

Finiteness of certain products of algebraic groups over a finite field

By

Toshiro HIRANOUCHI*

Abstract

Let $G_1, \dots, G_n$ be smooth connected and commutative algebraic groups over a finite field F . We show the finiteness of the tensor product $(G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_n)(\text{Spec } F)$ of $G_1, \dots, G_n$ in the category of Mackey functors and also Ivorra-Rülling's K -group $T(G_1, \dots, G_n)(\text{Spec } F)$ associated with those algebraic groups as reciprocity functors. We apply this to prove that, for a product of open curves, the finiteness of the relative Chow group and an abelian fundamental group which classifies abelian coverings with bounded ramification along the boundary.

§ 1. Introduction

A Mackey functor over a perfect field F (a finite Mackey functor in the sense of [4], see Def. 2.1 for the definition) is a co- and contravariant functor from the category of étale schemes over F to the category of abelian groups. A smooth connected and commutative algebraic group G over the field F is regarded as a Mackey functor by the correspondence $x \mapsto G(x)$. Such an algebraic group G can be extended to a Nisnevich sheaf with transfers on the category of regular schemes over F with dimension ≤ 1 . Furthermore, it satisfies the following condition which is the so-called reciprocity law (Rosenlicht's theorem when the base field F is algebraically closed [12], Chap. III, Sect. 3, Thm. 1; [3], Prop. 2.2.2): For any open (=non-proper) regular connected curve C over F and a section $a \in G(C)$ there exists an effective Weil divisor $D = \sum_P n_P P$ on the smooth compactification $\overline{C}$ of C such that its support is the boundary $|D| = \overline{C} \setminus C$.

Received March 23, 2013. Revised June 21, 2013, August 7, 2013, August 16, 2013 and August 21, 2013.

2010 Mathematics Subject Classification(s): 19F05

Supported by KAKENHI 25800019

*Department of Mathematics, Graduate School of Science, Hiroshima University 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 Japan.

e-mail: hira@hiroshima-u.ac.jp

© 2014 Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.

and

$$\sum_{P \in C} v_P(f) \operatorname{Tr}_{P/x_C} s_P(a) = 0$$

for any $f \neq 0$ in the function field $F(C)$ of C with $f \equiv 1 \pmod{D}$, that is, $v_P(f-1) \geq n_P$ for any $P \in |D|$, where v_P is the valuation at P , $s_P : G(C) \rightarrow G(P)$ is the pull-back along the natural inclusion $P \hookrightarrow C$ and $\operatorname{Tr}_{P/x_C} : G(P) \rightarrow G(x_C)$ is the push-forward along the finite map $P \rightarrow x_C := \operatorname{Spec} H^0(\overline{C}, \mathcal{O}_{\overline{C}})$. F. Ivorra and K. R ulling [3] have introduced the notion of a *reciprocity functor* as a Nisnevich sheaf with transfers on the category of regular schemes over F with dimension ≤ 1 satisfying several axioms. One of the axioms is the reciprocity law as above. They have also introduced a “product” $T(\mathcal{M}_1, \dots, \mathcal{M}_n)$ associated with reciprocity functors $\mathcal{M}_1, \dots, \mathcal{M}_n$ in the category of reciprocity functors (for the precise definition of the “product”, see [3] Def. 4.2.3). By the very construction of the product T , as a Mackey functor, $T(\mathcal{M}_1, \dots, \mathcal{M}_n)$ is a quotient of the tensor product $\mathcal{M}_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{M}_n$ (for the definition, see (2.1) in the next section). Hence we have a canonical surjection

$$(\mathcal{M}_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{M}_n)(\operatorname{Spec} F) \twoheadrightarrow T(\mathcal{M}_1, \dots, \mathcal{M}_n)(\operatorname{Spec} F).$$

Although the tensor product $\overset{M}{\otimes}$ gives a structure of a symmetric monoidal category in the abelian category of Mackey functors, it is not known that whether this product T satisfies the associativity and then gives a monoidal structure or not. However, this product coincides with the K -group of homotopy invariant Nisnevich sheaves with transfers on the category of smooth schemes over F ([6]). In particular, we obtain an isomorphism

$$T(G_1, \dots, G_n)(\operatorname{Spec} F) \simeq K(F; G_1, \dots, G_n),$$

for semi-abelian varieties $G_1, \dots, G_n$ over F , where $K(F; G_1, \dots, G_n)$ is Somekawa’s K -group [13] which was limited on considering only semi-abelian varieties. For semi-abelian varieties $G_1, \dots, G_n$ over a *finite field* F , B. Kahn showed in [5] that

$$(1.1) \quad K(F; G_1, \dots, G_n) = (G_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} G_n)(\operatorname{Spec} F) = 0$$

if $n > 1$. Because of the isomorphism ([13], Thm. 1.4)

$$K(F; \overbrace{\mathbb{G}_m, \dots, \mathbb{G}_m}^n) \xrightarrow{\simeq} K_n^M(F),$$

where $K_n^M(F)$ is the Milnor K -group of the field F , these results generalize the classical fact that $K_n^M(F) = 0$ if F is a finite field and $n > 1$. For algebraic groups G_1, G_2 which may contain unipotent part, the product $(G_1 \overset{M}{\otimes} G_2)(\operatorname{Spec} F)$ may not be trivial. In this note, we shall show the following theorem.

