependence on the system size.

Numerical simulations are conducted for the XX model, where the spin dynamics governed by Eq. (4.1.3) are integrated using a fourth-order symplectic scheme with time increment $dt = 0.01$. Initial spin configurations are sampled at temperature T via the Monte Carlo method, and thermal averages are evaluated over 10^6 independent realizations. As shown in the inset of Fig. 1(a), the autocorrelation function $C_N^s(t)$ exhibits a rapid decay, and the running integral $\int_0^t ds C_N^s(s)$ becomes effectively stationary by $t = 20$. Accordingly, we employ $t = 20$ as the cutoff time in the evaluation of κ_N^s . The same sampling size of 10^6 is used consistently for all numerical data presented in this section.

A summary of the results is provided in Fig. 5.1. Figure 5.1(a) displays the system-size dependence of κ_N^s , with the inset highlighting the fast decay of the current correlation

at $\alpha = 1.3$, ensuring the convergence of the Green–Kubo integral. Such convergence is physically expected, as the system is non-integrable and resides on a lattice lacking continuous translational invariance, as seen in Sec. 4.2.

The main panel of Fig. 5.1(a) reveals anomalous behavior for $\alpha < 1.5$, whereas for larger values of α , the conductivity κ_N^s approaches a finite limit, consistent with normal spin diffusion. These observations indicate the existence of a diffusion transition. Importantly, the anomalous behavior does not originate from a long-time tail in the current correlation, as clearly evidenced in the inset of Fig. 5.1(a). Rather, it arises from a distinct mechanism: the anomalous enhancement of the current correlation amplitude, analogous to the case of energy diffusion. This conclusion is corroborated by the system-size dependence of the equal-time equilibrium correlation $C_N^s(0)$, presented in Fig. 5.1(b). The close qualitative agreement between the behaviors of $C_N^s(0)$ and κ_N^s demonstrates that the anomalous spin diffusion in this regime is governed by the amplitude of the equal-time current correlation.

Taken together, these findings provide compelling evidence that the equal-time spin current correlation constitutes a key diagnostic quantity for characterizing the nature of spin diffusion. This structure of diffusion anomaly is completely analogous to the case of energy diffusion.

5.3 Upper bound for the equal-time spin current correlations

In this section, we investigate the one-dimensional long-range interacting XXZ model defined in Eq. (5.1.2). This model satisfies the continuity relations

$$\partial_t u_{a,n} = - \left(\mathcal{J}_{n+1}^a - \mathcal{J}_n^a \right), \quad (5.3.1)$$

$$\mathcal{J}_n^a = - \sum_{\substack{i \geq n, j < n \\ |i-j| < N/2}} t_{i \leftarrow j}^a, \quad (5.3.2)$$

for $a = s, \epsilon$. Here, $u_{a,n}$ denotes the locally conserved quantities, with $u_{s,n} = S_n^z$ and $u_{\epsilon,n} = \hat{h}_n$. The transmission operator $t_{i \leftarrow j}^a$ is defined as

$$t_{i \leftarrow j}^s = 2 \{ S_i^z, \hat{h}_j \} = \frac{J^2}{2} \sum_{\{a,b\}=\{i,j\}} \varepsilon_{ab} \frac{S_a^x S_b^y}{d_{a,b}^\alpha}, \quad (5.3.3)$$

$$t_{i \leftarrow j}^\epsilon = \{ \hat{h}_i, \hat{h}_j \} = \sum_{m=1}^N \sum_{\sigma \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2}}{4} \sum_{\{a,b,c\}=\{i,j,m\}} \tilde{\varepsilon}_{abc} \frac{S_a^{\sigma_1} S_b^{\sigma_2} S_c^{\sigma_3}}{d_{a,c}^\alpha d_{b,c}^\alpha},$$

with $J_x = J_y = J$. The symbol ε_{ab} denotes the Levi–Civita symbol,

$$\varepsilon_{ab} = \begin{cases} +1, & (a, b) = (i, j), \\ -1, & (a, b) = (j, i), \end{cases} \quad (5.3.4)$$

and $\tilde{\varepsilon}_{abc}$ is defined by

$$\tilde{\varepsilon}_{abc} = \begin{cases} +1, & (a, b, c) = (i, j, m), (m, j, i), (i, m, j), \\ -1, & (a, b, c) = (j, i, m), (j, m, i), (m, i, j), \end{cases} \quad (5.3.5)$$

which differs from the standard Levi-Civita tensor. The notation $\sum_{\sigma \in P_{\text{cyc}}^{xyz}}$ represents the sum over cyclic permutations (x, y, z) , (y, z, x) , and (z, x, y) .

Symmetries of the Hamiltonian imply

$$\langle S_i^x \rangle = \langle S_i^y \rangle = \langle S_i^z \rangle = 0, \quad (5.3.6)$$

$$\langle S_i^\sigma S_j^\tau \rangle = \delta_{\sigma, \tau} \langle S_i^\sigma S_j^\sigma \rangle, \quad (\sigma, \tau \in \{x, y\}), \quad (5.3.7)$$

which follow from the invariance of the Hamiltonian under the canonical transformation

$$(S_i^x, S_i^y, S_i^z) \longrightarrow (-S_i^x, -S_i^y, S_i^z), \quad i = 1, \dots, N.$$

From Eqs. (5.3.6) and (5.3.7), it follows immediately that

$$\langle \mathcal{J}_n^a \mathcal{J}_0^{a'} \rangle = \delta_{a, a'} \langle \mathcal{J}_n^a \mathcal{J}_0^a \rangle, \quad (5.3.8)$$

i.e., cross correlations between distinct conserved quantities vanish.

Using these symmetry relations together with Theorem 1, we obtain the following result.

Theorem 4. *For the one-dimensional long-range XXZ model on the infinite line, and for inverse temperature $\beta < \beta_c$, the equal-time spin and energy current correlations satisfy*

$$\left| \langle \mathcal{J}_n^a \mathcal{J}_0^a \rangle \right| \leq c'_a n^{2-2\alpha}, \quad (1 < \alpha < 2), \quad (5.3.9)$$

and

$$\left| \langle \mathcal{J}_n^a \mathcal{J}_0^a \rangle \right| \leq c''_a n^{-\alpha}, \quad (\alpha > 2), \quad (5.3.10)$$

where c'_a and c''_a ($a = s, \epsilon$) are constants independent of n .

The energy-current case has already been established in Theorem 2; the proof for the spin current proceeds analogously. By invoking Theorem 1 and employing an appropriate diagrammatic expansion, one arrives at the bounds stated above. Complete details are provided in Sec. 5.4. Although Theorem 4 is rigorously established only at high temperatures, it is widely expected to extend to general finite temperatures, as Theorem 1 captures physically robust behavior. Furthermore, Theorems 4 and 5 below naturally extend to quantum spin systems.

Theorem 4 plays a central role in elucidating universal features of spin transport. For spin diffusion, it yields the upper bound

$$C_N^s(0) < c' N^{3-2\alpha} + \text{const.}, \quad (5.3.11)$$

in excellent agreement with the numerical results displayed in Fig. 4.2(b). Consequently, one concludes that the condition $\alpha > 3/2$ suffices for normal diffusion, regardless of the specific model details or the particular conserved quantity under consideration. As discussed later, Theorem 4 also provides a theoretical foundation for anomalous diffusion behaviors described by fluctuating hydrodynamics.

5.4 Proof of Theorem 4

Proof of Theorem 4. The equal-time spin current correlation $\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle$ is given as follows:

$$\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle = \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \frac{J^4}{4} \sum_{\substack{\{a,b\}=\{k,\ell\} \\ \{a',b'\}=\{k',\ell'\}}} \varepsilon_{ab} \varepsilon_{a'b'} \frac{\langle S_a^x S_b^y S_{a'}^x S_{b'}^y \rangle}{d_{a,b}^\alpha d_{a',b'}^\alpha}. \quad (5.4.1)$$

The term $1/(d_{a,b}^\alpha d_{a',b'}^\alpha)$ in (5.4.1) is illustrated diagrammatically in Fig.5.2. In Fig.5.2, the vertices of the diagram are mapped to the local areas which corresponds to local quantities within the ensembles (i.e., local spin variables). The edges of the diagram represent the power-law components of the distance between these areas, indicating the connections among them.

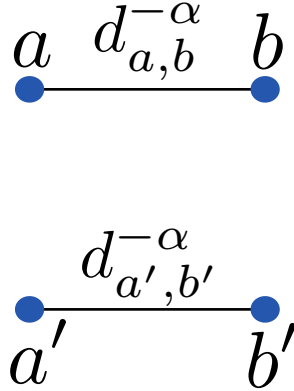


Figure 5.2: The diagrammatic representation of the term $d_{a,b}^{-\alpha} d_{a',b'}^{-\alpha}$ in (5.4.1)

The 4-body correlation $\langle S_a^x S_b^y S_{a'}^x S_{b'}^y \rangle$ can be decomposed into cumulant products as follows:

$$\langle S_a^x S_b^y S_{a'}^x S_{b'}^y \rangle = \langle S_a^x S_b^y S_{a'}^x S_{b'}^y \rangle_c + \langle S_a^x S_{a'}^x \rangle_c \langle S_b^y S_{b'}^y \rangle_c, \quad (5.4.2)$$

where we use the relation (5.3.6) and (5.3.7). Next, we use Theorem 1 as follows:

$$\langle S_a^x S_b^y S_{a'}^x S_{b'}^y \rangle_c \leq c_1 \|S_a^x\| \dots \|S_{b'}^y\| \sum_{\substack{\{j_1, \dots, j_4\} = \\ \{a, b, a', b'\} \\ \text{:connected}}} \frac{1}{d_{j_1, j_2}^\alpha d_{j_2, j_3}^\alpha d_{j_3, j_4}^\alpha}, \quad (5.4.3)$$

$$\langle S_{i_p}^\sigma S_{i_q}^\sigma \rangle_c \leq c_2 \|S_{i_p}^\sigma\| \|S_{i_q}^\sigma\| \frac{1}{d_{i_p, i_q}^\alpha}, \quad (5.4.4)$$

where $\sum_{\substack{\{j_1, \dots, j_m\} \\ \text{:connected}}}$ denotes taking the summation over the vertices, which are connected from j_1 to j_m along the Hamilton path.

By using Theorem 1, (5.4.3) and (5.4.4), we obtain the following upper bound of $|\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle|$:

$$\begin{aligned} |\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle| &\leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \frac{J^4}{4} \sum_{\substack{\{a,b\}=\{k,\ell\} \\ \{a',b'\}=\{k',\ell'\}}} \frac{1}{d_{a,b}^\alpha d_{a',b'}^\alpha} \left\{ |\langle S_a^x S_b^y S_{a'}^x S_{b'}^y \rangle_c| + |\langle S_a^x S_{a'}^x \rangle_c \langle S_b^y S_{b'}^y \rangle_c| \right\} \\ &\leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \frac{J^4}{4} \sum_{\substack{\{a,b\}=\{k,\ell\} \\ \{a',b'\}=\{k',\ell'\}}} \frac{1}{d_{a,b}^\alpha d_{a',b'}^\alpha} \left\{ \sum_{\substack{\{i_1, \dots, i_4\} = \\ \{a,b,a',b'\} \\ \text{:connected}}} \frac{c_1}{d_{i_1,i_2}^\alpha d_{i_2,i_3}^\alpha d_{i_3,i_4}^\alpha} + \frac{c_2}{d_{a,a'}^\alpha d_{b,b'}^\alpha} \right\} \quad (5.4.5) \end{aligned}$$

Let us first consider the contribution of the first term in (5.4.5):

$$\sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \frac{J^4}{4} \sum_{\substack{\{a,b\}=\{k,\ell\} \\ \{a',b'\}=\{k',\ell'\}}} \frac{1}{d_{a,b}^\alpha d_{a',b'}^\alpha} \sum_{\substack{\{i_1, \dots, i_4\} = \\ \{a,b,a',b'\} \\ \text{:connected}}} \frac{c_1}{d_{i_1,i_2}^\alpha d_{i_2,i_3}^\alpha d_{i_3,i_4}^\alpha} \quad (5.4.6)$$

Fig. 5.3 comprehensively covers all the cases of clustering linkage in (5.4.6). Here, diagrams with identical topology are regarded as equivalent.

The methods of clustering linkage can be classified into two cases: (1) the left two graphs: when there are two or more connected components linking the upper and lower graphs in Fig.5.3, and (2) the most right graph: when there is only one connected component.

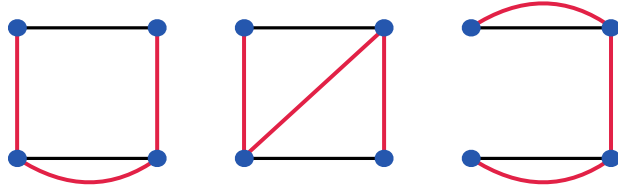


Figure 5.3: The diagrammatic representation of the contribution of the first term in (5.4.5) to the upper bound of $|\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle|$. It can be classified into two cases: (1) the left two graphs: when there are two or more connected components linking the upper and the lower graphs in Fig.5.3, and (2) the most right graphs: when there is only one connected component.

In case (1) in Fig. 5.3, we compute the upper bound of the contribution to $|\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle|$ of the graph leftmost as a typical example.

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \frac{J^4}{4} \sum_{\substack{\{a,b\}=\{k,\ell\} \\ \{a',b'\}=\{k',\ell'\}}} \frac{1}{d_{a,b}^\alpha d_{a',b'}^\alpha} \frac{c_1}{d_{a,b}^\alpha d_{b,b'}^\alpha d_{b',a'}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \frac{J^4}{4} \sum_{\substack{\{a,b\}=\{k,\ell\} \\ \{a',b'\}=\{k',\ell'\}}} \frac{c_1}{d_{a,b}^\alpha d_{b,b'}^\alpha d_{b',a'}^\alpha d_{a',a}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{J^4}{4} c_1 u^2 \frac{1}{d_{k,\ell'}^{2\alpha}} \leq c' n^{2-2\alpha}, \tag{5.4.7}
\end{aligned}$$

where $c' = J^4 c_1 u^2 / \{4(1-2\alpha)(2-2\alpha)\}$. From the first line to the second line, we use (4.4.14). From the second line to the third line, we also use Lemma 4. The same inequality can also be obtained for the graph in the middle of Fig.5.3.

In case (2) in Fig. 5.3, we compute the upper bound of the contribution to $\langle \mathcal{J}_n \mathcal{J}_0 \rangle$ of the graph leftmost.

In this case, (k, ℓ) and (k', ℓ') are connected via a single edge. We construct closed loops for both (k, ℓ) and (k', ℓ') . Furthermore, considering the dominant contribution in power-law clustering, the ordering relation between k, ℓ, k', ℓ' is as follows:

$$\forall u \in \{k, \ell\}, \forall v \in \{k', \ell'\}, u > v. \tag{5.4.8}$$

We define $p, p' \in \{1, \dots, N\}$ as $p = \min\{k, \ell\}$, $p' = \max\{k', \ell'\}$. Here, $p = \ell$ and $p' = k'$, which implies that k, ℓ, k', ℓ' are arranged as $k \geq n > \ell > k' \geq 0 > \ell'$. Since we obtain the following inequality for the contribution of the case (2) in Fig.5.3:

$$\begin{aligned}
& \sum_{\substack{k, \ell, k', \ell': \\ k \geq n > \ell > k' \geq 0 > \ell'}} \frac{J^4}{4} \sum_{\substack{\{a,b\}=\{k,\ell\} \\ \{a',b'\}=\{k',\ell'\}}} \frac{1}{d_{a,b}^\alpha d_{a',b'}^\alpha} \sum_{\substack{\{i_1, \dots, i_4\} = \\ \{a,b,a',b'\} \\ \text{:connected}}} \frac{c_1}{d_{i_1,i_2}^\alpha d_{i_2,i_3}^\alpha d_{i_3,i_4}^\alpha} \\
& \leq \mathcal{O}(1) \sum_{\substack{k, \ell, k', \ell' \\ :k \geq n > \ell > k' \geq 0 > \ell'}} \frac{1}{d_{k,\ell}^{2\alpha}} \frac{1}{d_{\ell,k'}^\alpha} \frac{1}{d_{k',\ell'}^{2\alpha}} \tag{5.4.9}
\end{aligned}$$

The corresponding schematic diagram is presented in Fig. 5.4.

