

Topological immersions and PL immersions

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1. Introduction, definitions and notation

In this paper we compute the homotopy groups $\Pi_i(V_{q,m}^{\text{TOP}}, V_{q,m}^{\text{PL}})$ where $V_{q,m}^{\text{TOP}}$ (resp. $V_{q,m}^{\text{PL}}$) is the topological analogue (resp. the PL analogue) of the stiefel manifold, using the result of Kirby-Siebenmann [7]. And we consider the relation between topological immersions and PL immersions of PL manifolds.

Theorem 1.

- 1) If $q-m > 3$, $\Pi_k(V_{q,m}^{\text{TOP}}, V_{q,m}^{\text{PL}}) = 0$ for $0 \leq k \leq m$.
- 2) If $q-m = 1$ $q \geq 1$ or $q-m = 2$ $q \geq 6$,

$$\Pi_k(\text{TOP}_q, \text{PL}_q) \longrightarrow \Pi_k(V_{q,m}^{\text{TOP}}, V_{q,m}^{\text{PL}})$$

is isomorphic for $0 \leq k \leq m$, and

$$\Pi_{m+1}(\text{TOP}_q, \text{PL}_q) \longrightarrow \Pi_{m+1}(V_{q,m}^{\text{TOP}}, V_{q,m}^{\text{PL}})$$

is surjective.

Corollary 2.

If $q-m = 1$ $q \geq 5$ or $q-m = 2$ $q \geq 6$,

$$\Pi_k(V_{q,m}^{\text{TOP}}, V_{q,m}^{\text{PL}}) = \begin{cases} 0 & k \neq 3 \\ z_2 & k = 3 \end{cases}$$

for $k \leq m+1$.

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Theorem 3.

1) Let Q, M be PL manifolds with $\dim. q$ and m respectively, and $f : M \longrightarrow Q$ be a topological immersion. Then, if $q-m \geq 3$, there is a topological regular homotopy which takes f to a PL immersion. If $q-m = 1$ $q \geq 5$ or $q-m = 2$ $q \geq 6$, there is a topological regular homotopy which takes f to a PL immersion if and only if the obstruction $c(f) \in H^3(M, Z_2)$ vanishes.

2) Suppose that $f_0, f_1 : M \longrightarrow Q$ are PL immersions, there is a topological regular homotopy which takes f_0 to f_1 , and $q-m = 1$, $q \geq 5$ or $q-m = 2$ $q \geq 6$. Then there is a PL regular homotopy which takes f_0 to f_1 if the obstruction $d(f_0, f_1) \in H^2(M, Z_2)$ vanishes.

In this paper, immersions and embeddings are assumed to be locally flat.

Definition.

Topological (resp. PL) stiefel $V_{q,m}^{TOP}$ (resp. $V_{q,m}^{PL}$) is a Kan complex whose k -simplex f is a topological (resp. PL) embedding

$$f : \Delta^k \times R^m \longrightarrow \Delta^k \times R^q$$

such that $f|(\Delta^k \times 0) = \text{id}$, $\text{pr}_1 \circ f = \text{pr}_1$ and f is locally flat uniformly with respect to Δ^k (i.e. for any $x \in \Delta^k$, there is a neighbourhood $U \subset \Delta^k$ of x , and homeomorphisms $g_1 : U \times R^m \longrightarrow U \times R^m$, $g_2 : U \times R^q \longrightarrow U \times R^q$, where $\text{pr}_1 \circ g_\alpha = \text{pr}_1$ ($\alpha=1,2$), such that the diagram

$$\begin{array}{ccc} U \times R^m & \xrightarrow{f|_{U \times R^m}} & U \times R^q \\ g_1 \uparrow & & \uparrow g_2 \\ U \times R^m & \hookrightarrow & U \times R^q \end{array}$$

commutes.)

Definition.

Let M', Q be topological (resp. PL) manifolds, $N \subset M \subset M'$ be locally flat proper submanifolds, $Q : N \longrightarrow Q$ be a topological (resp. PL) immersion. $I_{M', \theta}^{\text{TOP}}(M, Q)$ (resp. $I_{M', \theta}^{\text{PL}}(M, Q)$) is a Kan complex of topological (resp. PL) M' -immersions of M in Q whose restrictions on N is θ . $R_{M', \theta}^{\text{TOP}}(M, Q)$ (resp. $R_{M', \theta}^{\text{PL}}(M, Q)$) is a complex of its representations. When $N = \emptyset$, we omit the subscript θ .

B^n denotes n -dim. ball. $\hat{B}^n$ and ∂B^n denote the interior of B^n and boundary of B^n respectively.

Definition.

$\tilde{I}_{B^k \times R^{m-k}}(B^k, R^q)$ is a complex of topological $B^k \times R^{m-k}$ -immersions of B^k in R^q which is PL on a neighbourhood of $\partial B^k \times R^{m-k}$ in $B^k \times R^{m-k}$. $\tilde{R}_{B^k \times R^{m-k}}(B^k \times R^q)$ is a complex of its representations.

