

Hanner type inequalities and duality

九州工業大学・工学部 加藤幹雄 (Mikio KATO)

Department of Mathematics, Kyushu Institute of Technology

e-mail: katom@tobata.isc.kyutech.ac.jp

岡山県立大学・情報工学部 高橋泰嗣 (Yasuji TAKAHASHI)

Department of System Engineering, Okayama Prefectural University

e-mail: takahasi@cse.oka-pu.ac.jp

北九州工業高等専門学校 山田康隆 (Yasutaka YAMADA)

Kitakyushu College of Technology

e-mail: yamada@kct.ac.jp

We shall first discuss two kinds of Hanner type inequalities with a weight in a Banach space X in connection with sharp uniform smoothness and convexity: the first kind of inequalities will characterize the 2-uniform smoothness and 2-uniform convexity of X , and the other the p -uniform smoothness and q -uniform convexity of X . Next we shall present a duality theorem on a "general" Hanner type inequality with "several weights", which is valid for both kinds of the above inequalities. Finally the best value of the weight constant in these inequalities for L_p -spaces will be determined.

Let X be a Banach space and X^* its dual space. Let S_X be the unit sphere of X . Let $1 \leq p, q, r, \dots \leq \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1/r + 1/r' = \dots = 1$.

1. Hanner's inequalities for L_p (Hanner [3], 1956)

(i) If $1 < p \leq 2$, for all f, g in L_p

$$\|f + g\|_p^p + \|f - g\|_p^p \geq \left| \|f\|_p + \|g\|_p \right|^p + \left| \|f\|_p - \|g\|_p \right|^p \quad (\text{H1})$$

(ii) If $2 \leq p < \infty$, for all f, g in L_p

$$\|f + g\|_p^p + \|f - g\|_p^p \leq \left| \|f\|_p + \|g\|_p \right|^p + \left| \|f\|_p - \|g\|_p \right|^p \quad (\text{H2})$$

2. Definition (i) The modulus of convexity of X :

$$\delta_X(\varepsilon) := \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in S_X, \|x - y\| = \varepsilon \right\} \quad \text{for } 0 \leq \varepsilon \leq 2.$$

(ii) X is uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon > 0$.

(iii) X is q -uniformly convex ($2 \leq q < \infty$) if there exists $C > 0$ such that $\delta_X(\varepsilon) \geq C\varepsilon^q$ for all $\varepsilon > 0$.

3. Remark (i) If $1 \leq q < 2$ no Banach space is q -uniformly convex (cf. [2]; for a proof see e.g., [11, esp., p. 268]).

(ii) Let $2 \leq q \leq q_1 < \infty$. Then if X is q -uniformly convex, X is q_1 -uniformly convex.

(iii) L_q ($2 \leq q < \infty$) is q -uniformly convex (by Clarkson's inequality of (q, q) -type).

(iv) L_p ($1 < p \leq 2$) is p' -uniformly convex ($p' \geq 2$) (by Clarkson's inequality of (p, p') -type). But in fact, L_p ($1 < p \leq 2$) is 2-uniformly convex by Hanner's inequality (H1).

For convenience of the reader we see (iii) and the latter statement of (iv) in the general Banach space setting.

Proof of (iii). Let $2 \leq q < \infty$. Assume that Clarkson's inequality of (q, q) -type holds in X :

$$(\|x + y\|^q + \|x - y\|^q)^{1/q} \leq 2^{1/q'} (\|x\|^q + \|y\|^q)^{1/q}.$$

Let $x, y \in S_X$ and $\|x - y\| = \varepsilon$. Then

$$\|x + y\|^q + \varepsilon^q \leq 2^{q/q'} 2 = 2^{q(1/q' + 1/q)} = 2^q,$$

whence

$$\left\| \frac{x + y}{2} \right\|^q + \left(\frac{\varepsilon}{2} \right)^q \leq 1.$$

Therefore

$$\left(\frac{\varepsilon}{2} \right)^q \leq 1 - \left\| \frac{x + y}{2} \right\|^q \leq q \left(1 - \left\| \frac{x + y}{2} \right\| \right).$$

Consequently we have

$$1 - \left\| \frac{x + y}{2} \right\| \geq \frac{1}{q} \left(\frac{\varepsilon}{2} \right)^q,$$

from which it follows that

$$\delta_X(\varepsilon) \geq \frac{1}{q2^q} \varepsilon^q,$$

or X is q -uniformly convex.