Theorem 1.1 (Thm. 2.3). *Let $G_1, \dots, G_n$ be smooth commutative and connected algebraic groups over a finite field F . Then*

$$T(G_1, \dots, G_n)(\operatorname{Spec} F), \quad \text{and} \quad (G_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} G_n)(\operatorname{Spec} F)$$

are finite.

As an application of (1.1), the class field theory of a product of projective smooth curves over a finite field, a special case of the higher dimensional class field theory of S. Bloch, K. Kato and S. Saito (e.g., [7]) is deduced from the classical (unramified) class field theory (= class field theory of curves over a finite field) and Lang's theorem: the reciprocity map on a normal variety over a finite field has dense image. In Section 3, we will pursue related results on the (ramified) class field theory of a product of open (=non-proper) curves as a byproduct of the above theorem. In particular, we obtain a finiteness of the relative Chow group $\operatorname{CH}_0(X, D)$ for a product of smooth curves $X = X_1 \times \dots \times X_n$ over a finite field and an effective Weil divisor D on the product $\overline{X} = \overline{X}_1 \times \dots \times \overline{X}_n$ of the smooth compactification $\overline{X}_i$ of X_i with support $|D| \subset \overline{X} \setminus X$ (Thm. 3.1).

Throughout this note, we mean by an *algebraic group* a smooth connected and commutative group scheme over a field.

Acknowledgments. A part of this note was written during a stay of the author at the Duisburg-Essen university. He thanks the institute for its hospitality. The author learned most of what he knows about relative Chow groups and Albanese varieties from Henrik Russell. Finally, we would also like to thank the anonymous referee for his/her detailed and quick reading and for useful comments and suggestions.

§ 2. Mackey functors and reciprocity functors

Throughout this section, F is a perfect field.

Definition 2.1. A Mackey functor M over F is a contravariant functor from the category of étale schemes over F to the category of abelian groups equipped with a covariant structure for finite morphisms such that $M(x_1 \sqcup x_2) = M(x_1) \oplus M(x_2)$ and if

$$\begin{array}{ccc} x' & \xrightarrow{g'} & x \\ f' \downarrow & & \downarrow f \\ y' & \xrightarrow{g} & y \end{array}$$

is a Cartesian diagram with g and g' are finite, then the induced diagram

$$\begin{array}{ccc} M(x') & \xrightarrow{g'^*} & M(x) \\ f'^* \uparrow & & \uparrow f^* \\ M(y') & \xrightarrow{g_*} & M(y) \end{array}$$

commutes.

We call a morphism $x \rightarrow \operatorname{Spec} F$ a *finite point* if $x = \operatorname{Spec} E$ for some finite field extension of F . For Mackey functors $M_1, \dots, M_n$, the product $M_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} M_n$ called the *Mackey product* is defined as follows. For any finite point $x \rightarrow \operatorname{Spec} F$,

$$(2.1) \quad (M_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} M_n)(x) := \left(\bigoplus_{y \rightarrow x: \text{ finite}} M_1(y) \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} M_n(y) \right) / R(x)$$

where $y \rightarrow x$ runs through all finite points over x , and $R(x)$ is the subgroup generated by elements of the following form: For any morphism $j : y' \rightarrow y$ of finite points over x , and if $a'_{i_0} \in M_{i_0}(y')$ and $a_i \in M_i(y)$ for $i \neq i_0$, then

$$(2.2) \quad j^*(a_1) \otimes \dots \otimes a'_{i_0} \otimes \dots \otimes j^*(a_n) - a_1 \otimes \dots \otimes j_*(a'_{i_0}) \otimes \dots \otimes a_n \in R(x),$$

where j^* and j_* are the pull-back and the push-forward along j respectively. We write $\{a_1, \dots, a_n\}_{y/x}$ for the image of $a_1 \otimes \dots \otimes a_n \in M_1(y) \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} M_n(y)$ in the product $(M_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} M_n)(x)$. Using this symbol, the above relation (2.2) defining $R(x)$ above gives the following equation which is often called the *projection formula*:

$$(2.3) \quad \{j^*(a_1), \dots, a'_{i_0}, \dots, j^*(a_n)\}_{y'/x} = \{a_1, \dots, j_*(a'_{i_0}), \dots, a_n\}_{y/x}.$$

The Mackey product (2.1) satisfies the following properties:

- (M1) The Mackey product $\overset{M}{\otimes}$ gives a tensor product in the abelian category of the Mackey functors. Its unit is the constant Mackey functor $\mathbb{Z}$. In particular, the product commutes with the direct sum $\oplus$ and satisfies the associativity: $M_1 \overset{M}{\otimes} M_2 \overset{M}{\otimes} M_3 \xrightarrow{\cong} (M_1 \overset{M}{\otimes} M_2) \overset{M}{\otimes} M_3; \{a_1, a_2, a_3\}_{y/x} \mapsto \{\{a_1, a_2\}_{y/y}, a_3\}_{y/x}$.
- (M2) The product $-\overset{M}{\otimes} M$ is right exact for any Mackey functor M .
- (M3) For any finite point $j : x' \rightarrow x$, the push-forward $j_* : (M_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} M_n)(x') \rightarrow (M_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} M_n)(x)$ along j is given by $j_*(\{a_1, \dots, a_n\}_{y'/x'}) = \{a_1, \dots, a_n\}_{y'/x}$ on symbols.

Lemma 2.2. *Let G be a unipotent (smooth and commutative) algebraic group over F and A a semi-abelian variety over F . If F is a perfect field of characteristic $p > 0$, we have $G \overset{M}{\otimes} A = 0$.*

Proof. The unipotent group G has a composition series:

$$0 = G^r \subset \cdots \subset G^1 \subset G,$$

each G^i/G^{i+1} being isomorphic to $\mathbb{G}_a$. By the right exactness (M2), it is enough to show $(\mathbb{G}_a \overset{M}{\otimes} A) = 0$. By (M3) above, the assertion is reduced to showing $\{a, b\}_{x/x} = 0$ for any $a \in \mathbb{G}_a(x), b \in A(x)$. There exists a finite point $j : x' \rightarrow x$ such that $j^*(b) = pb'$ for some $b' \in A(x')$. Since the trace map (= the push-forward map on $\mathbb{G}_a$) $j_* = \text{Tr}_{x'/x} : \mathbb{G}_a(x') \rightarrow \mathbb{G}_a(x)$ is surjective, we obtain

$$\begin{aligned} \{a, b\}_{x/x} &= \{\text{Tr}_{x'/x}(a'), b\}_{x/x} \quad \text{for some } a' \in \mathbb{G}_a(x') \\ &= \{a', j^*b\}_{x'/x} \quad \text{by the projection formula (2.3)} \\ &= \{a', pb'\}_{x'/x} \\ &= 0. \end{aligned}$$

The assertion follows from this. □

A point x is a morphism $x = \text{Spec } k \rightarrow \text{Spec } F$ where k is a finitely generated field extension over F . Let $\text{Reg}^{\leq 1}$ be the category of regular schemes over F which are separated and of finite type over some point and Ab is the category of abelian groups. A *reciprocity functor* is a Nisnevich sheaf with transfers $\mathcal{M} : (\text{Reg}^{\leq 1} \text{Cor})^{\text{op}} \rightarrow \text{Ab}$ satisfying several axioms ([3], Def. 1.5.1), where $\text{Reg}^{\leq 1} \text{Cor}$ is the category with objects the objects of $\text{Reg}^{\leq 1}$ and morphisms are given by correspondences. In particular, $\mathcal{M}$ has the transfer $\text{Tr}_{X/Y} := f_* : \mathcal{M}(X) \rightarrow \mathcal{M}(Y)$ for a finite flat morphism $f : X \rightarrow Y$. As an example, an algebraic group G over F is a reciprocity functor which is given by $X \mapsto G(X)$. For a finite flat map $X \rightarrow Y$, The transfer map $\text{Tr}_{X/Y} : G(X) \rightarrow G(Y)$ equals the usual trace if $G = \mathbb{G}_a$ and the norm map if $G = \mathbb{G}_m$. Note that any reciprocity functor gives a Mackey functor by restricting to the category of finite points over F . The “product” $T(\mathcal{M}_1, \dots, \mathcal{M}_n)$ for reciprocity functors $\mathcal{M}_1, \dots, \mathcal{M}_n$ is a reciprocity functor and satisfies some functorial properties. In particular, there are functorial isomorphisms

$$T(\mathcal{M}_1, \dots, \mathcal{M}_i, \dots, \mathcal{M}_j, \dots, \mathcal{M}_n) \simeq T(\mathcal{M}_1, \dots, \mathcal{M}_j, \dots, \mathcal{M}_i, \dots, \mathcal{M}_n)$$

and

$$T(\mathcal{M}_1, \dots, \mathcal{M}_i \oplus \mathcal{M}_i', \dots, \mathcal{M}_n) \simeq T(\mathcal{M}_1, \dots, \mathcal{M}_i, \dots, \mathcal{M}_n) \oplus T(\mathcal{M}_1, \dots, \mathcal{M}_i', \dots, \mathcal{M}_n).$$

