

On certain non-Kähler manifolds.

-- Introduction to the theory of rigid analytic spaces --

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In my talk, I gave a few words about non-Kähler analogue of rigid analytic spaces. Since I am writing the paper on analytic threefolds with non-trivial Albanese tori, which was the main topic of my talk, here, I give a short introduction to the theory of rigid analytic spaces.

The origin of rigid analytic space is the theory of Tate's elliptic curves and Tate also developed the general theory of rigid analytic spaces ([T]). After him there are many literatures on this subject. As I shall explain below it seems to me that there is a possibility to develop the theory to a different direction, namely the study of compact rigid analytic spaces after compact complex manifolds. From our view point, many fundamental theorems such as the Riemann-Roch theorem, resolution of singularities are missing and the theory is still a primitive stage.

§0. Introduction.

There are certain similarities between a compact non-Kähler manifold M and a complete non-singular algebraic variety V in characteristic $p > 0$. For example, V may carry a regular m -form which is not d -closed and it may happen that $h^{m,n}(V) \neq h^{n,m}(V)$ where $h^{m,n}(V) = \dim_k H^n(V, \Omega_V^m)$. Let us consider more closely non-singular projective surfaces. The following properties are well-known.

A1) There exists a non-singular projective surface S which carries a regular 1-form ω with $d\omega \neq 0$.

A2) There exists a non-singular projective surface S with $h^{1,0}(S) \neq h^{0,1}(S)$.

The first Betti number $b_1(S)$ of S is equal to $2\dim \text{ALB}(S)$. Hence b_1 is always even.

A4) Put $q = h^{0,1}(S)$, $P_g = h^{0,2}(S)$, $\Delta = 2q - b_1$. Then we have

$$0 \leq \Delta \leq 2P_g.$$

Moreover, there exists a surface S with $\Delta = 2P_g \geq 2$.

On the other hand, a compact analytic surface M has the following properties.

N1) Every holomorphic 1-form is d -closed.

N2) $h^{1,0}(M) = h^{0,1}(M)$ if b_1 is even and $h^{1,0}(M) + 1 = h^{0,1}(M)$ if b_1 is odd.

N3) $b_1(M) = h^{1,0}(M) + h^{0,1}(M) \geq 2\dim \text{ALB}(M)$.

N4) $\Delta = 2q - b_1 = 0$ if b_1 is even and $\Delta = 1$ if b_1 is odd.

Thus, a non-singular projective surface and a non-Kähler analytic surface have different properties. Hence, we cannot expect that there are some relations between non-singular projective surfaces and non-Kähler surfaces. But if we drop the non-singularity assumption, it may be possible to find a certain relation. In the following we shall give some examples which suggest us such a relation. For that purpose, we need to consider compact rigid analytic spaces. In the following first we recall the basic definitions and facts on rigid analytic spaces.

§1. Affinoid algebras and affinoid spaces.

In what follows, by k we always mean a complete non-Archimedean valuation field with a multiplicative valuation $|\cdot|$. That is, the valuation $|\cdot|$ satisfies the following.

$$U1) \quad |a| \geq 0 \text{ for each } a \in k. \quad |a| = 0 \text{ iff } a = 0.$$

$$U2) \quad |a \cdot b| = |a| |b| \quad \text{for all } a, b \in k.$$

$$U3) \quad |a + b| \leq \max \{|a|, |b|\} \text{ for all } a, b \in k.$$

$d(a, b) = |a - b|$ defines a metric on k and k is totally disconnected with respect to the metric.

Put

$$k^{\circ} = \{ a \in k \mid |a| \leq 1 \}, \quad k^{\circ\circ} = \{ a \in k \mid |a| < 1 \}.$$

Then k° is a ring and $k^{\circ\circ}$ is its maximal ideal. The field $\bar{k} = k^{\circ}/k^{\circ\circ}$ is of characteristic $p = \min \{ m \in \mathbb{Z} \mid m > 0, |m| < 1 \}$.

Put

$$T_n = k\langle z_1, z_2, \dots, z_n \rangle$$

$$= \{ \sum_{\alpha_i \geq 0} a_\alpha z_1^{\alpha_1} \dots z_n^{\alpha_n} \mid \lim_{|\alpha| \rightarrow +\infty} a_\alpha = 0 \},$$

where $z_1, \dots, z_n$ are variables. T_n is a ring consisting of power series which convergent on $\{(z_1, \dots, z_n) \in k^n \mid |z_i| \leq 1, 1 \leq i \leq n\}$.

Define a semi-norm $\| \cdot \|$ on T_n by

$$\|f\| = \max |a_\alpha|, \quad f = \sum_{\alpha} a_\alpha z^\alpha$$

Then, $\| \cdot \|$ is indeed a norm and $(T_n, \| \cdot \|)$ is a Banach algebra.

Definition 1.1. An affinoid algebra A over k is a k algebra with a finite k algebra homomorphism $\rho : T_n \rightarrow A$. (That is A is a finite T_n module by ρ .)

The following theorem is fundamental.

Theorem 1.2 1) T_n is Noetherian.

2) T_n is UFD.

3) Every ideal in T_n is closed with respect to $\| \cdot \|$.