Proof of the latter assertion of (iv). Let $1 < p \leq 2$. We have to show the following: If Hanner's inequality (H1),

$$\|x + y\|^p + \|x - y\|^p \geq \left| \|x\| + \|y\| \right|^p + \left| \|x\| - \|y\| \right|^p,$$

holds in X , then X is 2-uniformly convex. Assume (H1). Then

$$\begin{aligned} \left(\frac{\|x+y\|^2 + \|x-y\|^2}{2} \right)^{1/2} &\geq \left(\frac{\|x+y\|^p + \|x-y\|^p}{2} \right)^{1/p} \\ &\geq \left(\frac{\|x\| + \|y\|^p + \left| \|x\| - \|y\| \right|^p}{2} \right)^{1/p} \\ &\geq \left(\frac{\|x\| + \gamma\|y\|^2 + \left| \|x\| - \gamma\|y\| \right|^2}{2} \right)^{1/2}, \end{aligned}$$

where $\gamma = \sqrt{(p-1)/(2-1)} = \sqrt{p-1}$ ([6, Corollary 1.e.15]). Therefore

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &\geq \left| \|x\| + \gamma\|y\| \right|^2 + \left| \|x\| - \gamma\|y\| \right|^2 \\ &= 2[\|x\|^2 + \gamma^2\|y\|^2]. \end{aligned}$$

Put here $x+y=u, x-y=v$. Then

$$\|u\|^2 + \|v\|^2 \geq 2 \left[\left\| \frac{u+v}{2} \right\|^2 + (p-1) \left\| \frac{u-v}{2} \right\|^2 \right].$$

Now let $u, v \in S_X$ and $\|u-v\| = \epsilon$. Then

$$2 \geq 2 \left[\left\| \frac{u+v}{2} \right\|^2 + (p-1) \left(\frac{\epsilon}{2} \right)^2 \right],$$

whence

$$(p-1) \left(\frac{\epsilon}{2} \right)^2 \leq 1 - \left\| \frac{u+v}{2} \right\|^2 \leq 2 \left(1 - \left\| \frac{u+v}{2} \right\| \right).$$

Therefore

$$\frac{p-1}{8} \epsilon^2 \leq 1 - \left\| \frac{u+v}{2} \right\|.$$

Consequently we have

$$\delta_X(\epsilon) \geq \frac{p-1}{8} \epsilon^2,$$

or X is 2-uniformly convex, as is desired.

4. Definition (i) The modulus of smoothness of X is defined by

$$\rho_X(\tau) := \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in S_X \right\} \quad \text{for } \tau > 0$$

- (ii) X is uniformly smooth if $\rho_X(\tau)/\tau \rightarrow 0$ as $\tau \rightarrow 0$.
- (iii) X is p -uniformly smooth ($1 < p \leq 2$) if there exists $K > 0$ such that $\rho_X(\tau) \leq K\tau^p$ for all $\tau > 0$.

5. Remark (i) No Banach space is p -uniformly smooth for $2 < p < \infty$.

(ii) Let $1 < p_1 \leq p \leq 2$. Then if X is p -uniformly smooth, X is p_1 -uniformly smooth.

(iii) L_p ($1 < p \leq 2$) is p -uniformly smooth.

(iv) L_q ($2 \leq q < \infty$) is 2-uniformly smooth.

The first kind of Hanner type inequalities

6. Theorem (Yamada-Takahashi-Kato [13]) Let $1 < p, s, t < \infty$. Then the following are equivalent.

(i) X is 2-uniformly convex.

(ii) There exists $\gamma > 0$ for which

$$\|x + y\|^p + \|x - y\|^p \geq \left| \|x\| + \|\gamma y\| \right|^p + \left| \|x\| - \|\gamma y\| \right|^p \quad (1)$$

holds in X .

(iii) There exists $\gamma > 0$ for which

$$\left(\frac{\|x + y\|^s + \|x - y\|^s}{2} \right)^{1/s} \geq \left(\frac{\left| \|x\| + \|\gamma y\| \right|^t + \left| \|x\| - \|\gamma y\| \right|^t}{2} \right)^{1/t} \quad (2)$$

holds in X .

According to Remark 3 (iv) the Hanner type inequalities (1) and (2) hold in L_r , $1 < r \leq 2$.

7. Theorem ([13]) Let $1 < p, s, t < \infty$. Then the following are equivalent.

(i) X is 2-uniformly smooth.

(ii) There exists $\gamma > 0$ for which

$$\|x + y\|^p + \|x - y\|^p \leq \left| \|x\| + \|\gamma y\| \right|^p + \left| \|x\| - \|\gamma y\| \right|^p \quad (3)$$

holds in X .

(iii) There exists $\gamma > 0$ for which

$$\left(\frac{\|x + y\|^s + \|x - y\|^s}{2} \right)^{1/s} \leq \left(\frac{\left| \|x\| + \|\gamma y\| \right|^t + \left| \|x\| - \|\gamma y\| \right|^t}{2} \right)^{1/t} \quad (4)$$

holds in X .

The above Hanner type inequalities (3) and (4) hold in L_r , $2 \leq r < \infty$.