There is a surjective map as Nisnevich sheaves $T(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3) \rightarrow T(T(\mathcal{M}_1, \mathcal{M}_2), \mathcal{M}_3)$. However, it is not known whether this map becomes an isomorphism. By the very construction, for any finite point x over F , the product $T(\mathcal{M}_1, \dots, \mathcal{M}_n)(x)$ evaluated at x is a quotient of the Mackey product $(\mathcal{M}_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{M}_n)(x)$. More precisely, the product $T(\mathcal{M}_1, \dots, \mathcal{M}_n)$ is defined to be the Nisnevich sheafification $\mathcal{L}_{\text{Nis}}^\infty$ of a quotient $\mathcal{L}^\infty$ of the product $\mathcal{M}_1 \otimes \dots \otimes \mathcal{M}_n$ whose underlying Mackey functor is the Mackey product $\mathcal{M}_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{M}_n$. However, an isomorphism $\mathcal{L}^\infty(x) \xrightarrow{\simeq} \mathcal{L}_{\text{Nis}}^\infty(x)$ exists since any Nisnevich covering of $\text{Spec } F$ refines a trivial covering.

Theorem 2.3. *Let $G_1, \dots, G_n$ be algebraic groups over a finite field F . Then the group $(G_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} G_n)(x)$ is finite for any finite point x over F . In particular, the product $T(G_1, \dots, G_n)(x)$ is also finite.*

Proof. The case of $n = 1$, there is nothing to show. So we assume $n > 1$. First we show that there is a surjection $(U_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} U_n)(x) \twoheadrightarrow (G_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} G_n)(x)$ by induction on n . We consider the case $n = 2$. The algebraic group G_i has a decomposition

$$0 \rightarrow U_i \rightarrow G_i \rightarrow A_i \rightarrow 0$$

with unipotent part U_i and a semi-abelian variety A_i over F . From the right exactness (M2), we obtain the following exact sequences

$$(2.4) \quad \begin{array}{ccccccc} U_1 \overset{M}{\otimes} U_2 & & & A_1 \overset{M}{\otimes} U_2 & & & \\ \downarrow & & & \downarrow & & & \\ U_1 \overset{M}{\otimes} G_2 & \longrightarrow & G_1 \overset{M}{\otimes} G_2 & \longrightarrow & A_1 \overset{M}{\otimes} G_2 & \longrightarrow & 0 \\ \downarrow & & & \downarrow & & & \\ U_1 \overset{M}{\otimes} A_2 & & & A_1 \overset{M}{\otimes} A_2 & & & \\ \downarrow & & & \downarrow & & & \\ 0 & & & 0 & & & . \end{array}$$

By Kahn's theorem (1.1), we have $A_1 \overset{M}{\otimes} A_2 = 0$. Lemma 2.2 implies $U_1 \overset{M}{\otimes} A_2 = A_1 \overset{M}{\otimes} U_2 = 0$. Hence we obtain $U_1 \overset{M}{\otimes} U_2 \twoheadrightarrow G_1 \overset{M}{\otimes} G_2$. For $n > 2$, consider a decomposition

$$0 \rightarrow U_n \rightarrow G_n \rightarrow A_n \rightarrow 0$$

as above. The right exactness (M2) again gives an exact sequence,

$$U_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} U_{n-1} \overset{M}{\otimes} U_n \rightarrow U_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} U_{n-1} \overset{M}{\otimes} G_n \rightarrow U_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} U_{n-1} \overset{M}{\otimes} A_n \rightarrow 0.$$

Lemma 2.2 induces $U_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} U_{n-1} \overset{M}{\otimes} U_n \twoheadrightarrow U_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} U_{n-1} \overset{M}{\otimes} G_n$. By induction hypothesis, we have $U_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} U_{n-1} \twoheadrightarrow G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_{n-1}$ and the claim follows from this. Thus it is enough to show the assertion for unipotent groups $G_i = U_i$. Take a composition series of the unipotent group $G := G_i$:

$$0 = G^r \subset \cdots \subset G^1 \subset G,$$

each $G^i/G^{i+1} \simeq \mathbb{G}_a$. By the right exactness (M2), we may assume $G_i = \mathbb{G}_a$ for all i and $x = \text{Spec } F$ without loss of generality. We show the finiteness of $(\mathbb{G}_a)^{\overset{M}{\otimes} n}(x)$ by induction on n . We assume that $(\mathbb{G}_a)^{\overset{M}{\otimes} (n-1)}(x)$ is finite. The group $(\mathbb{G}_a)^{\overset{M}{\otimes} n}(x)$ has a structure of an F -vector space given by $a\{a_1, \dots, a_n\}_{x'/x} := \{j^*(a)a_1, \dots, a_n\}_{x'/x}$, for any $a \in F$ and a symbol $\{a_1, \dots, a_n\}_{x'/x}$ on a finite point $j : x' \rightarrow x = \text{Spec } F$. Consider a subspace $I(x)$ of $(\mathbb{G}_a)^{\overset{M}{\otimes} n}(x)$ generated by the elements of the form