From (5.4.9), we obtain the upper bound for the contribution to $|\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle|$ in the case (2)

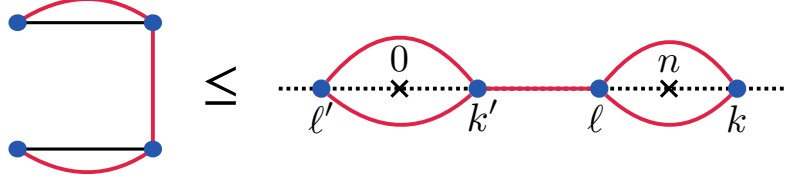


Figure 5.4: Typical examples of the diagrammatic derivation for the upper bound in the case Figure.5.3 (2)

of Fig. 5.3:

$$\begin{aligned}
& \sum_{\substack{k, \ell, k', \ell' \\ :k \geq n > \ell > k' \geq 0 > \ell'}} \frac{1}{d_{k, \ell}^{2\alpha}} \frac{1}{d_{\ell, k'}^\alpha} \frac{1}{d_{k', \ell'}^{2\alpha}} = \sum_{\substack{k, p, p', \ell' \\ :k \geq n > \ell > k' \geq 0 > \ell'}} \frac{1}{(d_{k, n} + d_{n, \ell})^{2\alpha}} \frac{1}{d_{\ell, k'}^\alpha} \frac{1}{(d_{k', 0} + d_{0, \ell'})^{2\alpha}} \\
& \leq \sum_{\substack{\ell, k' \\ :n > \ell > k' \geq 0}} \sum_{x, y=1}^{N/2} \frac{1}{(d_{n, \ell} + x)^{2\alpha}} \frac{1}{d_{\ell, k'}^\alpha} \frac{1}{(d_{k', 0} + y)^{2\alpha}} \leq \frac{1}{(1 - 2\alpha)^2} \sum_{\substack{\ell, k' \\ :n > \ell > k' \geq 0}} \frac{1}{d_{n, \ell}^{2\alpha-1}} \frac{1}{d_{\ell, k'}^\alpha} \frac{1}{d_{k', 0}^{2\alpha-1}} \\
& \leq \frac{1}{(1 - 2\alpha)^2} \sum_{\substack{\ell, k' \\ :n > \ell > k' \geq 0}} \frac{1}{d_{n, \ell}^\alpha} \frac{1}{d_{\ell, k'}^\alpha} \frac{1}{d_{k', 0}^\alpha} \leq \frac{u^2}{(1 - 2\alpha)^2} \frac{1}{d_{n, 0}^\alpha} = \frac{u^2}{(1 - 2\alpha)^2} n^{-\alpha}. \tag{5.4.10}
\end{aligned}$$

This bound decays faster than $n^{2-2\alpha}$ when $\alpha \leq 2$. Hence, the term shown in Fig.5.3 (2) does not constitute the leading term for $\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle$ for $1 < \alpha \leq 2$. However, this term should be dominant for $\alpha > 2$.

Next, we consider the contribution of the second term in (5.4.5):

$$\sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \frac{J^4}{4} \sum_{\substack{\{a, b\} = \{k, \ell\} \\ \{a', b'\} = \{k', \ell'\}}} \frac{1}{d_{a, b}^\alpha d_{a', b'}^\alpha} \frac{c_2}{d_{a, a'}^\alpha d_{b, b'}^\alpha} \tag{5.4.11}$$

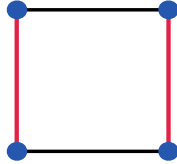


Figure 5.5: The diagrammatic representation of the contribution of the second term in (5.4.5) to the upper bound of $|\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle|$

Fig. 5.5 comprehensively covers all the case of clustering linkage in (5.4.11). Here, diagrams with identical topology are regarded as equivalent. We obtain the following upper bound for (5.4.11):

$$\sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \frac{J^4}{4} \sum_{\substack{\{a, b\} = \{k, \ell\} \\ \{a', b'\} = \{k', \ell'\}}} \frac{1}{d_{a, b}^\alpha d_{a', b'}^\alpha} \frac{c_2}{d_{a, a'}^\alpha d_{b, b'}^\alpha} \leq \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{J^4}{4} c_2 u^2 \frac{1}{d_{k, \ell'}^{2\alpha}} \leq c'' n^{2-2\alpha}, \tag{5.4.12}$$

where $c'' = J^4 c_2 u^2 / \{4(1 - 2\alpha)(2 - 2\alpha)\}$. To derive the first inequality, we also use Lemma 4.

Based on the above discussion, it has been demonstrated that all terms in (5.4.5) can be bounded as follows:

$$|\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle| < \mathcal{O}(1)n^{2-2\alpha} + \mathcal{O}(1)n^{-\alpha} \quad (5.4.13)$$

Hence, for $1 < \alpha \leq 2$, the equal-time current correlation $|\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle|$ is bounded by $\mathcal{O}(1)n^{2-2\alpha}$, while for $\alpha > 2$, $|\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle|$ is bounded by $\mathcal{O}(1)n^{-\alpha}$. □

5.5 Fluctuating hydrodynamics and Lévy diffusion

We investigate the anomalous transport regime within the framework established by Theorem 4. Here, we develop fluctuating hydrodynamics of energy and spin for long-range interacting systems, formulated in a continuum description on the infinite line. The dynamics of conserved fields is modeled by

$$\frac{\partial}{\partial t} u_a(x, t) = \frac{\partial}{\partial x} \left[\sum_{a'} \int dx' D_{a,a'}(x - x') \frac{\partial}{\partial x'} u_{a'}(x', t) + \xi_a(x, t) \right], \quad (5.5.1)$$

where $u_a(x, t)$ denotes the coarse-grained density of local conserved quantities labeled by $a = s$ (spin) and $a = \epsilon$ (energy). The stochastic term $\xi_a(x, t)$ has zero mean, and its correlations are given by

$$\langle\langle \xi_a(x, t) \xi_{a'}(x', t') \rangle\rangle = 2\Lambda_{a,a'}^{-1} D_{a,a'}(x - x') \delta_{a,a'} \delta(t - t'), \quad (5.5.2)$$

with $\Lambda_{a,a'}$ the inverse susceptibility matrix.

The derivation of fluctuating hydrodynamic equation (5.5.1) is presented in detail in Sec. 3.2, and it extends naturally to systems with multiple conserved quantities, such as long-range interacting spin chains possessing both energy and spin conservation. A key feature is the nonlocal diffusion kernel $D_{a,a'}(x - x')$, which encodes how the long-range couplings modify the transport. In short-range models the kernel decays rapidly, allowing the approximation $D_{a,a'}(x - x') \simeq \mathfrak{D}_{a,a'} \delta(x - x')$. By contrast, in long-range systems $D_{a,a'}(x - x')$ typically retains an algebraic tail, fundamentally altering the hydrodynamic behavior.

Within this framework, we focus on the space-time equilibrium correlation function

$$C_{a,a'}(x, t) := \langle\langle \delta u_a(x, t) \delta u_{a'}(0, 0) \rangle\rangle, \quad (5.5.3)$$

where $\delta u_a(x, t) = u_a(x, t) - \bar{u}_a$, and $\bar{u}_a$ denotes the long-time stationary value of the conserved field. The evolution equation for $C_{a,a'}(x, t)$ follows directly from (5.5.1),

$$\partial_t C_{a,a'}(x, t) = \partial_x \sum_{a''} \int dx' D_{a,a''}(x - x') \partial_{x'} C_{a'',a'}(x', t). \quad (5.5.4)$$

Introducing the Fourier representation $A(x, t) = \frac{1}{2\pi} \int dk A(k, t) e^{ikx}$, we obtain

$$\partial_t C_{a,a'}(k, t) = -k^2 \sum_{a''} D_{a,a''}(k) C_{a'',a'}(k, t). \quad (5.5.5)$$

Assume that the small-wavenumber scaling of the diffusion kernel is

$$D_{a,a'}(k) \sim k^{-\nu}, \quad (a, a' = s, \epsilon). \quad (5.5.6)$$

Then the long-time, large-distance behavior of the correlation function takes the scaling form

$$C_{a,a'}(x, t) \sim \begin{cases} t^{-1/(2-\nu)} f_{\text{Lévy}}(x/t^{1/(2-\nu)}), & \nu > 0, \\ t^{-1/2} f_{\text{Gauss}}(x/t^{1/2}), & \nu < 0, \end{cases} \quad (5.5.7)$$

where $f_{\text{Lévy}}$ and f_{Gauss} denote the Lévy and Gaussian scaling functions, respectively. When the diffusion kernel exhibits an algebraic tail in real space,

$$D_{a,a'}(x - x') \sim |x - x'|^{-\eta} \quad (|x - x'| \rightarrow \infty), \quad (5.5.8)$$

the exponent ν is related to η through $\nu = 1 - \eta$. Consequently, $\eta > 1$ corresponds to normal diffusion, whereas $\eta < 1$ marks the onset of anomalous (Lévy) diffusion driven by the long-range interactions.

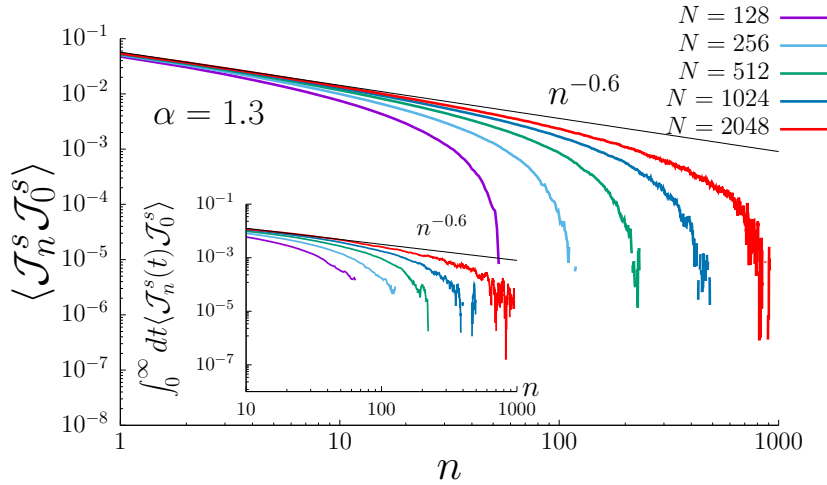


Figure 5.6: The correlation function $\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle$ for the XXZ model with the parameter $\alpha = 1.3$ and $T = 10$ ($k_B \equiv 1$) in the main panel. The inset shows the Green-Kubo like formula. The solid line is the theoretical line (5.5.13), i.e., $n^{-(2\alpha-2)}$. The equal-time correlation $\langle \mathcal{J}_n \mathcal{J}_0 \rangle$ explains the power law exponent, $\eta = 2\alpha - 2$.

To establish a microscopic connection between the nonlocal diffusion kernel and the underlying lattice dynamics, we return to the discrete formulation. On the lattice the kernel $D_{a,a'}(n - n')$ admits an effective Green-Kubo representation [254],

$$D_{a,a'}(n - n') = \Lambda_{a,a'}^{-1} \int_0^\infty dt \langle \mathcal{J}_n^a(t) \mathcal{J}_{n'}^{a'} \rangle, \quad (5.5.9)$$

so that the equal-time contribution to the integral can be constrained using Theorem 4. Owing to the symmetry relation (5.3.8), cross-terms vanish and one has

$$D_{a,a'}(n - n') = \delta_{a,a'} D_{a,a}(n - n'). \quad (5.5.10)$$

Consequently, the continuum fluctuating-hydrodynamic equation (5.5.1) can be written in a discrete form appropriate for the lattice as

$$\frac{\partial}{\partial t} u_a(n, t) = \nabla_n \left[\sum_{n'} D_{a,a}(n - n') \nabla_{n'} u_a(n', t) + \xi_a(n, t) \right], \quad (a = s, \epsilon), \quad (5.5.11)$$

where ∇_n denotes the discrete gradient. The corresponding space-time correlation function in the lattice representation satisfies the diagonal form $C_{a,a'}(n, t) = \delta_{a,a'} C_{a,a}(n, t)$ and evolves according to

$$\partial_t C_{a,a}(n, t) = \nabla_n \left[\sum_{n'} D_{a,a}(n - n') \nabla_{n'} C_{a,a}(n', t) \right]. \quad (5.5.12)$$

Fig. 5.6 presents numerical data for the equal-time spin-current correlation $\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle$ computed for the transverse Ising model at $\alpha = 1.3$. The inset reports the corresponding Green-Kubo-like integral for the same parameter set. The time-dependent correlator $\langle \mathcal{J}_n^s(t) \mathcal{J}_0^s \rangle$ decays rapidly in time, so that the time integral converges (data not shown). When plotted on logarithmic axes, the equal-time data approach a clear power-law regime as the system size is increased, up to finite-size deviations at the largest separations. The extracted power-law exponent η coincides with the theoretical upper bound given in Eq. (5.3.9), namely $\eta = 2\alpha - 2 = 0.6$ for $\alpha = 1.3$. The inset reproduces the same exponent, indicating that the equal-time contribution already determines the scaling of the nonlocal kernel.

Taken together, these observations provide strong support for the scaling ansatz (5.5.7) across different models and conserved quantities. In particular, the data imply the relation

$$\nu = 3 - 2\alpha, \quad (5.5.13)$$

which links the small- k exponent of the effective diffusion kernel, $D(k) \sim k^{-\nu}$, to the microscopic interaction exponent α . This scaling behavior is also analogous to that of energy diffusion.

To validate the scaling prediction, we next compute the full space-time correlation function for the XX model across a range of long-range exponents α . In the Hamiltonian formulation, the relevant correlation function is

$$C_{ss}(n, t) := \langle S_n^z(t) S_0^z(0) \rangle, \quad (5.5.14)$$

which probes the spreading of spin fluctuations in real space and time. Fig. 5.7 summarizes the numerical results. Panel (a) shows the correlation profiles for $\alpha = 1.3$, while panel (b)

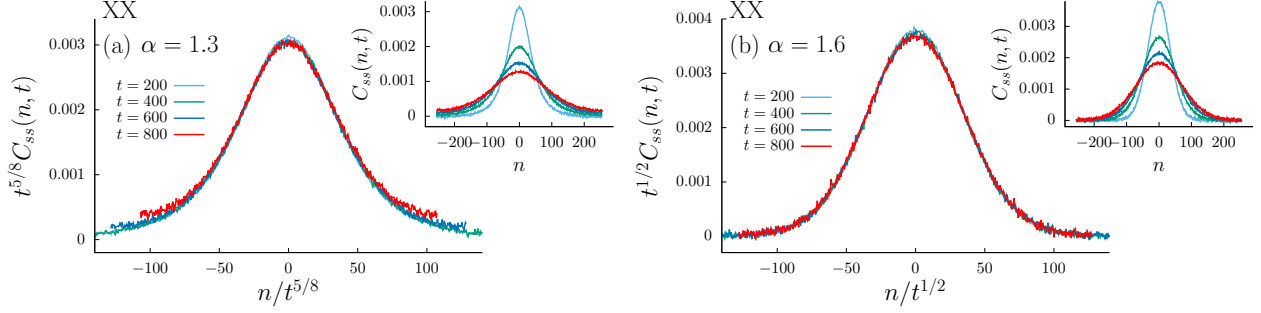


Figure 5.7: Space-time spin correlation function $C^s(n, t) := \langle S_n^z(t) S_0^z \rangle$ for the XX model (??) with the parameters $J = 1$. $T = 10$ ($k_B \equiv 1$). System size is $N = 512$. In (a), we use the parameter $\alpha = 1.3$, while (b) for $\alpha = 1.6$. It is clear that $\alpha = 1.3$ case obeys the Lévy scaling, while $\alpha = 1.6$ shows the diffusive scaling.

displays those for $\alpha = 1.6$. The results clearly show that $\alpha = 1.3$ case obeys the Lévy scaling, while $\alpha = 1.6$ case shows the diffusive scaling, as suggested by the scaling (4.5.7) with (5.5.13).

These results confirm that the transition from Gaussian to Lévy diffusion occurs precisely at the theoretically predicted threshold $\alpha = \alpha_c = 3/2$, reinforcing the validity of fluctuating hydrodynamic framework developed above.

5.6 Higher dimensional case

Theorem 1 extends to higher spatial dimensions. Accordingly, we next consider a generic D -dimensional hypercubic lattice with periodic boundary conditions, containing N^D sites. The microscopic Hamiltonian is taken to be the spatially uniform classical Hamiltonian

$$H = \sum_{\mathbf{n} \in \mathbb{Z}_N^D} \hat{h}_{\mathbf{n}}, \quad (5.6.1)$$

where $\hat{h}_{\mathbf{n}}$ denotes the local Hamiltonian associated with lattice site $\mathbf{n} \in \mathbb{Z}_N^D$. As a concrete class of models we analyze the D -dimensional generalization of (??), namely the long-range XXZ model,

$$\hat{h}_{\mathbf{n}}^{(\text{XXZ})} = - \sum_{\substack{j \in \mathbb{Z}_N^D \\ j \neq \mathbf{n}}} \frac{1}{2d_{\mathbf{n},j}^\alpha} \left(J S_{\mathbf{n}}^x S_j^x + J S_{\mathbf{n}}^y S_j^y + J_z S_{\mathbf{n}}^z S_j^z \right). \quad (5.6.2)$$

Here $S_{\mathbf{n}}^\sigma(t)$ is the $\sigma \in \{x, y, z\}$ component of the classical spin at site $\mathbf{n}$ and time t . The model possesses two locally conserved densities, the energy $\hat{h}_{\mathbf{n}}^{(\text{XXZ})}$ and the z -spin density $S_{\mathbf{n}}^z$. The equation of motion for the local z -spin reads

$$\partial_t S_{\mathbf{n}}^z(t) = \sum_{\mathbf{m} \neq \mathbf{n}} t_{\mathbf{n} \leftarrow \mathbf{m}}^s, \quad (5.6.3)$$

where the microscopic spin transmission operator is defined by

$$t_{i \leftarrow j}^s := 2\{S_i^z, \hat{h}_j\}, \quad (5.6.4)$$

with $\{\cdot, \cdot\}$ the spin Poisson bracket.

