

A Theory of Superstructures

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1 Axiom

We shall consider a first order theory of a language $\mathcal{L}_\in = \{\in\}$ on the classical logic with equality "=", where the symbol $\in$ is a membership relation.

We adopt the following abbreviations:

$$\begin{aligned} \text{Set}(x) &\equiv \exists y \ y \in x, \\ \forall x \in y \ \varphi(x) &\equiv \forall x \ [x \in y \Rightarrow \varphi(x)], \\ \exists x \in y \ \varphi(x) &\equiv \exists x \ [x \in y \wedge \varphi(x)], \\ x \subseteq y &\equiv \forall z \in x \ z \in y, \\ x \not\subseteq y &\equiv \neg x \subseteq y, \\ x \subsetneq y &\equiv x \subseteq y \wedge \exists u \in y \ u \notin x, \\ \text{Trans}(x) &\equiv \forall y \in x \ y \subseteq x, \\ \forall x \subseteq y \ \varphi(x) &\equiv \forall x \ [x \subseteq y \Rightarrow \varphi(x)], \\ \text{Wo}_\subseteq(x) &\equiv \forall y \subseteq x \ [\text{Set}(y) \Rightarrow \exists u \in y \ \forall z \in y \ u \subseteq z] \wedge \forall z \in x \ \text{Set}(z), \\ \text{Mater}(x, y) &\equiv \forall z \in x \ \exists u \in y \ z \in u, \\ \exists! x \ \varphi(x) &\equiv \exists x \ \varphi(x) \wedge \forall x_1 \ \forall x_2 \ [\varphi(x_1) \wedge \varphi(x_2) \Rightarrow x_1 = x_2], \\ \exists! x \in y \ \varphi(x) &\equiv \exists! x \ [x \in y \wedge \varphi(x)]. \end{aligned}$$

We call $\text{Mater}(x, y)$ that x is a set of materials of y . In **ZF** set theory, $\text{Mater}(x, y)$ means that x is a subset of union of y .

A formula φ of $\mathcal{L}_\in$ is *restricted* or *bounded* if all quantifiers in φ are of either form $\forall x \in y$ or $\exists x \in y$.

Here is an axiom system of a theory of Superstructures.

1. Extensionality of nonempty sets:

$$\forall x \ \forall y \ [\text{Set}(x) \wedge x \subseteq y \wedge y \subseteq x \Rightarrow x = y].$$

2. Pair:

$$\forall x \forall y \exists u [x \in u \wedge y \in u].$$

3. Transitive superset:

$$\forall x \exists u [x \subseteq u \wedge \text{Trans}(u)].$$

4. Power:

$$\forall x \exists u \forall y \subseteq x \ y \in u.$$

5. Infinity:

$$\exists u [\text{Set}(u) \wedge \text{Wo}_{\subseteq}(u) \wedge \forall y \in u \exists v \in u \ y \subsetneq v].$$

6. Strong foundation:

$$\forall x [\text{Set}(x) \wedge \forall y \in x \exists u \in x \ \text{Mater}(u, y) \Rightarrow \exists u \in x \neg \text{Set}(u)].$$

7. Choice:

$$\forall x [\forall y \in x \exists u \in y \exists! v \in x \ u \in v \Rightarrow \exists w \forall y \in x \exists! u \in y \ u \in w].$$

8. Restricted separation: If $\varphi(y, z)$ is a restricted formula, then

$$\forall p \forall x \exists u \forall y [y \in u \Leftrightarrow y \in x \wedge \varphi(y, p)].$$

9. $\in$ -induction schema:

$$\forall x [\forall y \in x \ \psi(y) \Rightarrow \psi(x)] \Rightarrow \forall x \ \psi(x).$$

We denote 1–9 by **SS** and 1–8 by **SS**₀.

2 Universe

In this section, we consider the universe of **SS**₀, and cumulative hierarchy of **SS**.

By Infinity and Restricted separation, there is an a such that $\neg \text{Set}(a)$, and by Power, there is b such that

$$\forall x \subseteq a \ x \in b, \text{ or } \forall x [\neg \text{Set}(x) \Rightarrow x \in b].$$

By Restricted separation and Extensionality, there is a unique $\neg$ such that

$$\forall x [x \in \neg \Leftrightarrow \neg \text{Set}(x)].$$

By Pair and Restricted separation, there is an unordered pair c for every a and b such that

$$\forall x [x \in c \Leftrightarrow [c = a \vee c = b]].$$

We denote such c by $\{a, b\}$ and $\{a, a\}$ by $\{a\}$. We define an ordered pair $\langle a, b \rangle$ by $\{\{a\}, \{a, b\}\}$.

Let $\varphi(x)$ be a restricted formula and suppose $\exists x \in a \varphi(x)$. Then, by Restricted separation and Extensionality of nonempty sets, there is a unique b such that

$$\forall x [x \in b \Leftrightarrow x \in a \wedge \varphi(x)].$$

We denote such b by $\{x \in a \mid \varphi(x)\}$.

By Power, there is a b for each a

$$\forall x \subseteq a \ x \in b.$$

We denote $\{x \in b \mid x \subseteq a\}$ by $\mathcal{P}_\neg(a)$. Note that $\neg \subseteq \mathcal{P}_\neg(a)$ for every a .