4) For each ideal I of T_n , there exists a finite injective k algebra homomorphism $T_d \rightarrow T_n/I$, where $d = \text{Krull dim } T_n/I$.

5) For every maximal ideal $\mathfrak{m}$ of T_n , $T_n/\mathfrak{m}$ is a finite extension of k .

6) Every affinoid algebra A is isomorphic to T_n/I and has a structure of Banach algebra by the quotient norm on T_n/I induced by T_n . The norms on A given by different expressions T_n/I , T_m/J are equivalent.

7) Every k algebra homomorphism $f : A \rightarrow B$ of affinoid algebras are continuous with respect to the norm given in 6).

Definition 1.3 For an affinoid algebra A put

$$\mathrm{Sp}(A) = \{ \mathfrak{m} \mid \mathfrak{m} \text{ is a maximal ideal of } A \}.$$

The space $\mathrm{Sp}(A)$ is called an affinoid space.

For any point $x \in \mathrm{Sp}(A)$, by $f(x)$ we mean the image of $f \in A$ in $A/\mathfrak{m}_x$. Since $A/\mathfrak{m}_x$ is finite over k , the valuation $|\cdot|$ can be uniquely extended to that of $A/\mathfrak{m}_x$. Hence $|f(x)|$ is well-defined. The spectral seminorm $\|f\|_{\mathrm{sp}}$ of A is defined by

$$\|f\|_{\mathrm{sp}} = \sup_{x \in \mathrm{Sp}(A)} |f(x)|.$$

The seminorm $\|\cdot\|_{\mathrm{sp}}$ is norm iff the radical of A is zero.

By Theorem 1.2,6) affinoid algebra A has a Banach norm $\|\cdot\|$. The Banach norm $\|\cdot\|$ induces a norm on $A/\mathfrak{m}_x$ which is larger than or equal to $|\cdot|$, since on k , $\|\cdot\|$ induces the valuation $|\cdot|$. Therefore, we have the inequality

$$\|f\|_{\mathrm{sp}} \leq \|f\|, \text{ for any } f \in A.$$

For T_n , we can easily show that $\|f\|_{\mathrm{sp}} = \|f\|$.

More generally we have the following theorem.

Theorem 1.4. Let A be a reduced affinoid algebra. Then the spectral seminorm is a norm and is equivalent to the quotient norm introduced in Theorem 1.2,6).

We also have the following theorem.

Theorem 1.5. If A is a reduced affinoid algebra, then the integral closure of A in its total quotient ring is a finitely generated A module.

Let A be an affinoid algebra. Put

$$A^{\circ} = \{ f \in A \mid \|f\|_{\mathrm{sp}} \leq 1 \}, \quad A^{\circ\circ} = \{ f \in A \mid \|f\|_{\mathrm{sp}} < 1 \}.$$

Then A° is a ring and $A^{\circ\circ}$ is an ideal of A° . The quotient $\bar{A} = A^\circ / A^{\circ\circ}$ is k algebra. Let $\phi : A \rightarrow B$ be a k algebra homomorphism of affinoid algebras. Then we have $\phi(A^\circ) \subset B^\circ$, $\phi(A^{\circ\circ}) \subset B^{\circ\circ}$. Hence, it induces a $\bar{k}$ algebra homomorphism $\phi : \bar{A} \rightarrow \bar{B}$.

Theorem 1.6. The following conditions are equivalent.

- 1) $\phi : A \rightarrow B$ is finite.
- 2) $\phi : A^\circ \rightarrow B^\circ$ is integral.
- 3) $\bar{\phi} : \bar{A} \rightarrow \bar{B}$ is finite.

Any k algebra homomorphism $\phi : A \rightarrow B$ of affinoid algebras defines a map $\text{Sp}(B) \rightarrow \text{Sp}(A)$ by $x \mapsto \phi^{-1}(x)$.

This map is denoted by ϕ^a . A morphism of affinoid spaces is, by definition, a map induced by a k algebra homomorphism.

Now we introduce a topology on $X = \text{Sp}(A)$ by making $\{x \in X \mid |f(x)| \leq 1\}$, $f \in A$, the subbases of open sets. This topology is the weakest topology that $X \ni x \mapsto f(x) \in k^a =$ the algebraic closure of k is continuous for all $f \in A$. Then we can easily show that $\text{Sp}(T_n)$ is homeomorphic to $D^n =$

$\{(z_1, \dots, z_n) \in k^n \mid |z_i| \leq 1, 1 \leq i \leq n\}$ and $\text{Sp}(T_n/I)$ with $\sqrt{I} = I$ is homeomorphic to the closed subset $\{(z_1, \dots, z_n) \in D^n \mid g(z) = 0 \text{ for all } g \in I\}$ of D^n .

Example 1.7. On $\{(z_1, \dots, z_n) \in k^n \mid |\pi_i| < |z_i| < 1\}$,

$\pi_i \in k$, $|\pi_i| < 1$, we can introduce a structure of an affinoid space by $\text{Sp}(A)$ where

$$A = k\langle z_1, \dots, z_n, w_1, \dots, w_n \rangle / (w_1 z_1 - \pi_1, \dots, w_n z_n - \pi_n).$$

We often write A as $k\langle z_1, \dots, z_n, \frac{\pi_1}{z_1}, \dots, \frac{\pi_n}{z_n} \rangle$.