$$\{1_{x'}, a_2, \dots, a_n\}_{x'/x} - \{a_2 \cdots a_n, 1_{x'}, \dots, 1_{x'}\}_{x'/x}$$

for any finite point $x' \rightarrow x$, where $1_{x'} \in \mathbb{G}_a(x')$ is the unit. By identifying the canonical isomorphism $(\mathbb{G}_a)^{\overset{M}{\otimes} n} \simeq \mathbb{G}_a \overset{M}{\otimes} (\mathbb{G}_a)^{\overset{M}{\otimes} (n-1)}$ by (M1) and $j^*(1_x) = 1_{x'}$, we have

$$\begin{aligned} \{1_{x'}, a_2, \dots, a_n\}_{x'/x} &= \{j^*(1_x), \{a_2, \dots, a_n\}_{x'/x'}\}_{x'/x} \\ &= \{1_x, j_*\{a_2, \dots, a_n\}_{x'/x'}\}_{x/x} \quad \text{by the projection formula} \\ &= \{1_x, \{a_2, \dots, a_n\}_{x'/x}\}_{x/x} \quad \text{by (M3)}. \end{aligned}$$

By the induction hypothesis, the set of elements of this form is finite. On the other hand, the projection formula implies

$$\begin{aligned} \{a_2 \cdots a_n, 1_{x'}, \dots, 1_{x'}\}_{x'/x} &= \{a_2 \cdots a_n, j^*(1_x), \dots, j^*(1_x)\}_{x'/x} \\ &= \{j_*(a_2 \cdots a_n), 1_x, \dots, 1_x\}_{x/x}. \end{aligned}$$

Thus the subspace $I(x)$ is finite. Define $Q(x) := (\mathbb{G}_a)^{\overset{M}{\otimes} n}(x)/I(x)$ the quotient space. We denote by $\overline{\{a_1, \dots, a_n\}_{x'/x}}$ the image of $\{a_1, \dots, a_n\}_{x'/x}$ in $Q(x)$. Now we consider a subspace $S(x)$ of $Q(x)$ generated by symbols of the form $\overline{\{a_1, \dots, a_n\}_{x/x}}$. It is easy to see that the subspace $S(x)$ is finite. We show that any symbol $\overline{\{a_1, \dots, a_n\}_{x'/x}}$ in $Q(x)$ for a finite point $j : x' \rightarrow x$ is in $S(x)$. In fact,

$$\begin{aligned} \overline{\{a_1, \dots, a_n\}_{x'/x}} &= j_*(\overline{\{a_1, \dots, a_n\}_{x'/x'}}) \quad \text{by (M3)} \\ &= j_*(a_1 \overline{\{1_{x'}, a_2, \dots, a_n\}_{x'/x'}}) \quad \text{because of } \overline{\{a_1, \dots, a_n\}_{x'/x'}} \in Q(x') \\ &= j_*(a_1 \overline{\{a_2 \cdots a_n, 1_{x'}, \dots, 1_{x'}\}_{x'/x'}}) \\ &= \overline{\{a_1 \cdots a_n, 1_{x'}, \dots, 1_{x'}\}_{x'/x}} \quad \text{by (M3)} \\ &= \overline{\{j_*(a_1 \cdots a_n), 1_x, \dots, 1_x\}_{x/x}} \quad \text{by the projection formula.} \end{aligned}$$

Thus we obtain $Q(x) = S(x)$ and the assertion follows from it. $\square$

§ 3. Applications

Let X be a smooth (and connected) variety over a finite field F . Assume that there is a smooth compactification $\overline{X}$ of X , that is, a projective smooth variety which contains X as an open subscheme. Let D be an effective Weil divisor on $\overline{X}$ with support $|D| \subset \overline{X} \setminus X$. We define the relative Chow group of the pair (X, D) by

$$\mathrm{CH}_0(X, D) := \mathrm{Coker} \left(\mathrm{div} : \bigoplus_{\phi: C \rightarrow X} P_C(\overline{\phi}^* D) \rightarrow Z_0(X) \right),$$

where the direct sum runs over the normalization $\phi : C \rightarrow X$ of a curve in X , $Z_0(X)$ is the group of 0-cycles on X , $\overline{\phi} : \overline{C} \rightarrow \overline{X}$ is the extension of the map ϕ to the smooth compactification $\overline{C}$ of C , the map div is given by the divisor map on each curve C and

$$P_C(\overline{\phi}^* D) := \{f \in F(C)^\times \mid f \equiv 1 \bmod \overline{\phi}^* D + (\overline{C} \setminus C)_{\mathrm{red}}\}$$

(cf. [1], Set. 8.1, see also [9], Sect. 3.4 and 3.5). Putting $X_y := X \times_{\mathrm{Spec} F} y$ and denoting by D_y the pull-back of D to $\overline{X}_y := \overline{X} \times_x y$ for any finite point $y \rightarrow x = \mathrm{Spec} F$, the assignment

$$\mathcal{CH}_0(X, D) : y \mapsto \mathrm{CH}_0(X_y, D_y)$$

gives a Mackey functor $\mathcal{CH}_0(X, D)$.