To characterize isotropic spreading of spin fluctuations, we introduce the mean-square displacement

$$\sigma(t) := \mathcal{N}^{-1} \sum_{i=1}^D \sum_{\mathbf{n}} n_i^2 \langle S_{\mathbf{n}}^z(t) S_{\mathbf{0}}^z(0) \rangle, \quad (5.6.5)$$

where $\mathcal{N} = \sum_{\mathbf{n}} \langle S_{\mathbf{n}}^z(0) S_{\mathbf{0}}^z(0) \rangle$ normalizes the correlator [203]. In the normal-diffusion regime the long-time growth of $\sigma(t)$ defines the spin diffusion constant via the Green–Kubo relation,

$$\begin{aligned} \mathfrak{D}^s &= \lim_{t \rightarrow \infty} \frac{\sigma(t)}{2t} = \Lambda_{ss} \int_0^\infty dt C_N^{(s,D)}(t), \\ C_N^{(s,D)}(t) &= \frac{1}{N^{D-1}} \sum_{i=1}^D \sum_{n_i=1}^N \langle \mathcal{J}_{n_i|n_i-1}^{(s,i)}(t) \mathcal{J}_{0|-1}^{(s,i)}(0) \rangle, \end{aligned} \quad (5.6.6)$$

where Λ_{ss} denotes the spin susceptibility per unit volume. The current in the i th direction is defined as the net spin transmission across the hyperplane between $\mathbf{n} - \mathbf{e}_i$ and $\mathbf{n}$,

$$\mathcal{J}_{n_i|n_i-1}^{(s,i)} := \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell: \ell_i < n_i}} t_{\mathbf{k} \leftarrow \ell}^s, \quad (5.6.7)$$

so that $\mathcal{J}_{n_i|n_i-1}^{(s,i)}$ collects all microscopic transmissions crossing the $(D-1)$ -dimensional boundary orthogonal to the i th axis. A derivation of the Green–Kubo relation (5.6.6) in this geometric setting is given in Sec 4.6.1.

We now state the higher-dimensional bound for the equal-time total spin-current correlation.

Theorem 5. *Consider the D -dimensional long-range XXZ model (5.6.2) with $D \geq 2$. For interaction exponent $\alpha > D$ and inverse temperature $\beta < \beta_c$, the equal-time total spin-current correlation*

$$C_N^{(s,D)}(0) = \frac{1}{N^{D-1}} \sum_{i=1}^D \sum_{n_i=1}^N \langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \rangle$$

remains finite in the thermodynamic limit.

Under the physically motivated assumption that the amplitude of the equal-time total current correlation governs the asymptotic transport class, Theorem 5 implies that spin diffusion in spatial dimensions $D \geq 2$ is normal provided $\alpha > D$. The condition $\alpha > D$ ensures thermodynamic extensivity and finiteness of thermodynamic susceptibilities, which are prerequisites for the applicability of the Green–Kubo description. A detailed proof of Theorem 5 is given in Sec. 5.7.

5.7 Proof of Theorem 5

Proof of Theorem 5. $C_N^{(s,D)}(0)$ is expressed as follows:

$$C_N^{(s,D)}(0) = N^{1-D} \sum_{i=1}^D \sum_{n_i} \langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \rangle, \quad (5.7.1)$$

where $\mathcal{J}_{n_i|n_i-1}^{(s,i)}$ is the spin current in i -th direction, which is defined as

$$\mathcal{J}_{n_i|n_i-1}^{(s,i)} := \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell: \ell_i < n_i}} t_{\mathbf{k} \leftarrow \ell}^s, \quad t_{\mathbf{k} \leftarrow \ell}^s = 2\{S_{\mathbf{k}}^z, \hat{h}_{\ell}\} \quad (5.7.2)$$

Therefore, the equal-time current correlation is described as

$$\langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \rangle = \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell: \ell_i < n_i}} \sum_{\substack{\mathbf{k}': k'_i \geq 0 \\ \ell': \ell'_i < 0}} \langle t_{\mathbf{k} \leftarrow \ell}^s t_{\mathbf{k}' \leftarrow \ell'}^s \rangle. \quad (5.7.3)$$

In exactly the same manner as the one-dimensional case, utilizing Theorem 1 and the symmetry derived from the Hamiltonian, we obtain the following upper bound for the current correlation, except for the case (2) in Fig.5.3:

$$\left| \langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \rangle \right|_{\text{except Fig.5.3(2)}} \leq \mathcal{O}(1) \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell': \ell'_i < 0}} \frac{1}{d_{\mathbf{k}, \ell'}^{2\alpha}} \quad (5.7.4)$$

For $D \geq 2$, the right-handed side is bounded as follows:

$$\begin{aligned} \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell': \ell'_i < 0}} \frac{1}{d_{\mathbf{k}, \ell'}^{2\alpha}} &\leq \int_{\substack{x_i \geq n_i \\ y_i < 0}} dx_i dy_i \prod_{k \neq i} \int_{-N/2}^{N/2} dx_k dy_k \left\{ (x_i - y_i)^2 + \sum_{j \neq i} (x_j - y_j)^2 \right\}^{-2\alpha/2} \\ &\leq \Gamma_{D-1} \int_{-N/2}^{N/2} d^{D-1} \mathbf{u} \int_{\substack{x_i \geq n_i \\ y_i < 0}} dx_i dy_i \int_0^\infty d\xi \xi^{D-2} \{(x_i - y_i)^2 + \xi^2\}^{-2\alpha/2} \\ &\leq \frac{\Gamma_{D-1} B(\frac{D-1}{2}, \alpha - \frac{D-1}{2})}{2} N^{D-1} \int_0^{N/2} dx dy (n_i + x + y)^{D-1-2\alpha} \\ &\leq \frac{\Gamma_{D-1} B(\frac{D-1}{2}, \alpha - \frac{D-1}{2})}{2(D-2\alpha)(D+1-2\alpha)} N^{D-1} n_i^{D+1-2\alpha}, \end{aligned} \quad (5.7.5)$$

where $u_j := (x_j + y_j)/2$ for $j \neq i$, $\xi := \{\sum_{j \neq i} (x_j + y_j)^2\}^{1/2}$, Γ_{D-1} is the coefficient of $(D-2)$ -dimensional spherical surface area, and $B(x, y)$ is Beta function.

Next, we consider the contribution of case (2) in Fig.5.3. In this case, we construct closed loops for $(\mathbf{k}, \ell)$ and $(\mathbf{k}', \ell')$, and they are connected by a single edge. The corresponding schematic diagram is shown in Fig.5.8.

By definition of $\{\mathbf{k}, \ell, \mathbf{k}', \ell'\}$ in (5.7.3), $\mathbf{k}$ and ℓ are on opposite sides of the hyperplane $x_i = n_i$, and $\mathbf{k}'$ and ℓ' are on opposite sides of the hyperplane $x_i = 0$. Thus, the condition $(k_i - n_i)(\ell_i - n_i) \leq 0$ and $k'_i \ell'_i \leq 0$ are satisfied.

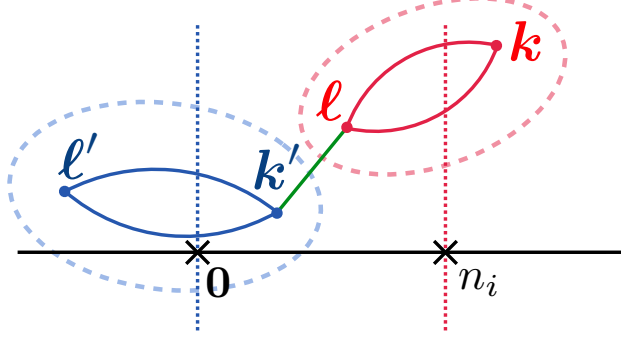


Figure 5.8: The schematic diagram of the case (2) in Fig.5.3 in D -dimensions.

In the same manner as one-dimensional case, the contributions of the case (2) in Fig.5.3 are bounded as follows:

$$\left| \left\langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \right\rangle \right|_{\text{Fig.5.3(2)}} \leq \mathcal{O}(1) \sum_{\mathbf{k}, \ell, \mathbf{k}', \ell'} \frac{1}{d_{\mathbf{k}, \ell}^{2\alpha}} \frac{1}{d_{\ell, \mathbf{k}'}^{\alpha}} \frac{1}{d_{\mathbf{k}', \ell'}^{2\alpha}} \quad (5.7.6)$$

Firstly, we evaluate the sum over $\mathbf{k}, \ell'$. For the sum over $\mathbf{k}$, the following inequality is obtained:

$$\begin{aligned} \sum_{\mathbf{k}: (k_i - n_i)(\ell_i - n_i) \leq 0} \frac{1}{d_{\mathbf{k}, \ell}^{2\alpha}} &\leq \int_{(k_i - n_i)(\ell_i - n_i) \leq 0} d^d \mathbf{k} \frac{1}{\{(k_i - \ell_i)^2 + \sum_{j \neq i} (k_j - \ell_j)^2\}^{2\alpha/2}} \\ &\leq \int_{(k_i - n_i)(\ell_i - n_i) \leq 0} dk_i \Gamma_{D-1} \int_0^\infty d\xi \xi^{D-2} \frac{1}{\{(k_i - \ell_i)^2 + \xi^2\}^{2\alpha/2}} \\ &\leq \Gamma' \int_{(k_i - n_i)(\ell_i - n_i) \leq 0} dk_i \frac{1}{(k_i - \ell_i)^{2\alpha - D + 1}} \\ &\leq \Gamma' \int_0^{N/2} dx \frac{1}{(x + n_i - \ell_i)^{2\alpha - D + 1}} \\ &\leq \Gamma'' \frac{1}{(n_i - \ell_i)^{2\alpha - D}} = \Gamma'' \frac{1}{d_{\bar{\mathbf{n}}_\ell, \ell}^{2\alpha - D}}, \end{aligned} \quad (5.7.7)$$

where $\Gamma' = \Gamma_{D-1} B(\frac{D-1}{2}, \alpha - \frac{D-1}{2})/2$, $\Gamma'' = \Gamma'/(2\alpha - D)$ and we denote $\bar{\mathbf{n}}_\ell$ as the foot of the perpendicular drop from ℓ to the hyperplane $x_i = n_i$, as in the leftmost part of Fig. 5.9.

Similarly, for the sum over ℓ' , the following inequality is obtained:

$$\sum_{\ell': k'_i \ell'_i \leq 0} \frac{1}{d_{\mathbf{k}', \ell'}^{2\alpha}} \leq \Gamma'' \frac{1}{d_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'}}^{2\alpha - D}} \quad (5.7.8)$$

where we denote $\bar{\mathbf{0}}_{\mathbf{k}'}$ as the foot of the perpendicular dropped from $\mathbf{k}'$ to the hyperplane $x_i = 0$, as in the leftmost part in Fig. 5.9.

Thus, we obtain the following upper bound:

$$\left| \left\langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \right\rangle \right|_{\text{Fig.5.3(2)}} \leq \sum_{\ell, \mathbf{k}'} \frac{1}{d_{\bar{\mathbf{n}}_\ell, \ell}^{2\alpha - D}} \frac{1}{d_{\ell, \mathbf{k}'}^{\alpha}} \frac{1}{d_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'}}^{2\alpha - D}} \leq \sum_{\ell, \mathbf{k}'} \frac{1}{d_{\bar{\mathbf{n}}_\ell, \ell}^{\alpha}} \frac{1}{d_{\ell, \mathbf{k}'}^{\alpha}} \frac{1}{d_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'}}^{\alpha}}, \quad (5.7.9)$$

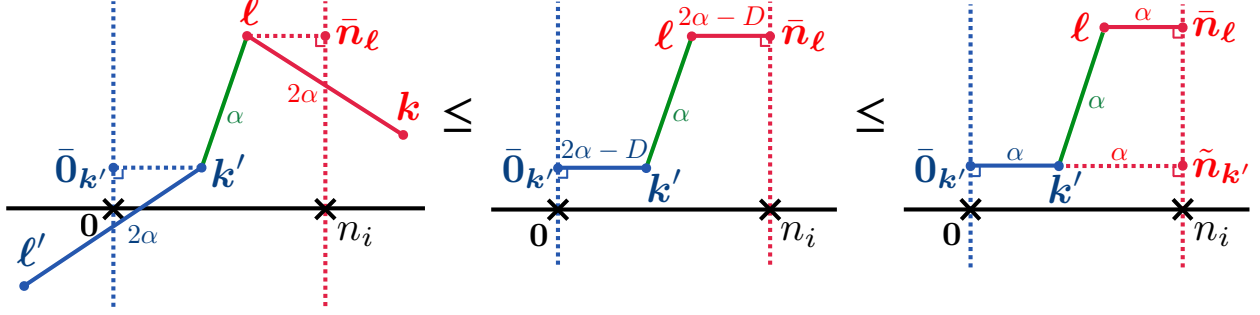


Figure 5.9: The diagrammatic derivation for the upper bound of the case (2) in Fig.5.3 in D -dimensions

where in the last inequality we use $\alpha > D$.

Next, to further evaluate this upper bound, we use Lemma 5 as follows:

$$\begin{aligned}
\left| \left\langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \right\rangle \right|_{\text{Fig.5.3(2)}} &\leq \sum_{\ell, k'} \frac{1}{d_{\bar{n}_\ell, \ell}^\alpha} \frac{1}{d_{\ell, k'}^\alpha} \frac{1}{d_{k', \bar{0}_{k'}}^\alpha} \\
&\leq \sum_{k'} \frac{1}{d_{k', \bar{0}_{k'}}^\alpha} 2^{\alpha-1} (u + \tilde{u}_{\bar{n}_\ell}) \sup_{\ell} \frac{1}{d_{\bar{n}_\ell, k'}^\alpha} \\
&= 2^{\alpha-1} (u + \tilde{u}_{\bar{n}_k}) \sum_{k'} \frac{1}{d_{\bar{n}_{k'}, k'}^\alpha} \frac{1}{d_{k', \bar{0}_{k'}}^\alpha} \\
&\leq 2^{\alpha-1} (u + \tilde{u}_{\bar{n}_k}) 2^{\alpha-1} (\tilde{u}_{\bar{0}_{k'}} + \tilde{u}_{\bar{n}_{k'}}) \sup_{k'} \frac{1}{d_{\bar{n}_{k'}, \bar{0}_{k'}}^\alpha} = \mathcal{O}(1) \frac{1}{n_i^\alpha}, \quad (5.7.10)
\end{aligned}$$

where $\tilde{u}_{\bar{n}_{k'}} = \sum_{k'} d_{\bar{n}_{k'}, k'}^{-\alpha} \leq \int_1^{N/2} d^d \mathbf{x} x_i^{-\alpha} \leq \mathcal{O}(1) N^{D-\alpha} < \infty$ for $\alpha > D$. Furthermore, let $\tilde{n}_{k'}$ be the $\bar{n}_{k'}$ that satisfies $\sup_{k'} d_{\bar{n}_{k'}, k'}^{-\alpha} = d_{\tilde{n}_{k'}, k'}^{-\alpha}$. As shown on the far right of Fig. 5.9, $\tilde{n}_{k'}$ coincides with the foot of the perpendicular dropped from k' to the hyperplane $x_i = n_i$. The condition of Lemma 5 is satisfied, as $\tilde{u}_{\bar{0}_{k'}} = \sum_{k'} d_{\bar{0}_{k'}, k'}^\alpha \leq \int_1^{N/2} d^d \mathbf{x} x_i^{-\alpha} < \infty$ and $\tilde{u}_{\tilde{n}_{k'}} = \sum_{k'} d_{\tilde{n}_{k'}, k'}^\alpha \leq \int_1^{N/2} d^d \mathbf{x} x_i^{-\alpha} < \infty$.

From (5.7.5) and (5.7.10), we obtain the following upper bound for $C_N^{(s,D)}(0)$:

$$\begin{aligned}
\left| C_N^{(s,D)}(0) \right| &\leq \mathcal{O}(1) N^{D+2-2\alpha} + \mathcal{O}(1) N^{2-D-\alpha} + \text{const.} \\
&\leq \mathcal{O}(1) N^{D+2-2\alpha} + \text{const.}, \quad (5.7.11)
\end{aligned}$$

for $D \geq 2$. Finally, we show that $C_N^{(D)}(0)$ converges to a finite value under the condition $\alpha > D/2 + 1$. With the assumption $\alpha > D (\geq 2)$, it immediately leads to $C_N^{(D)}(0) < \infty$ as long as $\alpha > D$. □

Chapter 6

Energy and Spin Diffusion in the Long-range Interacting Spin Systems in the Regime $\alpha < D$ with the Kac Factor

In the preceding chapters, we have restricted our analysis to the weak long-range interacting regime, $\alpha > D$. In the present chapter, by contrast, we address energy and spin diffusion in the strongly long-range interacting regime $0 \leq \alpha < D$. In this regime, the interaction is non-extensive, and the energy per site generally diverges in the thermodynamic limit. As a consequence, the theoretical framework and arguments developed so far are no longer directly applicable.