Especially we have

$$\{(z_1, \dots, z_n) \in k^n \mid |z_i| = 1, 1 \leq i \leq n\} \simeq$$

$$\text{Sp}(k\langle z_1, \dots, z_n, 1/z_1, \dots, 1/z_n \rangle).$$

§2. Rigid analytic spaces.

Since the topology on an affinoid space X above is totally disconnected, it is necessary to introduce a Grothendieck topology to obtain a good theory of analytic spaces.

Definition 2.1. An open subset $Y \subset X = \text{Sp}(A)$ of an affinoid space X is called an affinoid subdomain, if there exist an affinoid algebra A_Y and a k algebra homomorphism $\psi : A_Y \rightarrow A$ such that for any morphism $\phi : \text{Sp}(B) \rightarrow X$ with $\phi(\text{Sp}(B)) \subset Y$, there exists a unique k algebra homomorphism $\xi : A_Y \rightarrow B$

with $\phi = \psi^a \cdot \xi^a$,

$$\begin{array}{ccc} \text{Sp}(B) & \xrightarrow{\phi} & \text{Sp}(A) \\ & \searrow & \nearrow \\ & \text{Sp}(A_Y) & \end{array}$$

The following lemma is an easy consequence of the definition.

Lemma 2.2. Let $Y \subset X = \text{Sp}(B)$ be an affinoid subdomain.

Then the canonical morphism $\psi^a : \text{Sp}(A_Y) \rightarrow Y$ is bijective

Moreover, if $x \in \text{Sp}(A_Y)$, $y = \psi^a(x) \in Y \subset \text{Sp}(A)$, then there

are canonical isomorphisms

$$A/Y^n \xrightarrow{\sim} A_Y/X^n, n \geq 1,$$

and $Y = XA_Y$.

Corollary 2.3 $\varprojlim_n A/Y^n \xrightarrow{\sim} \varprojlim_n A_Y/X^n$.

By the above lemma, we can consider an affinoid subdomain as an affinoid space $\text{Sp}(A_Y)$.

Lemma 2.4. Let $Y_1 \subset X = \text{Sp}(A)$ be an affinoid subdomain and $Y_2 \subset Y_1$ an affinoid subdomain of Y_1 . Then Y_2 is an affinoid subdomain of X .

Lemma 2.5. Let $f : \text{Sp}(B) \longrightarrow \text{Sp}(A)$ be a morphism of affinoid spaces and Y is an affinoid subdomain of $\text{Sp}(A)$. Then $\widetilde{Y} = f^{-1}(Y)$ is an affinoid subdomain of $\text{Sp}(B)$ and $B_{\widetilde{Y}}$ is isomorphic to $B \widehat{\otimes}_A A_Y$ (the complete tensor product). In particular, if Y_1 and Y_2 are affinoid subdomains then so is $Y_1 \cap Y_2$.

Example 2.6 A rational domain $R \subset X = \text{Sp}(A)$ is, by definition, an open subset in X defined by

$$R = \{x \in X \mid |f_i(x)| \leq |f_0(x)|, 1 \leq i \leq n\},$$

where $f_0, f_1, \dots, f_n$ generate A .

A rational domain R is an affinoid subdomain. The affinoid algebra A_R is given by

$$A\langle X_1, \dots, X_n \rangle / (f_1 - X_1 f_0, \dots, f_n - X_n f_0)$$

with the canonical injection $\psi : A \longrightarrow A_R$, where by

$A\langle X_1, \dots, X_n \rangle$ we mean

$$A\langle X_1, \dots, X_n \rangle = \{ \sum a_\alpha X^\alpha \mid a_\alpha \in A, \lim_{|\alpha| \rightarrow \infty} \|a_\alpha\| = 0 \}.$$

If $A = T_n/I$, then

$$A\langle X_1, \dots, X_n \rangle \simeq k\langle Z_1, \dots, Z_n, X_1, \dots, X_n \rangle / I_k\langle \dots, Z_i, \dots, X_i, \dots \rangle.$$

We have the following lemma.

Lemma 2.7 1) If Y_1 and Y_2 are rational domains of $X = \text{Sp}(A)$,

then so is $Y_3 = Y_1 \cap Y_2$ and $A_{Y_3} \simeq A_{Y_1} \hat{\otimes}_A A_{Y_2}$.

2) If Y_1 is a rational domain of X and Y_2 is a rational domain of Y_1 , then Y_1 is a rational domain of X .

Now we introduce Grothendieck topologies on $X = \text{Sp}(A)$.

Let $\mathbf{F}$ be the set consisting of all affinoid subdomains

(resp. rational domains) of X . We consider the empty set $\emptyset$ as a rational domain of X . For any $U \in \mathbf{F}$ put

$$\text{Cov}(U) = \{ \underline{U} = \{U_i\}_{i \in I} \mid \#(I) < \infty, \bigcup U_i = U \},$$

where U_i is an affinoid subdomain (resp. rational domain) of

X . Then by virtue of Lemma 2.4 and Lemma 2.5 (resp. Lemma 2.7)

$(\mathbf{F}, \text{Cov})$ defines a Grothendieck topology of X . Namely,

$(\mathbf{F}, \text{Cov})$ has the following properties.