Theorem 3.1. *Let $X_1, \dots, X_n$ be smooth and geometrically connected curves over a finite field F and put $X := X_1 \times \cdots \times X_n$. For an effective Weil divisor D on $\overline{X} := \overline{X}_1 \times \cdots \times \overline{X}_n$ with support $|D| \subset \overline{X} \setminus X$, the kernel of the degree map $\mathrm{CH}_0(X, D)^0 := \mathrm{Ker}(\mathrm{deg} : \mathrm{CH}_0(X, D) \rightarrow \mathbb{Z})$ is finite.*

Recently, P. Deligne has showed the finiteness of the group $\mathrm{CH}_0(X, D)^0$ for a smooth variety X over a finite field and an effective Cartier divisor D with support in the boundary $\overline{X} \setminus X$ only assuming the existence of some normal compactification $\overline{X}$ of X as an application of his finiteness theorem for l -adic Galois representations of function fields ([1], Thm. 8.1).

Proof of Thm. 3.1. Let $p_i : \overline{X} = \overline{X}_1 \times \cdots \times \overline{X}_n \rightarrow \overline{X}_i$ be the projection. An irreducible component Z of the boundary

$$\overline{X} \setminus X = \bigcup_{i=1}^n \overline{X}_1 \times \cdots \times (\overline{X}_i \setminus X_i) \times \cdots \times \overline{X}_n.$$

has the form $Z = p_i^{-1}(P)$ for some $P \in \overline{X}_i \setminus X_i$. Therefore, for sufficiently large divisors D_i on $\overline{X}_i$, there is a surjection $\mathrm{CH}_0(X, p_1^* D_1 + \cdots + p_n^* D_n) \twoheadrightarrow \mathrm{CH}_0(X, D)$. Thus we may assume that D is a divisor of the form $p_1^* D_1 + \cdots + p_n^* D_n$. Now we consider the map

$$(3.1) \quad \psi : (\mathcal{CH}_0(X_1, D_1) \otimes^M \cdots \otimes^M \mathcal{CH}_0(X_n, D_n))(\mathrm{Spec} F) \rightarrow \mathrm{CH}_0(X, D)$$

defined by

$$\psi(\{[P_1], \dots, [P_n]\}_{y/x}) := (j_y)_* [P_1 \times_y \cdots \times_y P_n]$$

for a finite point $j_y : y \rightarrow x := \mathrm{Spec} F$ and the classes $[P_i]$ represented by closed points P_i of $(X_i)_y := X_i \times_x y$, where $[P_1 \times_y \cdots \times_y P_n]$ is the zero-cycle on X_y determined by P_i 's and the base change $X_y \rightarrow X$ to y is also denoted by j_y .

Well-definedness of ψ : First we have to prove that ψ annihilates the element of the form

$$\{j^*[P_1], \dots, [P'_{i_0}], \dots, j^*[P_n]\}_{y'/x} - \{[P_1], \dots, j_*[P'_{i_0}], \dots, [P_n]\}_{y/x}$$

for a map $j : y' \rightarrow y$ of finite points and a closed point P'_{i_0} of $(X_{i_0})_{y'}$ and closed points P_i of $(X_i)_y$ ($i \neq i_0$).

$$\begin{aligned} \psi(\{j^*[P_1], \dots, [P'_{i_0}], \dots, j^*[P_n]\}_{y'/x}) &= (j_{y'})_*(j^*[P_1] \times_{y'} \cdots \times_{y'} [P'_{i_0}] \times_{y'} \cdots \times_{y'} j^*[P_n]) \\ &= (j_y)_* \circ (j_*)([P_1 \times_y \cdots \times_y P'_{i_0} \times_y \cdots \times_y P_n]) \\ &= (j_y)_*([P_1] \times_y \cdots \times_y j_*[P'_{i_0}] \times_y \cdots \times_y [P_n]) \\ &= \psi(\{[P_1], \dots, j_*[P'_{i_0}], \dots, [P_n]\}_{y/x}). \end{aligned}$$