By Transitive superset, for every x , there is t such that $\text{Trans}(t) \wedge x \subseteq t$, define a transitive closure of x by:

$$\text{TC}(x) = \begin{cases} x & \text{if } x \in \neg \\ \{y \in t \mid \forall z \in \mathcal{P}_\neg(t) [\text{Trans}(z) \wedge x \subseteq z \Rightarrow y \in z]\} & \text{if } x \notin \neg \end{cases}.$$

When $a \not\subseteq \neg$, we denote the union $\{x \in \text{TC}(a) \mid \exists y \in a \ x \in y\}$ by $\bigcup a$. When $\{a, b\} \not\subseteq \neg$, we denote $\bigcup \{a, b\}$ by $a \cup b$.

As in **ZF**, we define maps, injections, surjections, bijections.

By Infinity, fixing α such that

$$\alpha \notin \neg \wedge \text{Wo}_\subseteq(\alpha) \wedge \forall y \in \alpha \exists v \in \alpha \ y \subsetneq v,$$

there is a unique $\subseteq$ -least element 0_α in α : $\forall x \in \alpha \ 0_\alpha \subseteq x$. For every $x \in a$, there is unique x' such that

$$\forall y \in \alpha [x' \subseteq y \Leftrightarrow x \subsetneq y].$$

We denote such x' by $x +_\alpha 1$. We can define a minimal unbounded well-ordered set N_α with order relation $\subseteq$ by

$$N_\alpha = \left\{ x \in \alpha \mid \forall y \in \alpha [0 \subsetneq y \wedge \forall z \in \alpha [z \subsetneq y \Rightarrow z +_\alpha 1 \subsetneq y] \Rightarrow x \subsetneq y] \right\}.$$

Then we have Restricted induction principle:

$$\varphi(0) \wedge \forall n \in \mathbb{N}_\alpha [\varphi(n) \Rightarrow \varphi(n +_\alpha 1)] \Rightarrow \forall n \in \mathbb{N}_\alpha \varphi(n),$$

where $\varphi(n)$ is restricted. Then we have that $\mathbb{N}_\alpha$ is unique up to isomorphism, so we denote a structure of natural numbers by $\langle \mathbb{N}, \leq, +1, 0 \rangle$.

Since $u \in y$ implies $\text{Mater}(y, u)$, we have, by Strong foundation, foundation principle:

$$\forall x [\text{Set}(x) \Rightarrow \exists y \in x \forall u \in x u \notin y].$$

We shall show dual foundation principle:

$$\forall x [\text{Set}(x) \Rightarrow \exists y \in x \forall u \in x y \notin u].$$

Suppose, on contrary, there is x such that $\text{Set}(x)$ and $\forall y \in x \exists u \in x y \in u$. Since $\text{Mater}(\text{TC}(x), \text{TC}(x))$, we have, by Strong foundation, there is $a \in \text{TC}(x)$ such that $a \in \{ \text{TC}(x) \}$, which is contradiction.

Let $\mathbb{N}$ be a structure of natural numbers. we define the predicate “ x has rank n ” by

$$\begin{aligned} \rho(n, x) &\equiv \bar{\rho}(n, \text{TC}(x) \cup \{x\}, x), \\ \bar{\rho}(n, t, x) &\equiv \exists f: t \rightarrow \mathbb{N} \left[\forall y \in t f(y) = \bigcup \{ k \in \mathbb{N} \mid k = 0 \vee \exists z \in y k = f(z) + 1 \} \right. \\ &\quad \left. \wedge n = f(x) \right]. \end{aligned}$$

Then every x has a unique rank.

In **SS**, applying the following $\psi(x)$ to $\in$ -induction schema, we have obtained cumulative hierarchy W_n . We cannot prove that there is W_n for every $n \in \mathbb{N}$.

$$\psi(x) \equiv \forall n \in \mathbb{N} [\rho(n, x) \Rightarrow \exists W_n \forall y [y \in W_n \Leftrightarrow \exists k \leq n \rho(k, y)]].$$

3 Models

We construct models for **SS** in **ZFC**. We say a model W is **ZF-standard** if the membership relation $\in$ of W is that of **ZFC**.

Given a set X , we define the iterated power set $V_n(X)$ over X recursively by

$$V_0(X) = X, \quad \text{and} \quad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The *superstructure* $V(X)$ is the union $\bigcup_{n < \omega} V_n(X)$. The set X is said to be a *base set* if $\emptyset \notin X$ and each element of X is disjoint from $V(X)$.

If X is a base set then $V(X)$ is a **ZF**-standard model for **SS**. In $V(X)$, we see $X \cup \{\emptyset\} = \mathbf{-}$ and $\mathcal{P}_{\mathbf{-}}(a) = \mathcal{P}(a) \cup \mathbf{-}$.

Let X and Y are infinite base sets, and let $j: V(X) \rightarrow V(Y)$ be a nontrivial bounded elementary embedding — $\langle V(X), V(Y), j \rangle$ is a nonstandard universe. Then the transitive closure W of $\text{ran } j$ within $V(Y)$ is a model for **SS**. In W , we see $j(\mathbb{N})$ is a structure of natural numbers if $\mathbb{N}$ is a structure of natural numbers in $V(X)$, and there is no W_ν for nonstandard $\nu \in j(\mathbb{N}) \setminus j''\mathbb{N}$.