1) $\emptyset \in \mathbf{F}$, $X \in \mathbf{F}$, if $U, V \in \mathbf{F}$, then $U \cap V \in \mathbf{F}$.

2) For any $U \in \mathbf{F}$, $\{U\} \in \text{Cov}(U)$.

3) If $\{U_i\}_{i \in I} \in \text{Cov}(U)$ and $V \subset U$, $V \in \mathbf{F}$, then $\{U_i \cap V\}_{i \in I} \in \text{Cov}(V)$.

4) If $\{U_j\}_{j \in J} \in \text{Cov}(U)$ and $\{U_{j_k}\}_{k \in K_J} \in \text{Cov}(U_j)$, $j \in J$, then $\{U_{j_k}\}_{j \in J, k \in K_J} \in \text{Cov}(U)$.

An element of $\mathbf{F}$ is called an admissible open set and an element of $\text{Cov}(U)$ is called an admissible covering of U . The Grothendieck topology given by affinoid subdomains and the one given by rational domains are essentially the same by virtue of the following theorem.

Theorem 2.8. (Gerritzen - Grauert) Any affinoid subdomain of an affinoid space X is a finite union of rational domains of X .

Let G_X be one of the Grothendieck topologies defined above. Then, to any admissible open set U , associating A_U we can define a presheaf O_X which we call the structure presheaf. Similarly for any finite A module M ,

$$U \rightsquigarrow M \otimes_A A_U$$

defines a presheaf $M^\sim$.

Theorem 2.9. (Tate) For any admissible covering $\mathcal{X}$ of X , we have

$$\check{H}^p(\mathcal{X}, M^\sim) = \begin{cases} M, & p = 0, \\ 0, & p \neq 0. \end{cases}$$

In particular, O_X and $M^\sim$ are sheaves of the Grothendieck topology.

Thus, on X we can introduce a ringed space structure (G-ringd space structure) (X, G_X, O_X) with respect to the Grothendieck topology. Moreover, by Corollary 2.3, for any

point $x \in X = \text{Sp}(A)$, we have

$$\mathcal{O}_{X,x} = \varinjlim A/\mathfrak{a}_x^n.$$

Hence, $\mathcal{O}_{X,x}$ is a local ring. Thus, $(X, G_X, \mathcal{O}_X)$ is a G -local ringed space.

Definition 2.10. Let X and Y be affinoid spaces. A morphism $f = (\phi, \psi) : (X, G_X, \mathcal{O}_X) \longrightarrow (Y, G_Y, \mathcal{O}_Y)$ of G -local ringed spaces is defined as follows.

- 1) $\phi : X \longrightarrow Y$ is a continuous map of the Grothendieck topologies. That is, for any admissible open set U of Y , $\phi^{-1}(U)$ is admissible in X and for any admissible covering $\underline{U}$ of U , $\phi^{-1}(\underline{U})$ is an admissible covering of $\phi^{-1}(U)$.
- 2) For any admissible open set U of Y , there is a k algebra homomorphism $\psi_U : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\phi^{-1}(U))$, which is compatible with restrictions. Moreover $\psi = \{ \psi_U \}$ induces a local homomorphism $\psi_x : \mathcal{O}_{Y,\psi(x)} \longrightarrow \mathcal{O}_{X,x}$.

One of the main reasons why we need to introduce the Grothendieck topology is the following.

Proposition 2.12. There is a one to one correspondence between the set of morphisms $X \longrightarrow Y$ of affinoid spaces and the set of morphisms $(X, G_X, \mathcal{O}_X) \longrightarrow (Y, G_Y, \mathcal{O}_Y)$ of G -local ringed spaces.

Definition 2.12. A morphism $f : \text{Sp}(B) \longrightarrow \text{Sp}(A)$ of affinoid spaces is a closed immersion (resp. finite), if the corresponding homomorphism $f^* : A \longrightarrow B$ is surjective (resp. finite).

Now a rigid analytic space is defined as follows.

Definition 2.12. A G -local ringed space (X, G_X, O_X) is called a rigid analytic space over k , if X has an admissible covering $\{X_i\}_{i \in I}$ ($\#(I)$ may be infinite) such that $(X_i, G_X|_{X_i}, O_X|_{X_i})$ is an affinoid space for each i . A morphism $f : (X, G_X, O_X) \longrightarrow (Y, G_Y, O_Y)$ of rigid analytic spaces is a G -local ringed space morphism.

Remark. Later we shall construct rigid analytic spaces X by patching together affinoid spaces X_i . There are many Grothendieck topologies which induce the same Grothendieck topology on X_i . Among them we can find the finest one, if the G -topologies satisfy certain conditions. (See, for example, [BGR], 9.1.) Therefore, in the following we do not give explicitly the G -topology.