Next we show that ψ annihilates the element $\{[P_1], \dots, \overset{i}{\mathrm{div}}(f), \dots, [P_n]\}_{y/x}$ for a finite point $j_y : y \rightarrow x$ and $f \in P_{(X_i)_y}((D_i)_y)$. Because of

$$\{[P_1], \dots, \mathrm{div}(f), \dots, [P_n]\}_{y/x} = (j_y)_*(\{[P_1], \dots, \mathrm{div}(f), \dots, [P_n]\}_{y/y}),$$

we may assume $y = x$ by the definition of ψ . Consider the product

$$P_1 \times_x \cdots \times_x P_{i-1} \times_x X_i \times_x P_{i+1} \times_x \cdots \times_x P_n = \bigcup_Q (X_i)_Q,$$

where the union runs over a point Q in $P_1 \times_x \cdots \times_x P_{i-1} \times_x P_{i+1} \times_x \cdots \times_x P_n$. For such a point Q , we denote by ϕ_Q the natural map $(X_i)_Q \rightarrow X$. Since the base change $j_Q : (X_i)_Q \rightarrow X_i$ is unramified, $j_Q^*(f) \in P_{(X_i)_Q}(\phi_Q^*(D))$. We obtain

$$\begin{aligned} \psi(\{[P_1], \dots, \overset{i}{\mathrm{div}}(f), \dots, [P_n]\}_{x/x}) &= \sum_Q j_Q^* \mathrm{div}(f) \\ &= 0. \end{aligned}$$

Surjectivity of ψ : We show that the map ψ is surjective. Take a cycle $[P]$ as a generator of $\mathrm{CH}_0(X, D)$ which is represented by a closed point P on X and is a finite point $j_P : P \rightarrow x = \mathrm{Spec} F$. By the definition of ψ , the push-forward map on the relative Chow group and the norm map on the Mackey product are compatible as in the following commutative diagram:

$$\begin{array}{ccc} (\mathcal{CH}_0(X_1, D_1) \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{CH}_0(X_n, D_n))(P) & \xrightarrow{\psi_P} & \mathrm{CH}_0(X_P, D_P) \\ (j_P)_* \downarrow & & \downarrow (j_P)_* \\ (\mathcal{CH}_0(X_1, D_1) \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{CH}_0(X_n, D_n))(x) & \xrightarrow{\psi} & \mathrm{CH}_0(X, D). \end{array}$$

Thus to show the surjectivity of ψ we may assume that P is an F -rational point. The point P is determined by maps $P \rightarrow X_i$. These maps give closed points P_i in X_i and $\psi(\{[P_1], \dots, [P_n]\}_{x/x}) = [P_1 \times_x \cdots \times_x P_n] = [P]$. Therefore ψ is surjective.

Finiteness of $\mathrm{CH}_0(X, D)$: By a theorem of F. K. Schmidt ([10], Sect. 8), there exists a degree 1 cycle in X . Hence, for each i , we have a decomposition $\mathcal{CH}_0(X_i, D_i) \simeq \mathbb{Z} \oplus J_{X_i, D_i}$ by the generalized Jacobian variety J_{X_i, D_i} of the pair (X_i, D_i) ([12]). According to this decomposition, we obtain

$$\mathcal{CH}_0(X_1, D_1) \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{CH}_0(X_n, D_n) \simeq \mathbb{Z} \oplus \bigoplus_{r=1}^n \bigoplus_{1 \leq i_1 < \cdots < i_r \leq n} (J_{X_{i_1}, D_{i_1}} \overset{M}{\otimes} \cdots \overset{M}{\otimes} J_{X_{i_r}, D_{i_r}})$$

by (M1). The surjection ψ (3.1) induces a surjection

$$\bigoplus_{r=1}^n \bigoplus_{1 \leq i_1 < \cdots < i_r \leq n} (J_{X_{i_1}, D_{i_1}} \overset{M}{\otimes} \cdots \overset{M}{\otimes} J_{X_{i_r}, D_{i_r}})(\mathrm{Spec} F) \twoheadrightarrow \mathrm{CH}_0(X, D)^0.$$

The left is finite by Theorem 2.3 and so is $\mathrm{CH}_0(X, D)^0$. $\square$

Let X be the product of curves over F , and D as in the above theorem (Thm. 3.1). For each normalization $\phi : C \rightarrow X$ of a curve in X , we have a divisor $D_C := \overline{\phi}^*(D) + (\overline{C} \setminus C)_{\mathrm{red}}$ on $\overline{C}$. The category of étale coverings of X with ramification bounded by the collection of Weil divisors $(D_C)_{\phi: C \rightarrow X}$ forms a Galois category and gives a fundamental group $\pi_1(X, D)$ ([2], Lem. 3.3). For each such C and a point $P \in \overline{C} \setminus C$ there is a canonical map $G_{C, P}^{\mathrm{ab}} := \mathrm{Gal}(F(C)_P^{\mathrm{ab}}/F(C)_P) \rightarrow \pi_1(X)^{\mathrm{ab}}$, where $F(C)_P^{\mathrm{ab}}$ is the maximal abelian extension of the completion $F(C)_P$ at P . By the very definition of the coverings, we have