As a first step toward a systematic investigation of the strongly long-range interacting regime $0 \leq \alpha < D$, we consider models equipped with the Kac normalization factor. The inclusion of the Kac factor restores thermodynamic stability, ensuring that the energy per site remains finite. Furthermore, within this framework, results analogous to the cumulant power-law clustering theorem can still be established, which allows us to apply essentially the same mathematical techniques as those employed in the weak long-range regime.

By arguments parallel to those presented in the previous sections, we can show that, in one-dimensional systems, the equal-time current correlations $\langle \mathcal{J}_n^a \mathcal{J}_0^a \rangle$ for both spin and energy transport ($a = s, \epsilon$) are upper-bounded by a constant independent of the separation n . In higher spatial dimensions, the corresponding equal-time total current correlations are likewise shown to converge to finite values in the thermodynamic limit.

On the other hand, the convergence of the time integral appearing in the Green–Kubo formula, as well as the system-size dependence of the associated relaxation time, remains poorly understood in this regime. Numerical investigations are severely affected by strong finite-size effects, preventing a clear assessment of the asymptotic behavior. As a result, the physical implications of the rigorous upper bounds obtained for the current correlations in

the strongly long-range interacting regime have not yet been fully clarified.

6.1 Models and setup

We begin by considering a uniform one-dimensional classical spin system described by the Hamiltonian

$$H = \sum_{n=1}^N \hat{h}_n, \quad (6.1.1)$$

where $\hat{h}_n$ denotes the local Hamiltonian at site n , and N is the total number of lattice sites. As a representative example, we focus on the long-range interacting XXZ model with Kac normalization, whose local Hamiltonian is given by

$$\hat{h}_n^{(\text{XXZ}_{\text{Kac}})} = - \sum_{r=1}^{N/2} \frac{1}{2\mathcal{N}_\alpha^{(1)} r^\alpha} \left[\sum_{\sigma=x,y} J(S_n^\sigma S_{n+r}^\sigma + S_{n-r}^\sigma S_n^\sigma) + J_z(S_n^z S_{n+r}^z + S_{n-r}^z S_n^z) \right]. \quad (6.1.2)$$

Here, $\hat{h}_n^{(\text{XXZ}_{\text{Kac}})}$ represents the local Hamiltonian of the long-range XXZ model equipped with the Kac factor. The parameter α characterizes the power-law decay of the interaction and is restricted to the strongly long-range regime $0 \leq \alpha < 1$.

The quantity $\mathcal{N}_\alpha^{(1)}$ denotes the Kac normalization factor in one dimension, defined as

$$\mathcal{N}_\alpha^{(1)} := \sum_{r=1}^{N/2} r^{-\alpha} = \mathcal{O}(N^{1-\alpha}), \quad (6.1.3)$$

which diverges algebraically with the system size for $\alpha < 1$. The inclusion of this normalization ensures the thermodynamic extensivity of the Hamiltonian even in the strongly long-range interacting regime $0 \leq \alpha < 1$, thereby rendering the energy per site finite in the thermodynamic limit.

6.2 Cumulant power-law clustering theorem for the long-range interacting systems in the regime $\alpha < D$ with the Kac factor

In this section, we study general D -dimensional k -local long-range interacting spin systems equipped with the Kac normalization. The Hamiltonian is defined in Eq. (2.1.1), and throughout this section we assume that the interaction structure satisfies the following decay condition:

$$J_{i,i'} := \sum_{Z: Z \ni \{i,i'\}} \|h_Z\| \leq \frac{g}{\mathcal{N}_\alpha^{(D)} d_{i,i'}^\alpha} = \frac{g}{\tilde{d}_{i,i'}^\alpha}, \quad (i \neq i'), \quad (6.2.1)$$

where $\mathcal{N}_\alpha^{(D)}$ denotes the Kac factor in D dimensions, defined by

$$\mathcal{N}_\alpha^{(D)} := \sum_{j \in \Lambda} d_{n,j}^{-\alpha} = \mathcal{O}(N^{D-\alpha}). \quad (6.2.2)$$

Here, we have introduced the rescaled distance

$$\tilde{d}_{i,i'} := (\mathcal{N}_\alpha^{(D)})^{1/\alpha} d_{i,i'}. \quad (6.2.3)$$

We consider D -dimensional lattice systems in the strongly long-range interacting regime $0 \leq \alpha < D$. Owing to the presence of the Kac normalization factor (6.2.2), the Hamiltonian remains thermodynamically extensive even in this regime, ensuring the finiteness of the energy density in the thermodynamic limit.

Under the above assumptions, we observe that the total interaction strength associated with a given site can be bounded as

$$\sum_{Z \ni i} \|h_Z\| \leq \sum_{j \in \Lambda} \sum_{Z \ni (i,j)} \|h_Z\| \leq g \tilde{u}, \quad (6.2.4)$$

where we have defined

$$\tilde{u} := 2^\alpha \sum_{j \in \Lambda} \tilde{d}_{i,j}^{-\alpha}. \quad (6.2.5)$$

In the regime $0 \leq \alpha < D$ with Kac factor, it is also possible to establish a power-law clustering property for equilibrium cumulants. Specifically, we obtain the following theorem.

Theorem 6. *Consider a k -local long-range interacting spin system with the Kac factor on a D -dimensional lattice. Let O_{X_j} be observables supported on disjoint sets X_j , satisfying $X_i \cap X_j = \emptyset$ for $i \neq j$. Under the conditions $0 \leq \alpha < D$ and $\beta < \tilde{\beta}_c$, the n -th order joint cumulant in equilibrium obeys the upper bound*

$$\langle O_{X_1} \cdots O_{X_n} \rangle_c \leq c 2^n n! \prod_{j=1}^n \|O_{X_j}\| \sum_P \prod_{i=1}^{n-1} \frac{|X_{P_i}| |X_{P_{i+1}}| e^{(|X_{P_i}| + |X_{P_{i+1}}|)/k}}{(\tilde{d}_{X_{P_i}, X_{P_{i+1}}})^\alpha}, \quad (6.2.6)$$

where $\tilde{\beta}_c = (ng\tilde{u}k)^{-1}W(1/2e^2)$ and W denotes the Lambert W function. Using the inequality $W(1/2e^2) < 1/(4e)$, we further obtain the bound $\tilde{\beta}_c < (4eng\tilde{u}k)^{-1}$. Here, c is a constant, and $\|O_X\|$ denotes the maximum absolute value of the observable O_X . The symbol P represents a Hamiltonian path on a graph whose vertices correspond to the supports $X_1, \dots, X_n$, with X_{P_i} denoting the support at the i -th step along the path. The distance $d_{X,Y}$ is defined as the minimal distance between the supports X and Y , and $|X|$ denotes the number of sites in the support X .

As in the weakly long-range interacting regime, this cumulant clustering theorem serves as a fundamental ingredient for deriving the upper bounds on equal-time current correlations, which are formulated below as Theorems 7 and 8.

Proof of Theorem 6. Theorem 1 can be proven by employing a connected cluster expansion in the high temperature regime, together with repeated applications of Lemma 4. For the proof of Theorem 6, the same strategy applies. Specifically, one replaces Lemma 4 with Lemma 6 stated below, and substitutes $d_{i,j}$ and u in the proof of Theorem 1 with $\tilde{d}_{i,j}$ and $\tilde{u}$, respectively. In this way, the argument can be straightforwardly extended to establish Theorem 6.

As in (2.4.56), the justification of the proof is ensured under the high-temperature condition $\beta < \tilde{\beta}_c$. □

Lemma 6. *Assuming that $0 \leq \alpha < D$, the following inequalities hold.*

$$\sum_k \frac{1}{\tilde{d}_{i,k}^\alpha} \frac{1}{\tilde{d}_{k,j}^\alpha} \leq \frac{\tilde{u}_\alpha}{\tilde{d}_{i,j}^\alpha}, \quad (6.2.7)$$

where $\tilde{d}_{i,j} = (\mathcal{N}_\alpha^{(D)})^{1/\alpha} d_{i,j}$ and $\tilde{u}_\alpha = 2^\alpha \sum_k \tilde{d}_{i,k}^{-\alpha} = 2^\alpha (\mathcal{N}_\alpha^{(D)})^{-1} \sum_k d_{i,k}^{-\alpha} (< \infty)$.

Proof of Lemma 6.

$$\begin{aligned} \sum_k \frac{1}{\tilde{d}_{i,k}^\alpha} \frac{1}{\tilde{d}_{k,j}^\alpha} &= \sum_k \frac{1}{\mathcal{N}_\alpha^{(D)} d_{i,k}^\alpha} \frac{1}{\mathcal{N}_\alpha^{(D)} d_{k,j}^\alpha} = \frac{1}{\mathcal{N}_\alpha^{(D)} d_{i,j}^\alpha} \sum_k \frac{d_{i,j}^\alpha}{\mathcal{N}_\alpha^{(D)} d_{i,k}^\alpha d_{k,j}^\alpha} \\ &\leq \frac{1}{\mathcal{N}_\alpha^{(D)} d_{i,j}^\alpha} \sum_k \frac{(d_{i,k} + d_{k,j})^\alpha}{\mathcal{N}_\alpha^{(D)} d_{i,k}^\alpha d_{k,j}^\alpha} \leq \frac{2^{\alpha-1}}{\mathcal{N}_\alpha^{(D)} d_{i,j}^\alpha} \sum_k \frac{d_{i,k}^\alpha + d_{k,j}^\alpha}{\mathcal{N}_\alpha^{(D)} d_{i,k}^\alpha d_{k,j}^\alpha} \\ &= \frac{2^{\alpha-1}}{\mathcal{N}_\alpha^{(D)} d_{i,j}^\alpha} \sum_k \frac{1}{\mathcal{N}_\alpha^{(D)}} \left(\frac{1}{d_{i,k}^\alpha} + \frac{1}{d_{k,j}^\alpha} \right) = \frac{\tilde{u}_\alpha}{\tilde{d}_{i,j}^\alpha} \end{aligned} \quad (6.2.8)$$

Here, we use $\{(x+y)/2\}^\alpha \leq (x^\alpha + y^\alpha)/2$, following from the convexity of $f(x) = x^\alpha$ and the convexity inequality. □

6.3 Upper bound for the equal-time current correlations in one-dimensional case

In this section, we consider the one-dimensional long-range interacting XXZ model with Kac factor (6.1.2). This model satisfies the following continuity equations

$$\partial_t u_{a,n} = -(\mathcal{J}_{n+1}^a - \mathcal{J}_n^a), \quad (6.3.1)$$

$$\mathcal{J}_n^a = - \sum_{\substack{i \geq n, j < n \\ |i-j| < N/2}} t_{i \leftarrow j}^a, \quad (6.3.2)$$

for $a = s, \epsilon$. Here $u_{a,n}$ is local conserved quantities, and we denote $u_{s,n} = S_n^z$ and $u_{\epsilon,n} = \hat{h}_n$. The transmission operator $t_{i \leftarrow j}^a$ is defined as

$$t_{i \leftarrow j}^s = 2\{S_i^z, \hat{h}_j\} = \frac{J^2}{2} \sum_{\{a,b\}=\{i,j\}} \varepsilon_{ab} \frac{S_a^x S_b^y}{\tilde{d}_{a,b}^\alpha}, \quad (6.3.3)$$

$$t_{i \leftarrow j}^\epsilon = \{\hat{h}_i, \hat{h}_j\} = \sum_{m=1}^N \sum_{\sigma \in P_{\text{cyc}}^{xyz}} \frac{J_{\sigma_1} J_{\sigma_2}}{4} \sum_{\{a,b,c\}=\{i,j,m\}} \tilde{\varepsilon}_{abc} \frac{S_a^{\sigma_1} S_b^{\sigma_2} S_c^{\sigma_3}}{\tilde{d}_{a,c}^\alpha \tilde{d}_{b,c}^\alpha},$$

where $J_x = J_y = J$. Here, ε_{ab} is Levi-Civita symbol, defined as (5.3.4) and $\tilde{\varepsilon}_{abc}$ is also defined as (5.3.5). The notation $\sum_{\sigma \in P_{\text{cyc}}^{xyz}}$ represents the sum over all cyclic permutations of (x, y, z) , i.e., $(x, y, z), (y, z, x), (z, x, y)$.

From the symmetry of the Hamiltonian, similarly as (5.3.6) and (5.3.7), the following properties hold:

$$\langle S_i^x \rangle = \langle S_i^y \rangle = \langle S_i^z \rangle = 0, \quad (6.3.4)$$

$$\langle S_i^\sigma S_j^\tau \rangle = \delta_{\sigma,\tau} \langle S_i^\sigma S_j^\sigma \rangle, \quad (6.3.5)$$

for $i, j = 1, \dots, N$ and $\sigma, \tau \in \{x, y\}$. From (6.3.4) and (6.3.5), we also easily obtain $\langle \mathcal{J}_n^a \mathcal{J}_{n'}^{a'} \rangle = \delta_{a,a'} \langle \mathcal{J}_n^a \mathcal{J}_{n'}^a \rangle$. For $\langle \mathcal{J}_n^a \mathcal{J}_{n'}^a \rangle$, using these properties and Theorem 6, we obtain the following theorem:

Theorem 7. *For the one-dimensional long-range interacting XXZ model with Kac factor (6.1.2) in the infinite line, under the condition $\beta < \tilde{\beta}_c$, the equal-time spin and energy current correlation $\langle \mathcal{J}_n^a \mathcal{J}_0^a \rangle$ ($a = s, \epsilon$) is upper-bounded as*

$$|\langle \mathcal{J}_n^a \mathcal{J}_0^a \rangle| \leq C_a, \quad (6.3.6)$$

for $0 \leq \alpha < 1$ and $\alpha \neq 1/2$. Here, C_a, C'_a ($a = s, \epsilon$) are constants that do not depend on n .

For $\alpha = 1/2$, the equal-time current correlation of N spins is upper-bounded as $|\langle \mathcal{J}_n^a \mathcal{J}_0^a \rangle| \leq \log(n + N)$. Therefore, this provides only a trivial constraint in the case $\alpha = 1/2$.

6.4 Proof of Theorem 7

Proof of Theorem 7. In the following, we provide the proof only for the spin current. The corresponding statement for the energy current can be established in an entirely analogous manner.

The equal-time current correlation $\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle$ is described as follows:

$$\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle = \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \frac{J^4}{4} \sum_{\substack{\{a,b\}=\{k,\ell\} \\ \{a',b'\}=\{k',\ell'\}}} \varepsilon_{ab} \varepsilon_{a'b'} \frac{\langle S_a^x S_b^y S_{a'}^x S_{b'}^y \rangle}{\tilde{d}_{a,b}^\alpha \tilde{d}_{a',b'}^\alpha}. \quad (6.4.1)$$

This expression (6.4.1) is obtained from (5.4.1) in the proof of Theorem 2, by replacing $d_{i,j}$ to $\tilde{d}_{i,j}$. Therefore, by applying Theorem 6 and Lemma 6 instead of Theorem 1 and Lemma 4, we can carry out the analogous argument as the proof of Theorem 2.

As seen in (5.4.7) and (5.4.12) for the proof of Theorem 2, the contribution to the upper bound of equal-time current correlation, except for the case (2) in Fig.5.3 is given as follows:

$$|\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle|_{\text{except Fig.5.3(2)}} \leq \mathcal{O}(1) \sum_{\substack{k \leq n \\ \ell' < 0}} \frac{1}{\tilde{d}_{k,\ell'}^{2\alpha}} \quad (6.4.2)$$

For $\alpha \neq 1/2$, the right-handed side of (6.4.2) is upper-bounded as

$$\begin{aligned} \sum_{\substack{k \leq n \\ \ell' < 0}} \frac{1}{\tilde{d}_{k,\ell'}^{2\alpha}} &\leq (\mathcal{N}_\alpha^{(1)})^{-2} \int_1^{N/2} dx \int_1^{N/2} dy \frac{1}{(n+x+y)^{2\alpha}} \\ &\leq (\mathcal{N}_\alpha^{(1)})^{-2} \frac{1}{(1-2\alpha)(2-2\alpha)} \left\{ (n+N)^{2-2\alpha} - 2(n+N/2)^{2-2\alpha} + n^{2-2\alpha} \right\} \\ &= \mathcal{O}(1), \end{aligned} \quad (6.4.3)$$

where we use $\mathcal{N}_\alpha^{(1)} = \mathcal{O}(N^{1-\alpha})$. On the other hand, for $\alpha = 1/2$, the right-handed side of (6.4.2) is upper-bounded as

$$\begin{aligned} \sum_{\substack{k \leq n \\ \ell' < 0}} \frac{1}{\tilde{d}_{k,\ell'}} &\leq (\mathcal{N}_{\alpha=1/2}^{(1)})^{-2} \int_1^{N/2} dx \int_1^{N/2} dy (n+x+y)^{-1} \\ &\leq (\mathcal{N}_{\alpha=1/2}^{(1)})^{-2} \{ (n+N) \log(n+N) - n \log n - N \} \\ &\leq \mathcal{O}(1) \log(n+N), \end{aligned} \quad (6.4.4)$$

where we use $\mathcal{N}_{\alpha=1/2}^{(1)} = \mathcal{O}(N^{1/2})$. From (6.4.4), we obtain only a trivial bound in the case of $\alpha = 1/2$.