Definition 2.13. 1) A morphism $f : (X, G_X, O_X) \longrightarrow (Y, G_Y, O_Y)$ of rigid analytic spaces is a closed immersion (resp. finite), if there exists an admissible affinoid covering $\{U_i\}_{i \in I}$ of X such that the induced morphism $f^{-1}(U_i) \longrightarrow U_i$ of affinoid spaces is a closed immersion (resp. finite) for each $i \in I$.
2) A morphism $f : X \longrightarrow Y$ of rigid analytic spaces is separated if the diagonal morphism $\Delta : X \longrightarrow X \times_X X$ is a closed immersion. If $Y = \text{Sp}(k)$, X is called a separated rigid analytic space (over k).

If X is separated and $U, V \subset X$ are affinoid subspaces of X , then $U \cap V$ is also an affinoid subspace of X .

As already mentioned above, the topology of an affinoid space $X = \text{Sp}(A)$ given by the Banach norm of A is totally disconnected and if the residue field $\bar{k}$ is not finite, X is not locally compact. But we can define a proper morphism.

Definition 2.14. Let $f : X = \text{Sp}(A) \longrightarrow Y = \text{Sp}(B)$ be a morphism of affinoid spaces and $U \subset X$ an affinoid subdomain of X . The open set U is called to be relatively compact in X over Y (we use the symbol $U \subsetneq_Y X$), if there exist a positive number ε and an affinoid generating system $\{f_1, \dots, f_n\}$ of A over B (that is, $\|f_i\| < 1$ and $B\langle T_1, \dots, T_n \rangle \longrightarrow A$ is surjective B algebra homomorphism) such that

$$T_i \longmapsto f_i$$

$$U \subset \{ x \in X \mid |f_1(x)| \leq 1-\varepsilon, \dots, |f_n(x)| \leq 1-\varepsilon \}.$$

Lemma 2.15. Let X_1, X_2 be affinoid spaces over an affinoid space Y and $U_i \subsetneq_{Y_i} X_i$, $i = 1, 2$. Then we have

$$U_1 \times_{Y_2} X_2 \subsetneq_{X_2} X_1 \times_{Y_2} X_2,$$

$$U_1 \times_Y U_2 \subsetneq_Y X_1 \times_Y X_2.$$

That is, relatively compactness is closed under a base change and a fibre product.

Definition 2.16. A morphism $f : X \longrightarrow Y$ of rigid analytic spaces is proper, if

- 1) f is separated.
 - 2) There exist an admissible affinoid covering $\{Y_i\}_{i \in I}$ of Y and a finite admissible affinoid coverings $\{X_{i,j}\}_{1 \leq j \leq n_i}$ and $\{X'_{i,j}\}_{1 \leq j \leq n_i}$ of $f^{-1}(Y_i)$ such that $X'_{i,j} \subseteq_{Y_i} X_{i,j}$.
- If $Y = \text{Sp}(k)$ and f is proper, X is called a compact rigid analytic space.

Note that the underlying topological space of a rigid analytic space is not necessarily locally compact, a proper morphism is not necessarily closed.

Definition 2.17. Let (X, G_X, O_X) be a rigid analytic space and $\underline{F}$ an O_X module. If there exist an admissible affinoid covering $\{U_i\}_{i \in I}$, $U_i = \text{Sp}(A_i)$, and a finite A_i module M_i for each $i \in I$ such that $\underline{F}|_{U_i} \simeq \tilde{M}_i$, then $\underline{F}$ is called to be coh_X^e . (If we need to specify the covering $\underline{U}$, we say $\underline{U}$ coherent.)

Theorem 2.18. (Kiehl). Let $\underline{F}$ be an O_X module over an affinoid space $X = \text{Sp}(A)$. The following conditions are equivalent.

- 1) $\underline{F}$ is $\underline{U}$ coherent for an admissible affinoid covering $\underline{U}$ of X .
- 2) $\underline{F}$ is $\underline{U}$ coherent for any admissible affinoid covering $\underline{U}$ of X .
- 3) There exist a finite A module M such that $\underline{F} \simeq \tilde{M}$.

Theorem 2.19. (Kiehl). Let $f : (X, G_X, O_X) \longrightarrow (Y, G_Y, O_Y)$

be a proper morphism of rigid analytic spaces and $\underline{F}$ a coherent O_X module. Then $R^i f_* \underline{F}$ is coherent O_Y module for any interger i .

Corollary 2.20. If X is a compact rigid analytic space over k , then

$$\dim_k H^i(X, \underline{F}) < \infty,$$

for any coherent O_X module $\underline{F}$ on X and integer i .

§3 Elementary examples.

In the following, for simplicity we assume that the field k is algebraically closed.

Example 3.1. A_k^n . Take an element $c \in k$ with $|c| > 1$.