The second kind of Hanner type inequalities

8. Theorem ([13]) Let $2 \leq q < \infty$, $1 \leq t \leq q$. Then the following are equivalent.

- (i) X is q -uniformly convex.
- (ii) There exists $\gamma > 0$ such that

$$\left(\|x + y\|^q + \|\gamma(x - y)\|^q \right)^{1/q} \leq \left(\left| \|x\| + \|y\| \right|^t + \left| \|x\| - \|y\| \right|^t \right)^{1/t} \quad (5)$$

for all $x, y \in X$.

The Hanner type inequality (5) holds in L_q ($2 \leq q < \infty$).

9. Theorem ([13]) Let $1 < p \leq 2$ and $p \leq s \leq \infty$. Then the following are equivalent.

- (i) X is p -uniformly smooth.
- (ii) There exists $\gamma > 0$ such that

$$\left(\|x + y\|^p + \|\gamma(x - y)\|^p \right)^{1/p} \geq \left(\left| \|x\| + \|y\| \right|^s + \left| \|x\| - \|y\| \right|^s \right)^{1/s} \quad (6)$$

for all $x, y \in X$.

The Hanner type inequality (6) holds in L_p ($1 < p \leq 2$).

Duality between Hanner type inequalities

According to Ball-Carlen-Lieb [1] Hanner's inequalities (H1) and (H2) are equivalent. This is extended as follows.

10. Theorem ([13]) Let $1 < s, t < \infty$, $1/s + 1/s' = 1/t + 1/t' = 1$ and $\alpha, \beta, \gamma > 0$. Then the following are equivalent.

- (i) For all $x, y \in X$

$$\left(\|\alpha(x + y)\|^s + \|\beta(x - y)\|^s \right)^{1/s} \geq \left(\left| \|x\| + \|\gamma y\| \right|^t + \left| \|x\| - \|\gamma y\| \right|^t \right)^{1/t} \quad (7)$$

(ii) For all $x^*, y^* \in X^*$

$$\left(\|\alpha^{-1}(x^* + y^*)\|^{s'} + \|\beta^{-1}(x^* - y^*)\|^{s'} \right)^{1/s'} \leq \left(\left| \|x^*\| + \|\gamma^{-1}y^*\| \right|^{t'} + \left| \|x^*\| - \|\gamma^{-1}y^*\| \right|^{t'} \right)^{1/t'} \quad (8)$$

11. Corollary Let $1 < s, t, p < \infty$, $1/s + 1/s' = 1/t + 1/t' = 1/p + 1/q = 1$ and $\gamma > 0$.

(i) The inequality

$$\left(\frac{\|x + y\|^s + \|x - y\|^s}{2} \right)^{1/s} \geq \left(\frac{\left| \|x\| + \|\gamma y\| \right|^t + \left| \|x\| - \|\gamma y\| \right|^t }{2} \right)^{1/t} \quad (2)$$

holds in X if and only if

$$\left(\frac{\|x^* + y^*\|^{s'} + \|x^* - y^*\|^{s'}}{2} \right)^{1/s'} \leq \left(\frac{\left| \|x^*\| + \|\gamma^{-1}y^*\| \right|^{t'} + \left| \|x^*\| - \|\gamma^{-1}y^*\| \right|^{t'} }{2} \right)^{1/t'} \quad (4^*)$$

holds in X^* .

(ii) The inequality

$$\|x + y\|^p + \|x - y\|^p \geq \left| \|x\| + \|\gamma y\| \right|^p + \left| \|x\| - \|\gamma y\| \right|^p \quad (1)$$

holds in X if and only if

$$\|x^* + y^*\|^q + \|x^* - y^*\|^q \leq \left| \|x^*\| + \|\gamma^{-1}y^*\| \right|^q + \left| \|x^*\| - \|\gamma^{-1}y^*\| \right|^q \quad (3^*)$$

holds in X^* .

The best value of the weights for L_p -spaces

12. Theorem ([13]) Let $1 < p \leq 2$ and $1 < s, t < \infty$. Then the Hanner type inequality (2) holds in L_p :

$$\left(\frac{\|x + y\|^s + \|x - y\|^s}{2} \right)^{1/s} \geq \left(\frac{\left| \|x\| + \|\gamma y\| \right|^t + \left| \|x\| - \|\gamma y\| \right|^t }{2} \right)^{1/t}$$

The best value of γ is

$$\gamma = \min \left\{ 1, \sqrt{\frac{p-1}{t-1}}, \sqrt{\frac{s-1}{t-1}} \right\}$$

13. Theorem ([13]) Let $2 \leq p < \infty$ and $1 < s, t < \infty$. Then the Hanner type inequality (4) holds in L_p :

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2} \right)^{1/s} \leq \left(\frac{\|x\| + \|\gamma y\|^t + \|x\| - \|\gamma y\|^t}{2} \right)^{1/t}$$

The best value of γ is

$$\gamma = \max \left\{ 1, \sqrt{\frac{p-1}{t-1}}, \sqrt{\frac{s-1}{t-1}} \right\}$$

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