Similarly as (5.4.9) for the proof of Theorem 2, the contribution from the case (2) in Fig.5.3 is obtained as follows:

$$\begin{aligned} |\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle|_{\text{Fig.5.3(2)}} &\leq \mathcal{O}(1) \sum_{\substack{k,\ell,k',\ell' \\ :k \geq n > \ell > k' \geq 0 > \ell'}} \frac{1}{\tilde{d}_{k,\ell}^{2\alpha}} \frac{1}{\tilde{d}_{\ell,k'}^\alpha} \frac{1}{\tilde{d}_{k',\ell'}^{2\alpha}} \\ &= \mathcal{O}(1) (\mathcal{N}_\alpha)^{-5} \sum_{\substack{k,\ell,k',\ell' \\ :k \geq n > \ell > k' \geq 0 > \ell'}} \frac{1}{d_{k,\ell}^{2\alpha}} \frac{1}{d_{\ell,k'}^\alpha} \frac{1}{d_{k',\ell'}^{2\alpha}} \end{aligned} \quad (6.4.5)$$

The right-handed side of (6.4.5) is upper-bounded as follows:

$$\begin{aligned}
& \sum_{\substack{k,\ell,k',\ell' \\ :k \geq n > \ell > k' \geq 0 > \ell'}} \frac{1}{\tilde{d}_{k,\ell}^{2\alpha}} \frac{1}{\tilde{d}_{\ell,k'}^\alpha} \frac{1}{\tilde{d}_{k',\ell'}^{2\alpha}} = (\mathcal{N}_\alpha^{(1)})^{-5} \sum_{\substack{k,\ell,k',\ell' \\ :k \geq n > \ell > k' \geq 0 > \ell'}} \frac{1}{\tilde{d}_{k,\ell}^{2\alpha}} \frac{1}{d_{\ell,k'}^\alpha} \frac{1}{\tilde{d}_{k',\ell'}^{2\alpha}} \\
&= (\mathcal{N}_\alpha^{(1)})^{-5} \sum_{\substack{k,\ell,k',\ell' \\ :k \geq n > \ell > k' \geq 0 > \ell'}} \frac{1}{(d_{k,n} + d_{n,\ell})^{2\alpha}} \frac{1}{d_{\ell,k'}^\alpha} \frac{1}{(d_{k',0} + d_{0,\ell'})^{2\alpha}} \\
&\leq (\mathcal{N}_\alpha^{(1)})^{-5} \sum_{\substack{\ell,k' \\ :n > \ell > k' \geq 0}} \int_1^{N/2} dx \int_1^{N/2} dy \frac{1}{(d_{n,\ell} + x)^{2\alpha}} \frac{1}{d_{\ell,k'}^\alpha} \frac{1}{(d_{k',0} + y)^{2\alpha}} \quad (6.4.6)
\end{aligned}$$

To estimate the integral in (6.4.6), we consider the two cases, $1/2 < \alpha < 1$ and $0 \leq \alpha < 1/2$. Firstly, for $1/2 < \alpha < 1$, we have

$$\begin{aligned}
& (\mathcal{N}_\alpha^{(1)})^{-5} \sum_{\substack{\ell,k' \\ :n > \ell > k' \geq 0}} \int_1^{N/2} dx \int_1^{N/2} dy \frac{1}{(d_{n,\ell} + x)^{2\alpha}} \frac{1}{d_{\ell,k'}^\alpha} \frac{1}{(d_{k',0} + y)^{2\alpha}} \\
&\leq \frac{(\mathcal{N}_\alpha^{(1)})^{-5}}{(1-2\alpha)^2} \sum_{\substack{\ell,k' \\ :n > \ell > k' \geq 0}} \frac{1}{\tilde{d}_{n,\ell}^{2\alpha-1}} \frac{1}{d_{\ell,k'}^\alpha} \frac{1}{\tilde{d}_{k',0}^{2\alpha-1}} \leq \frac{(\mathcal{N}_\alpha^{(1)})^{-5}}{(1-2\alpha)^2} \sum_{\substack{\ell,k' \\ :n > \ell > k' \geq 0}} \frac{1}{\tilde{d}_{n,\ell}^{2\alpha-1}} \frac{1}{d_{\ell,k'}^{2\alpha-1}} \frac{1}{\tilde{d}_{k',0}^{2\alpha-1}} \\
&= \frac{(\mathcal{N}_\alpha^{(1)})^{(6\alpha-3)/\alpha-5}}{(1-2\alpha)^2} \sum_{\substack{\ell,k' \\ :n > \ell > k' \geq 0}} \frac{1}{\tilde{d}_{n,\ell}^{2\alpha-1}} \frac{1}{\tilde{d}_{\ell,k'}^{2\alpha-1}} \frac{1}{\tilde{d}_{k',0}^{2\alpha-1}} \\
&\leq \frac{(\mathcal{N}_\alpha^{(1)})^{1-3/\alpha}}{(1-2\alpha)^2} \frac{(\tilde{u}_{2\alpha-1})^2}{\tilde{d}_{n,0}^{2\alpha-1}} = \frac{(\tilde{u}_{2\alpha-1})^2}{(1-2\alpha)^2} \frac{(\mathcal{N}_\alpha^{(1)})^{-2-2/\alpha}}{n^{2\alpha-1}} = \mathcal{O}(N^{(2+2\alpha)(\alpha-1)}) n^{1-2\alpha}, \quad (6.4.7)
\end{aligned}$$

where to derive the last line, we use Lemma 6. Secondly, for $0 \leq \alpha < 1/2$, we have

$$\begin{aligned}
& (\mathcal{N}_\alpha)^{-5} \sum_{\substack{\ell,k' \\ :n > \ell > k' \geq 0}} \int_1^{N/2} dx \int_1^{N/2} dy \frac{1}{(d_{n,\ell} + x)^{2\alpha}} \frac{1}{d_{\ell,k'}^\alpha} \frac{1}{(d_{k',0} + y)^{2\alpha}} \\
&\leq \frac{(\mathcal{N}_\alpha^{(1)})^{-5}}{(1-2\alpha)^2} \sum_{\substack{\ell,k' \\ :n > \ell > k' \geq 0}} \frac{1}{(N/2 + d_{n,\ell})^{2\alpha-1}} \frac{1}{d_{\ell,k'}^\alpha} \frac{1}{(d_{k',0} + N/2)^{2\alpha-1}} \\
&= \frac{(\mathcal{N}_\alpha^{(1)})^{-5}}{(1-2\alpha)^2} \sum_{\substack{\ell,k' \\ :n > \ell > k' \geq 0}} \frac{1}{(d_{n+N/2,\ell})^{2\alpha-1}} \frac{1}{d_{\ell,k'}^\alpha} \frac{1}{(d_{k',-N/2})^{2\alpha-1}} \\
&\leq \frac{(\mathcal{N}_\alpha^{(1)})^{(6\alpha-3)/\alpha-5}}{(1-2\alpha)^2} \sum_{\substack{\ell,k' \\ :n > \ell > k' \geq 0}} \frac{1}{(\tilde{d}_{n+N/2,\ell})^{2\alpha-1}} \frac{1}{\tilde{d}_{\ell,k'}^{2\alpha-1}} \frac{1}{(\tilde{d}_{k',-N/2})^{2\alpha-1}} \\
&\leq \frac{(\mathcal{N}_\alpha^{(1)})^{1-3/\alpha}}{(1-2\alpha)^2} \frac{(\tilde{u}_{2\alpha-1})^2}{\tilde{d}_{N/2,-N/2}^{2\alpha-1}} = \frac{(\tilde{u}_{2\alpha-1})^2}{(1-2\alpha)^2} \frac{(\mathcal{N}_\alpha^{(1)})^{-2-2/\alpha}}{N^{2\alpha-1}} = \mathcal{O}(N^{-2\alpha(1-\alpha)-1}), \quad (6.4.8)
\end{aligned}$$

where in the last inequality, we also use Lemma 6. From (6.4.7) and (6.4.8), for $\alpha \neq 1/2$, the case (2) in Fig.5.3 does not contribute to the leading term for $\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle$ compared to (6.4.3).

From (6.4.3), (6.4.7) and (6.4.8), we finally obtain the following upper bound for $|\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle|$ in the case $\alpha \neq 1/2$:

$$|\langle \mathcal{J}_n^s \mathcal{J}_0^s \rangle| \leq \mathcal{O}(1). \quad (6.4.9)$$

□

6.5 Higher dimensional case

Theorem 6 also holds in higher spatial dimensions. We therefore proceed to analyze a generic D -dimensional lattice consisting of N^D sites. We focus on energy and spin diffusion that propagates isotropically in concentric patterns, and characterize the transport by the mean square displacement $\sigma(t)$ defined in Eq. (5.6.5). When the diffusion is normal, this quantity gives rise to the diffusion constant $\mathfrak{D}^a$ and the associated total current correlation function $C_N^{(a,D)}(t)$ ($a = s, \epsilon$) through the Green–Kubo formula (4.6.2) and (5.6.6). Within this framework, we establish the following theorem, which provides an upper bound on the equal-time total current correlation $C_N^{(a,D)}(0)$.

Theorem 8. *In $D (\geq 2)$ -dimensional long-range interacting spin systems with the Kac factor, satisfying $0 \leq \alpha < D$, $\alpha \neq D/2$, $\alpha \neq (D+1)/2$, and $\beta < \tilde{\beta}_c$, including the one-dimensional realization given in Eq. (6.1.2), the equal-time total current correlation $C_N^{(a,D)}(0)$ ($a = s, \epsilon$) remains finite.*

6.6 Proof of Theorem 8

Proof of Theorem 8. In the following, we provide the proof only for the spin current. The corresponding statement for the energy current can be established in an entirely analogous manner.

$C_N^{(s,D)}(0)$ is described as follows:

$$C_N^{(s,D)}(0) = N^{1-D} \sum_{i=1}^D \sum_{n_i} \langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \rangle, \quad (6.6.1)$$

where $\mathcal{J}_{n_i|n_i-1}^{(s,i)}$ is the spin current in i -th direction, which corresponds to (5.7.2) under replacing $d_{i,j}$ to $\tilde{d}_{i,j}$. Therefore, by applying Theorem 6 and Lemma 6 instead of Theorem 1 and Lemma 4, a similar argument can be developed to the proof of Theorem 3. In the same way as (5.7.4), we obtain the upper bound for the current correlation, except for the case (2) in

Fig.5.3 as follows:

$$\left| \left\langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \right\rangle \right|_{\text{except Fig.5.3(2)}} \leq \mathcal{O}(1) \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell': \ell'_i < 0}} \frac{1}{\tilde{d}_{\mathbf{k}, \ell'}^{2\alpha}} \quad (6.6.2)$$

For $D \geq 2$, the right-handed side of (6.6.2) is upper-bounded as

$$\begin{aligned} \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell': \ell'_i < 0}} \frac{1}{\tilde{d}_{\mathbf{k}, \ell'}^{2\alpha}} &= (\mathcal{N}_\alpha^{(D)})^{-2} \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell': \ell'_i < 0}} \frac{1}{\tilde{d}_{\mathbf{k}, \ell'}^{2\alpha}} \\ &\leq (\mathcal{N}_\alpha^{(D)})^{-2} \int_{\substack{x_i \geq n_i \\ y_i < 0}} dx_i dy_i \prod_{k \neq i} \int_{-N/2}^{N/2} dx_k dy_k \left\{ (x_i - y_i)^2 + \sum_{j \neq i} (x_j - y_j)^2 \right\}^{-2\alpha/2} \\ &\leq \Gamma_{D-1} (\mathcal{N}_\alpha^{(D)})^{-2} \int_{-N/2}^{N/2} d^{D-1} \mathbf{u} \int_{\substack{x_i \geq n_i \\ y_i < 0}} dx_i dy_i \int_0^\infty d\xi \xi^{D-2} \{ (x_i - y_i)^2 + \xi^2 \}^{-2\alpha/2} \\ &\leq \frac{\Gamma_{D-1} B(\frac{D-1}{2}, \alpha - \frac{D-1}{2})}{2} (\mathcal{N}_\alpha^{(D)})^{-2} N^{D-1} \int_0^{N/2} dx dy (n_i + x + y)^{D-1-2\alpha} \\ &\leq \begin{cases} \Gamma' (\mathcal{N}_\alpha^{(D)})^{-2} N^{D-1} n_i^{D+1-2\alpha}, & \text{for } (D+1)/2 < \alpha < D \\ \Gamma' (\mathcal{N}_\alpha^{(D)})^{-2} N^{2D-2\alpha}, & \text{for } 0 \leq \alpha < (D+1)/2, \alpha \neq D/2 \end{cases} \end{aligned} \quad (6.6.3)$$

where $u_j := (x_j + y_j)/2$ for $j \neq i$, $\xi := \{\sum_{j \neq i} (x_j + y_j)^2\}^{1/2}$, Γ_{D-1} is the coefficient of $(D-2)$ -dimensional spherical surface area, $B(x, y)$ is the Beta function, and we denote $\Gamma' = \Gamma_{D-1} B(\frac{D-1}{2}, \alpha - \frac{D-1}{2}) / (2(D-2\alpha)(D+1-2\alpha))$.

From (6.6.3), in both cases $(D+1)/2 < \alpha < D$ and $\alpha < (D+1)/2$, $\alpha \neq D/2$, we obtain the following upper bound for $C_N^{(s,D)}(0)$, except for the case (2) in Fig.5.3 :

$$\begin{aligned} \left| C_N^{(s,D)}(0) \right|_{\text{except Fig.5.3(2)}} &\leq N^{1-D} \sum_{i=1}^D \sum_{n_i} \left| \left\langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \right\rangle \right|_{\text{except Fig.5.3(2)}} \\ &\leq \mathcal{O}(N^{2-D}) + \mathcal{O}(1) = \mathcal{O}(1), \end{aligned} \quad (6.6.4)$$

where we use $\mathcal{N}_\alpha^{(D)} = \mathcal{O}(N^{D-\alpha})$ and $D \geq 2$.

Next, we consider the contribution of case (2) in Fig. 5.3. Similarly to (5.7.6), we obtain the following upper bound for the current correlation for case (2) in Fig. 5.3:

$$\left| \left\langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \right\rangle \right|_{\text{Fig.5.3(2)}} \leq \mathcal{O}(1) \sum_{\mathbf{k}, \ell, \mathbf{k}', \ell'} \frac{1}{\tilde{d}_{\mathbf{k}, \ell}^{2\alpha}} \frac{1}{\tilde{d}_{\ell, \mathbf{k}'}^\alpha} \frac{1}{\tilde{d}_{\mathbf{k}', \ell'}^{2\alpha}}, \quad (6.6.5)$$

where we take the summation over $\{\mathbf{k}, \ell, \mathbf{k}', \ell'\}$ such that we construct closed loop for $(\mathbf{k}, \ell)$ and $(\mathbf{k}', \ell')$, and they are connected by a single edge. Therefore, it satisfies that $k_i > n_i > \ell_i$ and $k'_i > 0 > \ell'_i$.

Firstly, we evaluate the sum over $\mathbf{k}, \ell'$. For the sum over $\mathbf{k}$, the following inequality is

obtained:

$$\begin{aligned}
& \sum_{\mathbf{k}: (k_i - n_i)(\ell_i - n_i) \leq 0} \frac{1}{\tilde{d}_{\mathbf{k}, \ell}^{2\alpha}} = (\mathcal{N}_\alpha^{(D)})^{-2} \sum_{\mathbf{k}: (k_i - n_i)(\ell_i - n_i) \leq 0} \frac{1}{d_{\mathbf{k}, \ell}^{2\alpha}} \\
& \leq (\mathcal{N}_\alpha^{(D)})^{-2} \int_{(k_i - n_i)(\ell_i - n_i) \leq 0} d^d \mathbf{k} \frac{1}{\{(k_i - \ell_i)^2 + \sum_{j \neq i} (k_j - \ell_j)^2\}^{2\alpha/2}} \\
& \leq (\mathcal{N}_\alpha^{(D)})^{-2} \int_{(k_i - n_i)(\ell_i - n_i) \leq 0} dk_i \Gamma_{D-1} \int_0^\infty d\xi \xi^{D-2} \frac{1}{\{(k_i - \ell_i)^2 + \xi^2\}^{2\alpha/2}} \\
& \leq \Gamma''(\mathcal{N}_\alpha^{(D)})^{-2} \int_{(k_i - n_i)(\ell_i - n_i) \leq 0} dk_i \frac{1}{(k_i - \ell_i)^{2\alpha - D + 1}} \\
& \leq \Gamma''(\mathcal{N}_\alpha^{(D)})^{-2} \int_0^{N/2} dx \frac{1}{(x + n_i - \ell_i)^{2\alpha - D + 1}} \\
& \leq \begin{cases} \Gamma'''(\mathcal{N}_\alpha^{(D)})^{-2} \frac{1}{(n_i - \ell_i)^{2\alpha - D}} = \Gamma'''(\mathcal{N}_\alpha^{(D)})^{-2} \frac{1}{d_{\bar{\mathbf{n}}_\ell, \ell}^{2\alpha - D}} & \text{for } D/2 < \alpha < D, \\ \Gamma'''(\mathcal{N}_\alpha^{(D)})^{-2} \frac{1}{(n_i + N/2 - \ell_i)^{2\alpha - D}} = \Gamma'''(\mathcal{N}_\alpha^{(D)})^{-2} \frac{1}{d_{\bar{\mathbf{n}}_\ell + \mathbf{N}/2, \ell}^{2\alpha - D}} & \text{for } 0 \leq \alpha < D/2, \end{cases}
\end{aligned} \tag{6.6.6}$$

where $\Gamma'' = \Gamma_{D-1} B(\frac{D-1}{2}, \alpha - \frac{D-1}{2})/2$, $\Gamma''' = \Gamma''/(2\alpha - D)$. For $D/2 < \alpha < D$, we denote $\bar{\mathbf{n}}_\ell$ as the foot of the perpendicular drop from ℓ to the hyperplane $x_i = n_i$, as in the leftmost part of Fig. 5.9. On the other hand, for $0 \leq \alpha < D/2$, we define $\bar{\mathbf{n}}_\ell + \mathbf{N}/2$ as the point obtained by extending the perpendicular drop from ℓ to the plane $x_i = n_i$ by an additional distance $N/2$.