For any interger $m \geq 0$, put

$$A_m = \{ \sum a_{i_1 \dots i_n} T_1^{i_1} \dots T_n^{i_n} \mid \lim_{i_1 \dots i_n} a_{i_1 \dots i_n} c^{(i_1 + \dots + i_n)} = 0 \}.$$

Then $A_m \simeq k\langle z_1/c^m, \dots, z_n/c^m \rangle$ and there are natural inclusions

$$A_0 \supset A_1 \supset A_2 \supset \dots \supset A_m \supset A_{m+1} \supset \dots$$

Moreover, we have

$$\text{Sp}(A_m) \simeq \{ z \in A_k^n \mid |z_i| \leq |c|^m, 1 \leq i \leq n \}.$$

Hence $A_k^n = \bigcup_{m=0}^{\infty} \text{Sp}(A_m)$. Taking $\{ \text{Sp}(A_m) \}$ as an admissible covering of A_k^n , we can introduce a structure of a rigid analytic space on A_k^n . Also we have another admissible covering. Put

$$U_0 = \{ z \in A_k^n \mid |z_i| \leq |c|, i = 1, \dots, n \}$$

$$U_m = \{ z \in A_k^n \mid |c|^m \leq |z_i| \leq |c|^{(m+1)}, i = 1, \dots, n \}$$

where $c \in k$ with $|c| > 1$, $m = 1, 2, \dots$. Then we have

$$U_0 = \text{Sp}(k\langle z_1/c, \dots, z_n/c \rangle),$$

$$U_m = \text{Sp}(k\langle z_1/c^{m+1}, \dots, z_n/c^{m+1}, c^m/z_1, \dots, c^m/z_n \rangle)$$

$$\bigcup_{m=0}^{\infty} U_m = A_k^n.$$

Example 3.2. An affine scheme. Let X be an affine scheme of finite type over k . Then X is a closed subscheme of an affine space, hence using the rigid analytic structure of the affine space, we consider X as a closed analytic subspace of the affine space.

Example 3.3. A scheme X of finite type over k . Since each affine subscheme of X carries a structure of a rigid analytic space, by patching together the rigid analytic structures of open affine subscheme of X , on X we can introduce a rigid analytic structure. If X is a separated scheme, the corresponding rigid analytic space is separated.

Example 3.4. Projective space P_k^n .

Put

$$X_i = \{ (z_0 : z_1 : \dots : z_n) \in P_k^n \mid |z_k| \leq |z_i|, \substack{0 \leq k \leq n \\ i = 0, 1, \dots, n} \},$$

Then we have

$$\begin{aligned} X_i &\simeq D^n = \{ (w_1, \dots, w_n) \mid |w_i| \leq 1, i=1, \dots, n \} \\ &\simeq \text{Sp}(k\langle z_0/z_i, \dots, z_{i-1}/z_i, z_{i+1}/z_i, \dots, z_n/z_i \rangle). \end{aligned}$$

If we put

$$A_{ij} = k\langle z_0/z_i, \dots, z_n/z_i, (z_j/z_i)^{-1} \rangle,$$

then we have

$$\begin{aligned} X_{ij} = X_i \cap X_j &\simeq \text{Sp}(A_{ij}). \text{ Then a } k \text{ algebra homo-} \\ \text{morphism } \begin{matrix} A_{ij} & \longrightarrow & A_{ji} \\ \downarrow \oplus & & \downarrow \oplus \\ z_k/z_i & & (z_k/z_j)(z_i/z_j)^{-1} \end{matrix} &\text{ induces an isomorphism } X_{ij} \simeq X_{ji}. \end{aligned}$$

In this way, P_k^n has a structure of rigid analytic space.

Moreover, P_k^n is compact. To show it, let us take an

element c of k with $|c| > 1$ and put

$$\tilde{X}_i = \{ (z_0 : z_1 : \dots : z_n) \in P_k^n \mid |z_k| \leq |cz_i|, \substack{0 \leq k \leq n \\ i = 0, 1, \dots, n} \},$$

Then $\tilde{X}_i$ is also an affinoid space and $\{ X_i \}_{0 \leq i \leq n}$ is

also a finite admissible covering of P_k^n . By definition,

we have

$$X_i \subset \tilde{X}_i.$$

Example 3.5. Any projective scheme of finite type over k has a natural rigid analytic structure which is the closed analytic subspace of a rigid analytic projective space.

Example 3.6. Tate's elliptic curve. Take an element $q \in k$ with $0 < |q| < 1$. The map $z \mapsto qz$ generates an infinite cyclic group $\langle q \rangle$ of automorphism of $k^\times$. Put $T = k^\times / \langle q \rangle$.

By a similar method as in Example 3.1, on $k^\times = \mathbb{A}_k^1 - \{0\}$ we can

introduce a structure of a rigid analytic space such that the map $z \mapsto qz$ defines an analytic automorphism of the rigid analytic structure. Take an element π with $q = \pi^2$.

We let X_1 (resp. X_2) be the image of $\{z \in k \mid |\pi| \leq |z| \leq 1\}$

(resp. $\{z \in k \mid |q| \leq |z| \leq |\pi|\}$) in T . Also by U_1 (resp. U_2), we mean

the image of $\{z \in k \mid |z| = 1\}$ (resp. $\{z \in k \mid |z| = |\pi|\}$) in T . Then, we

have $X_1 \cap X_2 = U_1 \cup U_2$ (disjoint union). We consider $X_1, X_2,$

U_1, U_2 as affinoid spaces. (For example $X_1 \simeq \text{Sp}(k\langle z, \pi/z \rangle),$

$U_2 \simeq \text{Sp}(k\langle z/\pi, \pi/z \rangle)$). Then T is a compact rigid analytic

space and $\{X_1, X_2\}$ is an admissible covering. Since $\{X_1, X_2\}$ is

an acyclic covering, we can easily calculate the cohomology

groups $H^i(T, \mathcal{O}_T)$ and obtain $\dim_k H^i(T, \mathcal{O}_T) = 1$ for $i = 0, 1$.