$$\mathrm{Coker} \left(\bigoplus_{\phi: C \rightarrow X} \bigoplus_{P \in \overline{C} \setminus C} G_{C, P}^{\mathrm{ab}, m_P(D_C)} \rightarrow \pi_1(X)^{\mathrm{ab}} \right) \twoheadrightarrow \pi_1(X, D)^{\mathrm{ab}},$$

where $G_{C,P}^{\text{ab},m}$ is the m -th (upper numbering) ramification subgroup of $G_{C,P}^{\text{ab}}$ ([11], Chap. IV, Sect. 3) and $m_P(D_C)$ is the multiplication of the divisor D_C at P . Using the idele theoretic description of the relative Chow group

$$\text{CH}_0(X, D) \simeq \text{Coker} \left(\bigoplus_{\phi: C \rightarrow X} F(C)^\times \rightarrow Z_0(X) \oplus \bigoplus_{\phi: C \rightarrow X} \bigoplus_{P \in \overline{C} \setminus C} F(C)_P^\times / U_{C,P}^{m_P(D_C)} \right),$$

where $U_{C,P}^m = 1 + \mathfrak{m}_{C,P}^m$ is the higher unit group, local class field theory (see e.g., [11], Chap. XV) induces a commutative diagram:

$$\begin{array}{ccc} Z_0(X)^0 & \longrightarrow & \text{CH}_0(X, D)^0 \\ \rho \downarrow & & \downarrow \rho_D \\ \pi_1(X)^{\text{ab},0} & \longrightarrow & \pi_1(X, D)^{\text{ab},0}. \end{array}$$

Here $Z_0(X)^0$ is the kernel of the degree map $\deg : Z_0(X) \rightarrow \mathbb{Z}$, the left vertical map ρ is the reciprocity map on X and $\pi_1(X, D)^{\text{ab},0}$ is the geometric part of the abelian fundamental group (= the kernel of the canonical map $\pi_1(X, D)^{\text{ab}} \rightarrow \pi_1(\text{Spec}(F))^{\text{ab}}$). The image of the reciprocity map $\rho : Z_0(X) \rightarrow \pi_1(X)^{\text{ab}}$ is known to be dense (due to Lang [8]) and the image of ρ_D in the above diagram is finite by Theorem 3.1. Therefore, the map ρ_D is surjective and we obtain the following finiteness result.

Corollary 3.2. *Let X and D be as in Theorem 3.1. Then, the geometric part of the abelian fundamental group $\pi_1(X, D)^{\text{ab},0}$ is finite.*

References

- [1] H. Esnault and M. Kerz, *A finiteness theorem for Galois representations of function fields over finite fields (after Deligne)*, Acta Math. Vietnam. **37** (2012), no. 4, 531–562.
- [2] S. Harada and T. Hiranouchi, *Smallness of fundamental groups for arithmetic schemes*, J. Number Theory **129** (2009), no. 11, 2702–2712.
- [3] F. Ivorra and K. R ulling, *K-groups of reciprocity functors*, arXiv:1209.1207.
- [4] B. Kahn, *The decomposable part of motivic cohomology and bijectivity of the norm residue homomorphism*, Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989), Contemp. Math., vol. 126, Amer. Math. Soc., Providence, RI, 1992, pp. 79–87.
- [5] ———, *Nullit  de certains groupes attach s aux vari t s semi-ab liennes sur un corps fini; application*, C. R. Acad. Sci. Paris S r. I Math. **314** (1992), no. 13, 1039–1042.
- [6] B. Kahn and T. Yamazaki, *Voevodsky’s motives and Weil reciprocity*, arXiv:1108.2764.
- [7] K. Kato and S. Saito, *Unramified class field theory of arithmetical surfaces*, Ann. of Math. (2) **118** (1983), no. 2, 241–275.

- [8] S. Lang, *Sur les séries L d'une variété algébrique*, Bull. Soc. Math. France **84** (1956), 385–407.
- [9] H. Russell, *Albanese varieties with modulus over a perfect field*, arXiv:0902.2533.
- [10] F. K. Schmidt, *Analytische Zahlentheorie in Körpern der Charakteristik p* , Math. Z. **33** (1931), no. 1, 1–32.
- [11] J.-P. Serre, *Corps locaux*, Deuxième édition, Actualités scientifiques et industrielles 1296, Publications de l'Institut Mathématique de l'Université de Nancago, No. VIII, Hermann, Paris, 1968.
- [12] ———, *Groupes algébriques et corps de classes*, Reprint of the second edition, Actualités Scientifiques et Industrielles 1264, Publications de l'Institut Mathématique de l'Université de Nancago, No. VII, Hermann, Paris, 1984.
- [13] M. Somekawa, *On Milnor K -groups attached to semi-abelian varieties*, K -Theory **4** (1990), no. 2, 105–119.