Similarly, for the sum over ℓ' , the following inequality is obtained:

$$\sum_{\ell': k'_i \ell'_i \leq 0} \frac{1}{\tilde{d}_{\mathbf{k}', \ell'}^{2\alpha}} \leq \begin{cases} \Gamma'''(\mathcal{N}_\alpha^{(D)})^{-2} \frac{1}{d_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'}}^{2\alpha - D}} & \text{for } D/2 < \alpha < D, \\ \Gamma'''(\mathcal{N}_\alpha^{(D)})^{-2} \frac{1}{d_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'} - \mathbf{N}/2}^{2\alpha - D}} & \text{for } 0 \leq \alpha < D/2, \end{cases} \tag{6.6.7}$$

where for $D/2 < \alpha < D$, we denote $\bar{\mathbf{0}}_{\mathbf{k}'}$ as the foot of the perpendicular drop from $\mathbf{k}'$ to the hyperplane $x_i = 0$, as in the leftmost part of Fig. 5.9. For $0 \leq \alpha < D/2$, we also denote $\bar{\mathbf{0}}_{\mathbf{k}'} - \mathbf{N}/2$ as the point obtained by extending the perpendicular drop from $\mathbf{k}'$ to the plane $x_i = 0$ by an additional distance $N/2$.

Substituting (6.6.6) and (6.6.7) into (6.6.5), we obtain the upper bound for $D/2 < \alpha < D$ as follows:

$$\begin{aligned}
\left| \left\langle \mathcal{J}_{n_i | n_i - 1}^{(s, i)} \mathcal{J}_{0 | -1}^{(s, i)} \right\rangle \right|_{\text{Fig. 5.3(2)}} & \leq (\Gamma''')^2 (\mathcal{N}_\alpha^{(D)})^{-5} \sum_{\ell, \mathbf{k}'} \frac{1}{d_{\bar{\mathbf{n}}_\ell, \ell}^{2\alpha - D}} \frac{1}{d_{\ell, \mathbf{k}'}^\alpha} \frac{1}{d_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'}}^{2\alpha - D}} \\
& \leq (\Gamma''')^2 (\mathcal{N}_\alpha^{(D)})^{-5} \sum_{\ell, \mathbf{k}'} \frac{1}{d_{\bar{\mathbf{n}}_\ell, \ell}^{2\alpha - D}} \frac{1}{d_{\ell, \mathbf{k}'}^{2\alpha - D}} \frac{1}{d_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'}}^{2\alpha - D}}.
\end{aligned} \tag{6.6.8}$$

For $0 \leq \alpha < D/2$, we also obtain the following upper bound:

$$\begin{aligned} \left| \left\langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \right\rangle \right|_{\text{Fig. 5.3(2)}} &\leq (\Gamma''')^2 (\mathcal{N}_\alpha^{(D)})^{-5} \sum_{\ell, \mathbf{k}'} \frac{1}{d_{\bar{\mathbf{n}}_\ell + \mathbf{N}/2, \ell}^{2\alpha-D}} \frac{1}{d_{\ell, \mathbf{k}'}^\alpha} \frac{1}{d_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'} - \mathbf{N}/2}^{2\alpha-D}} \\ &\leq (\Gamma''')^2 (\mathcal{N}_\alpha^{(D)})^{-5} \sum_{\ell, \mathbf{k}'} \frac{1}{d_{\bar{\mathbf{n}}_\ell + \mathbf{N}/2, \ell}^{2\alpha-D}} \frac{1}{d_{\ell, \mathbf{k}'}^{2\alpha-D}} \frac{1}{d_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'} - \mathbf{N}/2}^{2\alpha-D}}. \end{aligned} \quad (6.6.9)$$

Next, to further evaluate the upper bound (6.6.8) and (6.6.9), we present the following lemma, which corresponds to Lemma 5 by replacing $d_{i,j}$ with $\tilde{d}_{i,j}$.

Lemma 7. *Consider $\mathbf{i}, \mathbf{j}$ depend on $\mathbf{k}$, and assume that $\bar{u}_i := \sum_{\mathbf{k}} 1/\tilde{d}_{\mathbf{k}, \mathbf{i}(\mathbf{k})}^\alpha < \infty$ and $\bar{u}_j := \sum_{\mathbf{k}} 1/\tilde{d}_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha < \infty$ hold. Under the condition $0 \leq \alpha < D$, we obtain the following inequality:*

$$\sum_{\mathbf{k}} \frac{1}{\tilde{d}_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha} \frac{1}{\tilde{d}_{\mathbf{j}(\mathbf{k}), \mathbf{k}}^\alpha} \leq \frac{\bar{u}_\alpha}{\tilde{d}_{\mathbf{i}, \mathbf{j}}^\alpha}, \quad (6.6.10)$$

where $\tilde{d}_{\mathbf{i}, \mathbf{j}}^{-\alpha} := \sup_{\mathbf{k}} \tilde{d}_{\mathbf{i}(\mathbf{k}), \mathbf{j}(\mathbf{k})}^{-\alpha}$, and $\bar{u}_\alpha := 2^{\alpha-1}(\bar{u}_i + \bar{u}_j)$.

Proof of Lemma 7.

$$\begin{aligned} \sum_{\mathbf{k}} \frac{1}{\tilde{d}_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha} \frac{1}{\tilde{d}_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha} &= (\mathcal{N}_\alpha^{(D)})^{-2} \sum_{\mathbf{k}} \frac{1}{d_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha} \frac{1}{d_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha} = (\mathcal{N}_\alpha^{(D)})^{-2} \sum_{\mathbf{k}} \frac{1}{d_{\mathbf{i}(\mathbf{k}), \mathbf{j}(\mathbf{k})}^\alpha} \frac{d_{\mathbf{i}(\mathbf{k}), \mathbf{j}(\mathbf{k})}^\alpha}{d_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha d_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha} \\ &\leq (\mathcal{N}_\alpha^{(D)})^{-2} \frac{1}{d_{\mathbf{i}, \mathbf{j}}^\alpha} \sum_{\mathbf{k}} \frac{(d_{\mathbf{i}(\mathbf{k}), \mathbf{k}} + d_{\mathbf{k}, \mathbf{j}(\mathbf{k})})^\alpha}{d_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha d_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha} \leq 2^{\alpha-1} (\mathcal{N}_\alpha^{(D)})^{-2} \frac{1}{d_{\mathbf{i}, \mathbf{j}}^\alpha} \sum_{\mathbf{k}} \frac{d_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha + d_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha}{d_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha d_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha} \\ &= 2^{\alpha-1} \frac{1}{\mathcal{N}_\alpha^{(D)} d_{\mathbf{i}, \mathbf{j}}^\alpha} \sum_{\mathbf{k}} \left(\frac{1}{\mathcal{N}_\alpha^{(D)} d_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha} + \frac{1}{\mathcal{N}_\alpha^{(D)} d_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha} \right) \\ &= 2^{\alpha-1} \frac{1}{\tilde{d}_{\mathbf{i}, \mathbf{j}}^\alpha} \sum_{\mathbf{k}} \left(\frac{1}{\tilde{d}_{\mathbf{i}(\mathbf{k}), \mathbf{k}}^\alpha} + \frac{1}{\tilde{d}_{\mathbf{k}, \mathbf{j}(\mathbf{k})}^\alpha} \right) = \frac{\bar{u}_\alpha}{\tilde{d}_{\mathbf{i}, \mathbf{j}}^\alpha} \end{aligned} \quad (6.6.11)$$

Here, we use $\{(x+y)/2\}^\alpha \leq (x^\alpha + y^\alpha)/2$, following from the convexity of $f(x) = x^\alpha$ and the convexity inequality. $\square$

Firstly, we consider the case $D/2 < \alpha < D$. Applying Lemma 7 to (6.6.8), we obtain the

following upper bound:

$$\begin{aligned}
& \left| \left\langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \right\rangle \right|_{\text{Fig.5.3(2)}} \\
& \leq (\Gamma''')^2 (\mathcal{N}_\alpha^{(D)})^{-5} \sum_{\ell, \mathbf{k}'} \frac{1}{d_{\bar{\mathbf{n}}_\ell, \ell}^{2\alpha-D}} \frac{1}{d_{\ell, \mathbf{k}'}^{2\alpha-D}} \frac{1}{d_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'}}^{2\alpha-D}} \\
& = (\Gamma''')^2 (\mathcal{N}_\alpha^{(D)})^{(6\alpha-3D)/\alpha-5} \sum_{\ell, \mathbf{k}'} \frac{1}{\tilde{d}_{\bar{\mathbf{n}}_\ell, \ell}^{2\alpha-D}} \frac{1}{\tilde{d}_{\ell, \mathbf{k}'}^{2\alpha-D}} \frac{1}{\tilde{d}_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'}}^{2\alpha-D}} \\
& \leq (\Gamma''')^2 (\mathcal{N}_\alpha^{(D)})^{1-3D/\alpha} \sum_{\mathbf{k}'} \frac{1}{\tilde{d}_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'}}^{2\alpha-D}} 2^{2\alpha-D-1} (\tilde{u}_{2\alpha-D} + \bar{u}_{\bar{\mathbf{n}}_\ell}) \sup_{\ell} \frac{1}{\tilde{d}_{\bar{\mathbf{n}}_\ell, \mathbf{k}'}^{2\alpha-D}} \\
& = 2^{2\alpha-D-1} (\Gamma''')^2 (\tilde{u}_{2\alpha-D} + \bar{u}_{\bar{\mathbf{n}}_\ell}) (\mathcal{N}_\alpha^{(D)})^{1-3D/\alpha} \sum_{\mathbf{k}'} \frac{1}{\tilde{d}_{\bar{\mathbf{n}}_{\mathbf{k}'}, \mathbf{k}'}^{2\alpha-D}} \frac{1}{\tilde{d}_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'}}^{2\alpha-D}} \\
& \leq 2^{2(2\alpha-D-1)} (\Gamma''')^2 (\tilde{u}_{2\alpha-D} + \bar{u}_{\bar{\mathbf{n}}_\ell}) (\bar{u}_{\bar{\mathbf{0}}_{\mathbf{k}'}} + \bar{u}_{\bar{\mathbf{n}}_{\mathbf{k}'}}) (\mathcal{N}_\alpha^{(D)})^{1-3D/\alpha} \sup_{\mathbf{k}'} \frac{1}{\tilde{d}_{\bar{\mathbf{n}}_{\mathbf{k}'}, \bar{\mathbf{0}}_{\mathbf{k}'}}^{2\alpha-D}} \\
& = \mathcal{O}(1) (\mathcal{N}_\alpha^{(D)})^{1-3D/\alpha} \frac{(\mathcal{N}_\alpha^{(D)})^{-(2\alpha-D)/\alpha}}{n_i^{2\alpha-D}} = \mathcal{O}(1) (\mathcal{N}_\alpha^{(D)})^{-1-2D/\alpha} n_i^{D-2\alpha}, \tag{6.6.12}
\end{aligned}$$

where $\bar{u}_{\bar{\mathbf{n}}_{\mathbf{k}'}} = \sum_{\mathbf{k}'} \tilde{d}_{\bar{\mathbf{n}}_{\mathbf{k}'}, \mathbf{k}'}^{D-2\alpha} < \infty$ for $\alpha < D$. Furthermore, let $\tilde{\mathbf{n}}_{\mathbf{k}'}$ be the $\bar{\mathbf{n}}_{\mathbf{k}'}$ that satisfies $\sup_{\mathbf{k}'} \tilde{d}_{\tilde{\mathbf{n}}_{\mathbf{k}'}, \mathbf{k}'}^{D-2\alpha} = \tilde{d}_{\tilde{\mathbf{n}}_{\mathbf{k}'}, \mathbf{k}'}^{D-2\alpha}$. As shown on the far right of Fig. 5.9, $\tilde{\mathbf{n}}_{\mathbf{k}'}$ coincides with the foot of the perpendicular dropped from $\mathbf{k}'$ to the hyperplane $x_i = n_i$. The condition of Lemma 5 is satisfied, as $\bar{u}_{\bar{\mathbf{0}}_{\mathbf{k}'}} = \sum_{\mathbf{k}'} \tilde{d}_{\bar{\mathbf{0}}_{\mathbf{k}'}, \mathbf{k}'}^{D-2\alpha} < \infty$ and $\bar{u}_{\tilde{\mathbf{n}}_{\mathbf{k}'}} = \sum_{\mathbf{k}'} \tilde{d}_{\tilde{\mathbf{n}}_{\mathbf{k}'}, \mathbf{k}'}^{D-2\alpha} < \infty$.

Secondly, we consider the case $0 \leq \alpha < D/2$. Using the Lemma 7 and (6.6.9), we obtain the following upper bound:

$$\begin{aligned}
& \left| \left\langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \right\rangle \right|_{\text{Fig.5.3(2)}} \\
& \leq (\Gamma''')^2 (\mathcal{N}_\alpha^{(D)})^{-5} \sum_{\ell, \mathbf{k}'} \frac{1}{d_{\bar{\mathbf{n}}_\ell + \mathbf{N}/2, \ell}^{2\alpha-D}} \frac{1}{d_{\ell, \mathbf{k}'}^{2\alpha-D}} \frac{1}{d_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'} - \mathbf{N}/2}^{2\alpha-D}} \\
& = (\Gamma''')^2 (\mathcal{N}_\alpha^{(D)})^{(6\alpha-3D)/\alpha-5} \sum_{\ell, \mathbf{k}'} \frac{1}{\tilde{d}_{\bar{\mathbf{n}}_\ell + \mathbf{N}/2, \ell}^{2\alpha-D}} \frac{1}{\tilde{d}_{\ell, \mathbf{k}'}^{2\alpha-D}} \frac{1}{\tilde{d}_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'} - \mathbf{N}/2}^{2\alpha-D}} \\
& \leq (\Gamma''')^2 (\mathcal{N}_\alpha^{(D)})^{1-3D/\alpha} \sum_{\mathbf{k}'} \frac{1}{\tilde{d}_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'} - \mathbf{N}/2}^{2\alpha-D}} 2^{2\alpha-D-1} (\tilde{u}_{2\alpha-D} + \bar{u}_{\bar{\mathbf{n}}_\ell}) \sup_{\ell} \frac{1}{\tilde{d}_{\bar{\mathbf{n}}_\ell + \mathbf{N}/2, \mathbf{k}'}^{2\alpha-D}} \\
& = 2^{2\alpha-D-1} (\Gamma''')^2 (\tilde{u}_{2\alpha-D} + \bar{u}_{\bar{\mathbf{n}}_\ell}) (\mathcal{N}_\alpha^{(D)})^{1-3D/\alpha} \sum_{\mathbf{k}'} \frac{1}{\tilde{d}_{\bar{\mathbf{n}}_{\mathbf{k}'} + \mathbf{N}/2, \mathbf{k}'}^{2\alpha-D}} \frac{1}{\tilde{d}_{\mathbf{k}', \bar{\mathbf{0}}_{\mathbf{k}'} - \mathbf{N}/2}^{2\alpha-D}} \\
& \leq 2^{2(2\alpha-D-1)} (\Gamma''')^2 (\tilde{u}_{2\alpha-D} + \bar{u}_{\bar{\mathbf{n}}_\ell}) (\bar{u}_{\bar{\mathbf{0}}_{\mathbf{k}'}} + \bar{u}_{\tilde{\mathbf{n}}_{\mathbf{k}'}}) (\mathcal{N}_\alpha^{(D)})^{1-3D/\alpha} \sup_{\mathbf{k}'} \frac{1}{\tilde{d}_{\bar{\mathbf{n}}_{\mathbf{k}'} + \mathbf{N}/2, \bar{\mathbf{0}}_{\mathbf{k}'} - \mathbf{N}/2}^{2\alpha-D}} \\
& = \mathcal{O}(1) (\mathcal{N}_\alpha^{(D)})^{1-3D/\alpha} \frac{(\mathcal{N}_\alpha^{(D)})^{-(2\alpha-D)/\alpha}}{(n_i + N)^{2\alpha-D}} = \mathcal{O}(1) (\mathcal{N}_\alpha^{(D)})^{-1-2D/\alpha} (n_i + N)^{D-2\alpha}, \tag{6.6.13}
\end{aligned}$$

From (6.6.12) and (6.6.13), in both cases $0 \leq \alpha < D/2$ and $D/2 < \alpha < D$, we obtain the

following upper bound for $C_N^{(s,D)}(0)$, for case (2) in Fig.5.3:

$$\begin{aligned} \left| C_N^{(s,D)}(0) \right|_{\text{Fig.5.3(2)}} &\leq N^{1-D} \sum_{i=1}^D \sum_{n_i} \left| \left\langle \mathcal{J}_{n_i|n_i-1}^{(s,i)} \mathcal{J}_{0|-1}^{(s,i)} \right\rangle \right|_{\text{Fig.5.3(2)}} \\ &\leq \mathcal{O}(N^{1-\alpha-2D(D-\alpha)/\alpha}) + \mathcal{O}(1) = \mathcal{O}(1), \end{aligned} \quad (6.6.14)$$

for $D \geq 2$.