The rigid analytic space T is an elliptic curve. Put

$$p_1(z) = \sum_{n \in \mathbb{Z}} q^n / (1 - q^n z)^2, \quad p_1'(z) = \sum_{n \in \mathbb{Z}} q^{2n} z^2 / (1 - q^n z)^3$$

Then $p_1(qz) = p_1(z)$, $p'_1(qz) = p'_1(z)$ and they are meromorphic function on T . Put

$$s_k = \sum_{m \geq 1} m^k q^m / (1 - q^m),$$

$$p(z) = p_1(z) - 2s_1, \quad p'(z) = p'_1(z) + s_1.$$

Theorem 3.6. (Tate). p, p' satisfy the following equation.

$$(*) \quad p'^2 + pp' = p^3 + Bp + C,$$

where

$$B = -5s_3, \quad C = -(5s_3 + 7s_5)/12.$$

Moreover, the map

$$\begin{array}{ccc} T & \xrightarrow{\quad} & \mathbb{P}_k^2 \\ \downarrow \omega & & \downarrow \omega_k \\ [z] & \longmapsto & (1:p(z):p'(z)) \end{array}$$

defines an embedding of T as a cubic curve defined by the equation (*).

The j invariant of the cubic curve defined by (*) has the form

$$\begin{aligned} j &= (1 - 48B)^2 / \Delta, & \Delta &= q \prod_{m \geq 1} (1 - q^m)^{24} \\ &= 1/q + R(q) \\ &= 1/q + 744 + 196884q + \dots, \end{aligned}$$

where $R(q) \in \mathbb{Z}[[q]]$ (resp. $\mathbb{F}_p[[q]]$), if $\text{char}.k = 0$ (resp. $\text{char}.k = p > 0$).

The following proposition shows that not all elliptic are Tate's curve.

Proposition 3.7. Let k be a complete non-Archimedean valuation field (not necessarily algebraically closed) and E an elliptic curve defined over k . Then the following conditions are equivalent.

1) There exists a finite extension K/k such that

$$E \times_K k \simeq K^{\times} / \langle q \rangle.$$

2) $|j(E)| > 1$.

§4. Analytic reduction and formal schemes.

Let A be an affinoid algebra. Put

$$A^{\circ} = \{ f \in A \mid \|f\|_{\text{sp}} \leq 1 \}, \quad A^{\circ\circ} = \{ f \in A \mid \|f\|_{\text{sp}} < 1 \},$$

$$\bar{A} = A^{\circ} / A^{\circ\circ}.$$

If $A = T_n = k\langle X_1, \dots, X_n \rangle$, then $A^{\circ} = \{ \sum_{\alpha} a_{\alpha} X^{\alpha} \mid a_{\alpha} \in k^{\circ} \}$

$= k^{\circ}\langle X_1, \dots, X_n \rangle$, $A^{\circ\circ} = k^{\circ\circ}\langle X_1, \dots, X_n \rangle$. Since for any element

$g = \sum_{\alpha} b_{\alpha} X^{\alpha}$, $b_{\alpha} \in k^{\circ\circ}$ for sufficiently large $|\alpha|$, $\bar{A} \simeq$

$\bar{k}[X_1, \dots, X_n]$. Since any affinoid algebra is a quotient T_n/I ,

$\bar{A}$ is a $\bar{k}$ algebra of finite type. The analytic reduction $\bar{X}$

of an affinoid space $X = \text{Sp}(A)$ is, by definition, $\text{Spec}(\bar{A})$,

an algebraic k -scheme.

Lemma 4.1. There exists the reduction morphism $R : \text{Sp}(A)$

$\longrightarrow \text{Spec}(\bar{A})$, where for any maximal ideal $\mathfrak{m}$ of A , $R(\mathfrak{m})$

is the image of $\mathfrak{m} \cap A^\circ$ in $\bar{A}$. Moreover, R is surjective to the set of closed points.

Definition 4.2. Let (X, G_X, O_X) be a separated rigid analytic space. An admissible covering $\underline{U} = \{ U_i \}$ is called a pure covering, if $\underline{U}$ satisfies the following conditions.

1) For each i , U_i intersect with a finite number of U_j .

If $U_i \cap U_j \neq \emptyset$, then there exists an affine open set $V_{ij} \subset \bar{U}_i$

with $U_i \cap U_j = R_i^{-1}(V_{ij})$ where $R_i : U_i \rightarrow \bar{U}_i$ and $R_{ij} : U_{ij} \rightarrow \bar{U}_{ij}$

is the canonical reduction.

From the above definition, if $\{ \text{Sp}(A_i) \}$ is a pure covering of a rigid analytic space X , we can define a scheme $\bar{X}$ locally of finite type over $\bar{k}$, by patching together $\text{Spec}(\bar{A}_i)$.

The scheme $\bar{X}$ thus obtained is called the analytic reduction of X (with respect to the pure covering $\{ \text{Sp}(A_i) \}$). Note that

$\bar{X}$ depends on the choice of a pure covering.