From (6.6.4) and (6.6.14), we finally obtain that the equal-time total spin current correlation $C_N^{(s,D)}(0)$ is finite in the thermodynamic limit:

$$\left| C_N^{(s,D)}(0) \right| \leq \mathcal{O}(1). \quad (6.6.15)$$

□

Chapter 7

Conclusion

In this thesis, we have investigated the nature of energy and spin diffusion in long-range interacting spin systems.

Our analysis has focused on several prototypical models: the transverse Ising model and the XYZ model for energy transport, and the XXZ model for spin transport. These models are defined on a D -dimensional lattice with an interaction exponent $\alpha > D$, which ensures thermodynamic extensivity.

In one-dimensional systems, we have demonstrated that both normal and anomalous diffusion can occur. We have shown that the anomalous regime originates from an anomalous enhancement in the amplitude of equal-time current correlations.

To analyze such equal-time current correlations, we introduced the cumulant power-law clustering theorem (Theorem 1). This theorem extends the conventional clustering property to higher-order cumulants and is applicable to long-range interacting spin systems defined on D -dimensional lattices.

By applying the cumulant power-law clustering theorem to equal-time current correlations, we established a series of rigorous results that lead to a unified conclusion (Theorems 2 and 4): in one dimension, a sufficient condition for energy and spin diffusion to be normal is $\alpha > 3/2$, independently of the specific model or choice of parameters.

Furthermore, we constructed a theoretical framework of fluctuating hydrodynamics for long-range interacting systems (Sec. 3.2), which provides an effective description of the anomalous regime. Within this framework, the emergence of Lévy diffusion for $\alpha < 3/2$ is consistently captured, indicating that the above condition is optimal.

In higher dimensions, $D \geq 2$, our results show that diffusion remains normal whenever $\alpha > D$, which is the threshold required for thermodynamic extensivity (Theorems 3 and 5).

We have also addressed energy and spin diffusion in the strongly long-range interacting regime $0 \leq \alpha < D$ by introducing the Kac factor. In this setting, results analogous to the cumulant power-law clustering theorem can still be established (Theorem 6), allowing us to employ essentially the same mathematical techniques as in the weak long-range regime. In

one-dimensional systems, the equal-time current correlations are shown to be upper-bounded by a constant independent of the separation n (Theorem 7). In higher dimensions, the corresponding equal-time total current correlations are likewise proven to converge to finite values in the thermodynamic limit (Theorem 8).

On the other hand, the convergence of the time integral appearing in the Green–Kubo formula, as well as the system-size dependence of the associated relaxation time, remains poorly understood in this regime. Numerical studies are strongly affected by finite-size effects, making it difficult to reliably extract the asymptotic behavior. Consequently, the physical implications of the rigorous upper bounds obtained for current correlations in the strongly long-range interacting regime have yet to be fully clarified.

Overall, our analysis provides a systematic and exact framework for the study of transport phenomena in long-range interacting spin systems.

It would be of interest to extend this exact framework to energy transport in spring–particle systems with long-range interactions, as briefly discussed in Sec. 1.4. The behavior of energy and spin diffusion in the regime $\alpha < D$, both with and without the Kac factor, remains an open problem. Moreover, the cumulant power-law clustering theorem is valid for quantum systems as well, and it is therefore intriguing to examine whether similar diffusive phenomena arise in the quantum setting.

In deriving Theorem 1, we relied on the finiteness of the one-site energy ($gu < \infty$ and $\|O_{X_j}\| < \infty$). As a consequence, the theorem is not directly applicable to systems with unbounded local variables, namely those with $gu \rightarrow \infty$ and $\|O_{X_j}\| \rightarrow \infty$. Energy diffusion in systems with unbounded degrees of freedom, such as bosonic lattice models, therefore remains an important open problem.

Chapter A

Energy Diffusion in the Long-range Interacting XY Model

In this chapter, we present rigorous results on the upper bounds for energy current correlations in the context of energy diffusion in the long-range interacting XY model. Although these results are already contained in Theorem 2 and 3, which address the more general XYZ model, the XY case admits a substantially simpler diagrammatic computation compared to the XYZ model. For this reason, we include the analysis of the XY model in this chapter as an appendix.

A.1 Upper bound for the equal-time energy current correlation in one dimension

In this section, we consider the one-dimensional long-range interacting XY model, which is described by the following Hamiltonian

$$H = - \sum_{n=1}^N \sum_{r=1}^{N/2} \left[J_x \frac{S_n^x S_{n+r}^x}{r^\alpha} + J_y \frac{S_n^y S_{n+r}^y}{r^\alpha} \right]. \quad (\text{A.1})$$

The local energy at site n is defined as

$$\hat{h}_n := - \sum_{r=1}^{N/2} \sum_{\sigma=x,y} \frac{J_\sigma}{2} \left[\frac{S_n^\sigma S_{n+r}^\sigma}{r^\alpha} + \frac{S_n^\sigma S_{n-r}^\sigma}{r^\alpha} \right], \quad (\text{A.2})$$

and satisfies the following continuity equations

$$\partial_t \hat{h}_n = -(\mathcal{J}_{n+1}^\epsilon - \mathcal{J}_n^\epsilon), \quad (\text{A.3})$$

$$\mathcal{J}_n^\epsilon = - \sum_{\substack{i \geq n, j < n \\ |i-j| < N/2}} t_{i \leftarrow j}^\epsilon. \quad (\text{A.4})$$

The energy transmission operator $t_{i \leftarrow j}^\epsilon$ is defined as

$$t_{i \leftarrow j}^\epsilon = \{\hat{h}_i, \hat{h}_j\} = \sum_{m=1}^N \frac{J_x J_y}{4} \sum_{\{a,b,c\}=\{i,j,m\}} \tilde{\epsilon}_{abc} \frac{S_a^x S_b^y S_c^z}{d_{a,c}^\alpha d_{b,c}^\alpha}, \quad (\text{A.5})$$

where $\tilde{\epsilon}_{abc}$ is defined as follows:

$$\tilde{\epsilon}_{abc} = \begin{cases} +1, & \text{for } (a,b,c) = (i,j,m), (m,j,i), (i,m,j), \\ -1, & \text{for } (a,b,c) = (j,i,m), (j,m,i), (m,i,j), \end{cases} \quad (\text{A.6})$$

Note that $\tilde{\epsilon}_{abc}$ is different from Levi-Civita symbols.

From the symmetry of the Hamiltonian, we note that the following properties hold

$$\langle S_i^x \rangle = \langle S_i^y \rangle = \langle S_i^z \rangle = 0, \quad (\text{A.7})$$

$$\langle S_i^\sigma S_j^\tau \rangle = \delta_{\sigma,\tau} \langle S_i^\sigma S_j^\sigma \rangle \quad (\text{A.8})$$

$$\langle S_i^z S_j^z \mathcal{O} \rangle = \delta_{i,j} \langle (S_i^z)^2 \mathcal{O} \rangle, \quad (\text{A.9})$$

for $i, j = 1, \dots, n$ and $\sigma, \tau \in \{x, y, z\}$. The relation (A.7), (A.8) is shown from the fact that the Hamiltonian is invariant to the canonical transformation $(S_i^\sigma, S_i^\tau, S_i^v) \rightarrow (S_i^\sigma, -S_i^\tau, -S_i^v)$ for $\{\sigma, \tau, v\} = \{x, y, z\}$ and $i = 1, \dots, N$. The relation (A.9) is derived from the absence of components S_i^z in the Hamiltonian.

Using these properties and Theorem 1, we establish the XY case of Theorem 2:

Theorem A.1. *In the one-dimensional long-range interacting XY model, under the condition $\beta < \beta_c$, the equal-time current correlation is upper-bounded as*

$$|\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle| \leq C_1 n^{2-2\alpha}, \quad (\text{A.10})$$

for $\alpha > 1$.

Proof. The equal-time current correlation $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$ is given as follows:

$$\begin{aligned} \langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle &= \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a,b,c\}=\{k,\ell,m\} \\ \{a',b',c'\}=\{k',\ell',m'\}}} \tilde{\epsilon}_{abc} \tilde{\epsilon}_{a'b'c'} \frac{\langle S_a^x S_b^y S_c^z S_{a'}^x S_{b'}^y S_{c'}^z \rangle}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c}^\alpha d_{b',c}^\alpha} \\ &= \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a,b,c\}=\{k,\ell,m\} \\ \{a',b',c'\}=\{k',\ell',m'\}}} \tilde{\epsilon}_{abc} \tilde{\epsilon}_{a'b'c'} \delta_{c,c'} \frac{\langle S_a^x S_b^y S_{a'}^x S_{b'}^y (S_c^z)^2 \rangle}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c}^\alpha d_{b',c}^\alpha} \end{aligned} \quad (\text{A.11})$$

where we use the relation (A.9). The term $1/(d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c}^\alpha d_{b',c}^\alpha)$ in (A.11) is illustrated diagrammatically in Fig. A.1.

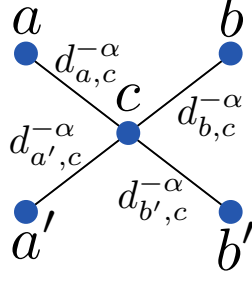


Figure A.1: The diagrammatic representation of the term $d_{a,c}^{-\alpha} d_{b,c}^{-\alpha} d_{a',c}^{-\alpha} d_{b',c}^{-\alpha}$ in (A.11). [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

The 5-body correlation $\langle S_a^x S_b^y S_{a'}^x S_{b'}^y (S_c^z)^2 \rangle$ can be decomposed into cumulant products as follows:

$$\begin{aligned} \langle S_a^x S_b^y S_{a'}^x S_{b'}^y (S_c^z)^2 \rangle &= \langle S_a^x S_b^y S_{a'}^x S_{b'}^y (S_c^z)^2 \rangle_c + \langle S_a^x S_b^y S_{a'}^x S_{b'}^y \rangle_c \langle (S_c^z)^2 \rangle \\ &\quad + \langle S_a^x S_{a'}^x (S_c^z)^2 \rangle_c \langle S_b^y S_{b'}^y \rangle_c + \langle S_a^x S_{a'}^x \rangle_c \langle S_b^y S_{b'}^y (S_c^z)^2 \rangle_c + \langle S_a^x S_{a'}^x \rangle_c \langle S_b^y S_{b'}^y \rangle_c \langle (S_c^z)^2 \rangle \end{aligned} \quad (\text{A.12})$$

where we use the relation (A.7), (A.8). Next, we use Theorem 1 as follows:

$$\langle S_a^x S_b^y S_{a'}^x S_{b'}^y (S_c^z)^2 \rangle_c \leq c_1 \|S_a^x\| \dots \| (S_c^z)^2 \| \sum_{\substack{\{j_1, \dots, j_5\} = \\ \{a, b, a', b', c\} \\ \text{:connected}}} \frac{1}{d_{j_1, j_2}^\alpha \dots d_{j_4, j_5}^\alpha}, \quad (\text{A.13})$$

$$\langle S_a^x S_b^y S_{a'}^x S_{b'}^y \rangle_c \leq c_1 \|S_a^x\| \dots \|S_{b'}^y\| \sum_{\substack{\{j_1, \dots, j_4\} = \\ \{a, b, a', b'\} \\ \text{:connected}}} \frac{1}{d_{j_1, j_2}^\alpha d_{j_2, j_3}^\alpha d_{j_3, j_4}^\alpha}, \quad (\text{A.14})$$

$$\langle S_{i_1}^\sigma S_{i_2}^\sigma (S_c^z)^2 \rangle_c \leq c_3 \|S_{i_1}^\sigma\| \|S_{i_2}^\sigma\| \| (S_c^z)^2 \| \sum_{\substack{\{j_1, j_2, j_3\} = \\ \{i_1, i_2, c\} \\ \text{:connected}}} \frac{1}{d_{j_1, j_2}^\alpha d_{j_2, j_3}^\alpha}, \quad (\text{A.15})$$

$$\langle S_{i_p}^\sigma S_{i_q}^\sigma \rangle_c \leq c_4 \|S_{i_p}^\sigma\| \|S_{i_q}^\sigma\| \frac{1}{d_{i_p, i_q}^\alpha}, \quad (\text{A.16})$$

where $\sum_{\substack{\{j_1, \dots, j_m\} \\ \text{:connected}}}$ denotes taking the summation over the vertices, which are connected from j_1 to j_m along the Hamilton path. By using (A.13) - (A.16), we obtain the following inequality

for $|\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle|$:

$$\begin{aligned}
|\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle| &\leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \\
&\times \left\{ \left| \langle S_a^x S_b^y S_{a'}^x S_{b'}^y (S_c^z)^2 \rangle_c \right| + \left| \langle S_a^x S_b^y S_{a'}^x S_{b'}^y \rangle_c \langle (S_c^z)^2 \rangle \right| + \left| \langle S_a^x S_{a'}^x (S_c^z)^2 \rangle_c \langle S_b^y S_{b'}^y \rangle_c \right| \right. \\
&\quad \left. + \left| \langle S_a^x S_{a'}^x \rangle_c \langle S_b^y S_{b'}^y (S_c^z)^2 \rangle_c \right| + \left| \langle S_a^x S_{a'}^x \rangle_c \langle S_b^y S_{b'}^y \rangle_c \langle (S_c^z)^2 \rangle \right| \right\} \\
&\leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \\
&\times \left\{ \sum_{\substack{\{i_1, \dots, i_5\} = \\ \{a, b, a', b', c\} \\ : \text{connected}}} \frac{c_1}{d_{i_1, i_2}^\alpha \dots d_{i_4, i_5}^\alpha} + \sum_{\substack{\{i_1, \dots, i_4\} = \\ \{a, b, a', b'\} \\ : \text{connected}}} \frac{c_2}{d_{i_1, i_2}^\alpha d_{i_2, i_3}^\alpha d_{i_3, i_4}^\alpha} \right. \\
&\quad \left. + \sum_{\substack{\{i_1, \dots, i_4\} = \{a, b, a', b'\} \\ \{i_1, i_2\} = \{a, a'\}, \{b, b'\} \\ : \text{connected}}} \sum_{\substack{\{j_1, j_2, j_3\} = \\ \{i_1, i_2, c\} \\ : \text{connected}}} \frac{2c_3}{d_{j_1, j_2}^\alpha d_{j_2, j_3}^\alpha} \frac{c_4}{d_{i_3, i_4}^\alpha} + \frac{(c_4)^2}{d_{a, a'}^\alpha d_{b, b'}^\alpha} \right\} \tag{A.17}
\end{aligned}$$

Let us first consider the contribution of the first term in (A.17):

$$\sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \sum_{\substack{\{i_1, \dots, i_5\} = \\ \{a, b, a', b', c\} \\ : \text{connected}}} \frac{c_1}{d_{i_1, i_2}^\alpha \dots d_{i_4, i_5}^\alpha} \tag{A.18}$$

Fig. A.2 enumerates all possible cases of clustering linkages in (A.18) and illustrates the derivation of the upper bound for each case. Here, diagrams with identical topology are regarded as equivalent.