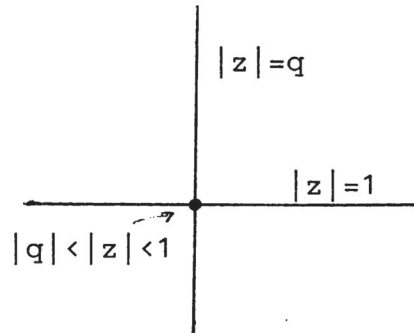
Example 4.3. $X = \{ z \in k \mid |q| \leq |z| \leq 1 \}$ where $q \in k$, $0 < |q| < 1$. Then we have

$$A = k\langle X, Y \rangle / (XY - q), \quad A^\circ = k^\circ\langle X, Y \rangle / (XY - q),$$

Hence, we have

$$\bar{A} \simeq \bar{k}[X,Y]/(XY).$$

Therefore, $\bar{X} = \text{Spec}(\bar{A})$ is a join of two affine lines.



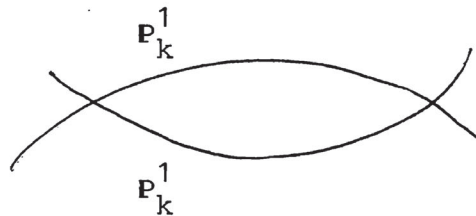
Example 4.4. $X = A_k^n$. The admissible covering

$\{U_i\}_{0 \leq i}$ is pure and $\bar{X}$ is an infinite union of A_k^n .



Example 4.5 $T = k^x / \langle q \rangle$. If we use the admissible covering

$\{X_1, X_2\}$, $\bar{T}$ is as follows.



The dependence of the analytic reduction on pure coverings is clarified, if we consider the relationship between rigid analytic spaces and formal schemes. Recall that for $A = T_n$

$$A^\circ = k^\circ \langle X_1, \dots, X_n \rangle, \quad A^{\circ\circ} = k^{\circ\circ} \langle X_1, \dots, X_n \rangle = \mathfrak{m} A^\circ,$$

where $\mathfrak{m} = k^{\circ\circ}$, the maximal ideal of the valuation ring k° .

Hence we have

$$A^\circ = \varprojlim_{\leftarrow} A^\circ / \mathfrak{m}^s A^\circ = \varprojlim_{\leftarrow} k^\circ[X_1, \dots, X_n] / \mathfrak{m}^s k^\circ[X_1, \dots, X_n].$$

Thus, to A° we can associate a formal scheme $\mathrm{Spf} A^\circ$ over $\mathrm{Spf} k^\circ$. More generally, for any affinoid algebra we have a formal scheme $\mathrm{Spf} A^\circ$ over $\mathrm{Spf} k^\circ$ and $\mathrm{Sp}(A)$ can be considered as the generic fibre of it. In general, to any rigid analytic space X of finite type over k (i.e., covered by a finite number of affinoid spaces), we can associate a formal scheme $\underline{X}$ over $\mathrm{Spf} k^\circ$ and X can be considered as the generic fibre of $\underline{X} \longrightarrow \mathrm{Spf} k^\circ$. Conversely, for a formal scheme $\underline{X} \longrightarrow \mathrm{Spf} k^\circ$ locally of finite type, we can construct a rigid analytic space X over k . The correspondence is not one to one, since the formal schemes depend on pure coverings. Moreover, if $\underline{X}'$ is obtained from the monoidal transformation of $\underline{X}$ along a closed subscheme whose ideal sheaf is contained in an ideal generated by an element

$q \in k$ with $|q| < 1$, then the rigid analytic spaces corresponding to $\underline{X}'$ and $\underline{X}$ are the same. The details can be found in [R].

§5. Non-Kähler rigid analytic surfaces.

In this section, we only give the simplest example. A compact analytic surface S with $b_1(S) = 2g+1$, $g \geq 1$ is an elliptic surface over a curve of genus g and obtained by means of logarithmic transformations from an elliptic bundle over the curve C . A rigid analytic analogue of such a surface is obtained as follows.

Let L be the total space of a line bundle over a complete non-singular curve C . Then $L' = L - 0$ -section has a structure of rigid analytic space and the fibre wise multiplication by $q \in k$ with $|q| < 1$ generates the infinite cyclic group $\langle q \rangle$ of analytic automorphisms of L' . The quotient space $S = L'/\langle q \rangle$ with a natural morphism $f : S \rightarrow C$ is an elliptic surface without singular fibres. If $\deg L \neq 0$, then the elliptic surface $f : S \rightarrow C$ is an analogue of a non-Kähler analytic surface with $b_1(S) = 2g(C)+1$. For example if $g(C) = 0$ (resp. $g(C) = 1$), then S is an analogue of a Hopf surface (resp. a Kodaira surface). The analytic reduction is a cycle of P^1 bundle over C and non-projective.

Moreover, if the curve C contains an affinoid space $\{ z \in k \mid |z| \leq \varepsilon \}$, then we can define a logarithmic transformation of $f : S \rightarrow C$ at $z = 0$ and obtain an elliptic surface with a multiple singular fibre over $z = 0$.

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