Corresponding to the case (1) in Fig. A.2, we obtain the following upper bound of the

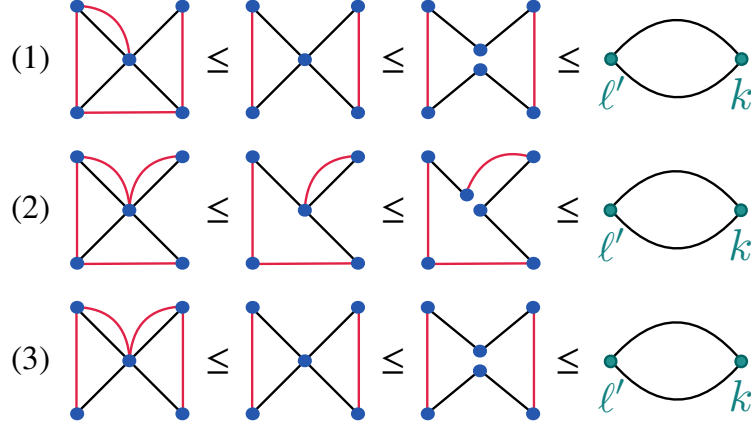


Figure A.2: The diagrammatic derivation for the upper bound of the contribution of the first term in (A.17). [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

contribution to $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$:

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \sum_{\substack{\{i_2, \dots, i_5\} = \\ \{a, b, a', b'\} \\ : \text{connected}}} \frac{c_1}{d_{c, i_2}^\alpha d_{i_2, i_3}^\alpha \dots d_{i_4, i_5}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \sum_{\substack{\{i_2, \dots, i_5\} = \\ \{a, b, a', b'\} \\ : \text{connected}}} \frac{c_1}{d_{i_2, i_3}^\alpha d_{i_4, i_5}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \sum_{\substack{c''=1 \\ \{i_2, \dots, i_5\} = \\ \{a, b, a', b'\} \\ : \text{connected}}}^N \frac{\delta_{c, c'} c_1}{d_{i_2, i_3}^\alpha d_{i_3, c}^\alpha d_{c, i_4}^\alpha d_{i_4, i_5}^\alpha d_{i_5, c''}^\alpha d_{c'', i_2}^\alpha} \\
& \leq \frac{J_x^2 J_y^2}{16} u^4 c_1 \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{1}{d_{k, \ell'}^{2\alpha}} \leq c^{(1)} n^{2-2\alpha}, \tag{A.19}
\end{aligned}$$

where $c^{(1)} = (J_x^2 J_y^2 / 16) u^4 c_1 / \{(1 - 2\alpha)(2 - 2\alpha)\}$.

Corresponding to the case (2) in Fig. A.2, we similarly obtain the following upper bound

of the contribution to $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$:

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \sum_{\substack{\{i_1, i_3, i_4, i_5\} = \\ \{a, b, a', b'\} \\ : \text{connected}}} \frac{c_1}{d_{i_1, c}^\alpha d_{i_3, c}^\alpha d_{i_4, c}^\alpha d_{i_5, c}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \sum_{\substack{\{i_1, i_3, i_4, i_5\} = \\ \{a, b, a', b'\} \\ : \text{connected}}} \frac{\delta_{c, c'}}{d_{i_1, c}^\alpha d_{i_2, c}^\alpha d_{i_4, c}^\alpha d_{i_5, c}^\alpha} \frac{c_1}{d_{i_1, c}^\alpha d_{i_3, c}^\alpha d_{i_4, c}^\alpha d_{i_5, c}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \sum_{c''=1}^N \sum_{\substack{\{i_1, i_3, i_4, i_5\} = \\ \{a, b, a', b'\} \\ : \text{connected}}} \frac{\delta_{c, c'} c_1}{d_{i_1, c}^\alpha d_{c, i_2}^\alpha d_{i_2, i_3}^\alpha d_{i_3, i_4}^\alpha d_{i_4, c''}^\alpha d_{c'', i_1}^\alpha} \\
& \leq \frac{J_x^2 J_y^2}{16} u^4 c_1 \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{1}{d_{k, \ell'}^{2\alpha}} \leq c^{(1)} n^{2-2\alpha}. \tag{A.20}
\end{aligned}$$

Corresponding to the case (3) in Fig. A.2, we similarly obtain the following upper bound of the contribution to $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$:

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \sum_{\substack{\{i_1, i_2, i_4, i_5\} = \\ \{a, b, a', b'\} \\ : \text{connected}}} \frac{c_1}{d_{i_1, i_2}^\alpha d_{i_2, c}^\alpha d_{c, i_4}^\alpha d_{i_4, i_5}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \sum_{\substack{\{i_1, i_2, i_4, i_5\} = \\ \{a, b, a', b'\} \\ : \text{connected}}} \frac{c_1}{d_{i_1, i_2}^\alpha d_{i_4, i_5}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \sum_{c''=1}^N \sum_{\substack{\{i_1, i_2, i_4, i_5\} = \\ \{a, b, a', b'\} \\ : \text{connected}}} \frac{\delta_{c, c'} c_1}{d_{i_1, i_2}^\alpha d_{i_2, c}^\alpha d_{c, i_4}^\alpha d_{i_4, i_5}^\alpha d_{i_5, c''}^\alpha d_{c'', i_1}^\alpha} \\
& \leq \frac{J_x^2 J_y^2}{16} u^4 c_1 \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{1}{d_{k, \ell'}^{2\alpha}} \leq c^{(1)} n^{2-2\alpha}. \tag{A.21}
\end{aligned}$$

From the above case analysis, we obtain the following upper bound for the contribution from the first term in Fig. A.2:

$$\sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \sum_{\substack{\{i_1, \dots, i_5\} = \\ \{a, b, a', b', c\} \\ : \text{connected}}} \frac{c_1}{d_{i_1, i_2}^\alpha \dots d_{i_4, i_5}^\alpha} \leq 3c^{(1)} n^{2-2\alpha} \tag{A.22}$$

Next, let us consider the contribution of the second term in (A.17):

$$\sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \sum_{\substack{\{i_1, \dots, i_4\} = \\ \{a, b, a', b'\} \\ : \text{connected}}} \frac{c_2}{d_{i_1, i_2}^\alpha d_{i_2, i_3}^\alpha d_{i_3, i_4}^\alpha} \quad (\text{A.23})$$

Fig. A.3 illustrates the method of clustering linkage in (A.23), which, when considering graphs with equivalent topology as identical, is restricted to a single case. Furthermore, it shows the derivation of the upper bound for this case.

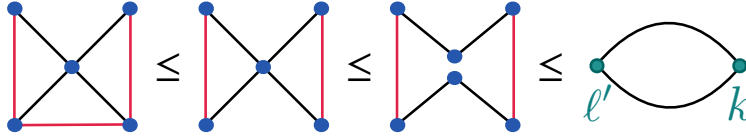


Figure A.3: The diagrammatic derivation for the upper bound of the contribution of the second term in (A.17). [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

In this case, we obtain the following upper bound of the contribution to $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$:

$$\begin{aligned} & \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \sum_{\substack{\{i_1, \dots, i_4\} = \\ \{a, b, a', b'\} \\ : \text{connected}}} \frac{c_2}{d_{i_1, i_2}^\alpha d_{i_2, i_3}^\alpha d_{i_3, i_4}^\alpha} \\ & \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \sum_{\substack{\{i_1, \dots, i_4\} = \\ \{a, b, a', b'\} \\ : \text{connected}}} \frac{c_2}{d_{i_1, i_2}^\alpha d_{i_3, i_4}^\alpha} \\ & \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \sum_{c''=1}^N \sum_{\substack{\{i_1, \dots, i_4\} = \\ \{a, b, a', b'\} \\ : \text{connected}}} \frac{\delta_{c, c'} c_2}{d_{i_1, i_2}^\alpha d_{i_2, c}^\alpha d_{c, i_3}^\alpha d_{i_3, i_4}^\alpha d_{i_4, c''}^\alpha d_{c'', i_1}^\alpha} \\ & \leq \frac{J_x^2 J_y^2}{16} u^4 c_2 \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{1}{d_{k, \ell'}^{2\alpha}} \leq c^{(2)} n^{2-2\alpha}, \end{aligned} \quad (\text{A.24})$$

where $c^{(2)} = (J_x^2 J_y^2 / 16) u^4 c_1 / \{(1 - 2\alpha)(2 - 2\alpha)\}$.

Next, let us consider the contribution of the third term in (A.17):

$$\sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \sum_{\substack{\{i_1, \dots, i_4\} = \{a, b, a', b'\} \\ \{i_1, i_2\} = \{a, a'\}, \{i_3, i_4\} = \{b, b'\} \\ : \text{connected}}} \sum_{\substack{\{j_1, j_2, j_3\} = \\ \{i_1, i_2, c\} \\ : \text{connected}}} \frac{2c_3}{d_{j_1, j_2}^\alpha d_{j_2, j_3}^\alpha} \frac{c_4}{d_{i_3, i_4}^\alpha} \quad (\text{A.25})$$

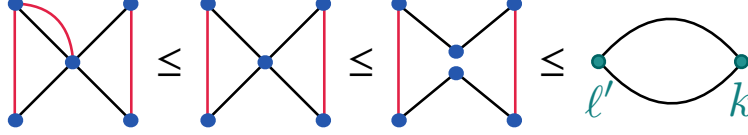


Figure A.4: The diagrammatic derivation for the upper bound of the contribution of the third term in (A.17). [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

Fig. A.4 illustrates the method of clustering linkage in (A.25), which, when considering graphs with equivalent topology as identical, is restricted to a single case. Furthermore, it shows the derivation of the upper bound for this case.

In this case, we similarly obtain the following upper bound of the contribution to $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$:

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \sum_{\substack{\{i_1, \dots, i_4\} = \{a, b, a', b'\} \\ \{i_1, i_2\} = \{a, a'\}, \{b, b'\}}} \sum_{\substack{\{j_1, j_2, j_3\} = \{i_1, i_2, c\} \\ \text{:connected}}} \frac{2c_3}{d_{j_1, j_2}^\alpha d_{j_2, j_3}^\alpha} \frac{c_4}{d_{i_3, i_4}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \sum_{\substack{\{i_1, \dots, i_4\} = \{a, b, a', b'\} \\ \{i_1, i_2\} = \{a, a'\}, \{b, b'\}}} \frac{2c_3}{d_{i_1, i_2}^\alpha} \frac{c_4}{d_{i_3, i_4}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \sum_{c''=1}^N \sum_{\substack{\{i_1, \dots, i_4\} = \{a, b, a', b'\} \\ \text{:connected}}} \frac{\delta_{c, c'} c_3 c_4}{d_{i_1, i_2}^\alpha d_{i_2, c}^\alpha d_{c, i_3}^\alpha d_{i_3, i_4}^\alpha d_{i_4, c''}^\alpha d_{c'', i_1}^\alpha} \\
& \leq \frac{J_x^2 J_y^2}{16} u^4 c_3 c_4 \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{1}{d_{k, \ell'}^{2\alpha}} \leq c^{(3)} n^{2-2\alpha}, \tag{A.26}
\end{aligned}$$

where $c^{(3)} = (J_x^2 J_y^2 / 16) u^4 c_3 c_4 / \{(1 - 2\alpha)(2 - 2\alpha)\}$.

Lastly, let us consider the contribution of the fourth term in (A.17):

$$\sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \frac{(c_4)^2}{d_{a, a'}^\alpha d_{b, b'}^\alpha} \tag{A.27}$$

Fig. A.5 illustrates the method of clustering linkage in (A.27), which, when considering graphs with equivalent topology as identical, is restricted to a single case. Furthermore, it shows the derivation of the upper bound for this case.

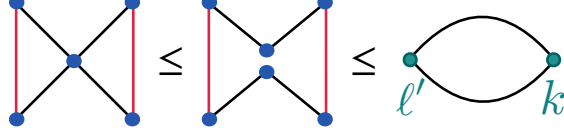


Figure A.5: The diagrammatic derivation for the upper bound of the contribution of the fourth term in (A.17). [This figure is reproduced with permission from H. Nishikawa and K. Saito, Phys. Rev. Lett. 135, 141702 (2025), Copyrighted by the American Physical Society.]

We similarly obtain the following upper bound of the contribution to $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle$:

$$\begin{aligned}
& \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \frac{\delta_{c, c'}}{d_{a, c}^\alpha d_{b, c}^\alpha d_{a', c}^\alpha d_{b', c}^\alpha} \frac{(c_4)^2}{d_{a, a'}^\alpha d_{b, b'}^\alpha} \\
& \leq \sum_{\substack{k \geq n \\ \ell < n}} \sum_{\substack{k' \geq 0 \\ \ell' < 0}} \sum_{m, m'=1}^N \frac{J_x^2 J_y^2}{16} \sum_{\substack{\{a, b, c\} = \{k, \ell, m\} \\ \{a', b', c'\} = \{k', \ell', m'\}}} \sum_{c''=1}^N \frac{\delta_{c, c'} (c_4)^2}{d_{a, a'}^\alpha d_{a', c}^\alpha d_{c, b'}^\alpha d_{b', b}^\alpha d_{b, c''}^\alpha d_{c'', a}^\alpha} \\
& \leq \frac{J_x^2 J_y^2}{16} u^4 (c_4)^2 \sum_{\substack{k \geq n \\ \ell' < 0}} \frac{1}{d_{k, \ell'}^{2\alpha}} \leq c^{(4)} n^{2-2\alpha}, \tag{A.28}
\end{aligned}$$

where $c^{(4)} = (J_x^2 J_y^2 / 16) u^4 (c_4)^2 / \{(1 - 2\alpha)(2 - 2\alpha)\}$.

From (A.22), (A.24), (A.26), (A.28), we demonstrate that all terms in (A.17) can be bounded as follows:

$$|\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle| \leq C_1 n^{2-2\alpha}, \tag{A.29}$$

where $C_1 = 3c^{(1)} + c^{(2)} + c^{(3)} + c^{(4)}$. Thus, we establish the claims of the theorem. $\square$

A.2 Upper bound for the equal-time total energy current correlation in higher dimensions

We consider the D dimensional long-range interacting XY model, which is described by the following Hamiltonian

$$H = - \sum_{i \neq j} \left[J_x \frac{S_i^x S_j^x}{d_{i, j}^\alpha} + J_y \frac{S_i^y S_j^y}{d_{i, j}^\alpha} \right]. \tag{B.1}$$

The local energy at site $\mathbf{n}$ is defined as

$$\hat{h}_{\mathbf{n}} := - \sum_{j \neq \mathbf{n}} \sum_{\sigma=x, y} \frac{J_\sigma}{2} \frac{S_{\mathbf{n}}^\sigma S_j^\sigma}{d_{\mathbf{n}, j}^\alpha}, \tag{B.2}$$

and the current $\mathcal{J}_{n_i|n_i-1}^{(\epsilon,i)}$ is described as follows

$$\mathcal{J}_{n_i|n_i-1}^{(\epsilon,i)} = - \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell: \ell_i < n_i}} t_{\mathbf{k} \leftarrow \ell} \quad (\text{B.3})$$

$$t_{\mathbf{k} \leftarrow \ell} = \{\hat{h}_{\mathbf{k}}, \hat{h}_{\ell}\} = \sum_{\mathbf{m}} \frac{J_x J_y}{4} \sum_{\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} = \{\mathbf{k}, \ell, \mathbf{m}\}} \tilde{\varepsilon}_{\mathbf{abc}} \frac{S_{\mathbf{a}}^x S_{\mathbf{b}}^y S_{\mathbf{c}}^z}{d_{\mathbf{a}, \mathbf{c}}^\alpha d_{\mathbf{b}, \mathbf{c}}^\alpha}, \quad (\text{B.4})$$

where $\sum_{\sigma \in P_{\text{cyc}}^{xyz}}$ is defined as sums over all circular permutation of the variables (x, y, z) , and $\tilde{\varepsilon}_{\mathbf{abc}}$ is defined as follows:

$$\tilde{\varepsilon}_{\mathbf{abc}} = \begin{cases} +1, & \text{for } (\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{k}, \ell, \mathbf{m}), (\mathbf{m}, \ell, \mathbf{k}), (\mathbf{k}, \mathbf{m}, \ell), \\ -1, & \text{for } (\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\ell, \mathbf{k}, \mathbf{m}), (\ell, \mathbf{m}, \mathbf{k}), (\mathbf{m}, \mathbf{k}, \ell). \end{cases} \quad (\text{B.5})$$

Note that $\tilde{\varepsilon}_{\mathbf{abc}}$ is different from Levi-Civita symbols.

Theorem A.2. *In the D -dimensional long-range interacting XY model, under the condition $\alpha > D/2 + 1$, $\alpha > D$ and $\beta < \beta_c$, $C_N^{(D)}(0)$ converges to the finite value independent of the system size N . It immediately follows that $C_N^{(D)}(0)$ is finite as long as $\alpha > D$.*

Proof. In exactly the same manner as the one-dimensional case, utilizing Theorem 1 and the symmetry derived from the Hamiltonian, the following upper bound for the current correlation can be obtained:

$$\left| \left\langle \mathcal{J}_{n_i|n_i-1}^{(\epsilon,i)} \mathcal{J}_{0|-1}^{(\epsilon,i)} \right\rangle \right| \leq \mathcal{O}(1) \sum_{\substack{\mathbf{k}: k_i \geq n_i \\ \ell': \ell'_i < 0}} \frac{1}{d_{\mathbf{k}, \ell'}^{2\alpha}} \quad (\text{B.6})$$

Thus, in exactly the same manner as Theorem 3.1, we establish the claim of the theorem. $\square$

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