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§2. Preliminary results

Lemma 5.

$$\begin{aligned} 1) \quad \Pi_0(\tilde{I}_{B^k \times R^{m-k}}(B^k, R^q)) &\cong \Pi_0(\tilde{R}_{B^k \times R^{m-k}}(B^k, R^q)) \\ &\cong \Pi_k(V_{q,m}^{TOP}, V_{q,m}^{PL}) \end{aligned}$$

for $0 \leq k \leq m$.

$$\begin{aligned} 2) \quad \Pi_i(\tilde{I}_{B^k \times R^{m-k}}(B^k, R^q), I_{B^k \times R^{m-k}}^{PL}(B^k, R^q)) \\ \cong \Pi_i(\tilde{R}_{B^k \times R^{m-k}}(B^k, R^q), R_{B^k \times R^{m-k}}^{PL}(B^k, R^q)) \\ \cong \Pi_{i+k}(V_{q,m}^{TOP}, V_{q,m}^{PL}) \end{aligned}$$

for $0 \leq k \leq m$, $1 \leq i$, $m < q$.

Proof.

1) is obtained from Kirby [7] and Kurata [10].

Proof of 2) is as follows. An isomorphism $\Pi_i(\tilde{R}_{B^k \times R^{m-k}}(B^k, R^q), R_{B^k \times R^{m-k}}^{PL}(B^k, R^q)) \longrightarrow \Pi_{i+k}(V_{q,m}^{TOP}, V_{q,m}^{PL})$ is obtained similarly with the proof of 1).

The differential

$$d : I_{B^k \times R^{m-k}}^{PL}(B^k, R^q) \longrightarrow R_{B^k \times R^{m-k}}^{PL}(B^k, R^q)$$

is homotopy equivalent, therefore it is sufficient to prove the homotopy equivalence of

$$d : \tilde{I}_{B^k \times R^{m-k}}(B^k, R^q) \longrightarrow \tilde{R}_{B^k \times R^{m-k}}(B^k, R^q).$$

Consider the homotopy equivalence between fibrations

$$\begin{array}{ccc}
 I_{2B^k \times R^{m-k}, i}^{TOP}(B^k, R^q) & \longrightarrow & R_{2B^k \times R^{m-k}, i}^{TOP}(B^k, R^q) \\
 \downarrow & & \downarrow \\
 I_{2B^k \times R^{m-k}}^{TOP}(B^k, R^q) & \longrightarrow & R_{2B^k \times R^{m-k}}^{TOP}(B^k, R^q) \\
 \downarrow i_* & & \downarrow i_* \\
 I_{2B^k \times R^{m-k}}^{TOP}(2B^k - \frac{1}{2}B^k, R^q) & \longrightarrow & R_{2B^k \times R^{m-k}}^{TOP}(2B^k - \frac{1}{2}B^k, R^q),
 \end{array}$$

where i_* is induced by the inclusion $i : (2B^k - \frac{1}{2}B^k) \times R^{m-k} \longrightarrow 2B^k \times R^{m-k} \subset R^q$. Note that $I_{2B^k \times R^{m-k}}^{PL}(2B^k - \frac{1}{2}B^k, R^q)$ and

$R_{2B^k \times R^{m-k}}^{PL}(2B^k - \frac{1}{2}B^k, R^q)$ are subcomplexes of $I_{2B^k \times R^{m-k}}^{TOP}(2B^k - \frac{1}{2}B^k, R^q)$

and $R_{2B^k \times R^{m-k}}^{TOP}(2B^k - \frac{1}{2}B^k, R^q)$ respectively. The restrictions

$$i_* : \tilde{I}_{2B^k \times R^{m-k}}(B^k, R^q) \longrightarrow I_{2B^k \times R^{m-k}}^{PL}(2B^k - \frac{1}{2}B^k, R^q), \text{ and}$$

$$i_* : \tilde{R}_{2B^k \times R^{m-k}}(B^k, R^q) \longrightarrow R_{2B^k \times R^{m-k}}^{PL}(2B^k - \frac{1}{2}B^k, R^q) \text{ are fibrations}$$

with fibres being $I_{2B^k \times R^{m-k}, i}^{TOP}(B^k, R^q)$ and $R_{2B^k \times R^{m-k}, i}^{TOP}(B^k, R^q)$

respectively, because

$$i_*^{-1}(I_{2B^k \times R^{m-k}}^{PL}(2B^k - \frac{1}{2}B^k, R^q)) = \tilde{I}_{2B^k \times R^{m-k}}(B^k, R^q) \text{ and}$$

$$i_*^{-1}(R_{2B^k \times R^{m-k}}^{PL}(2B^k - \frac{1}{2}B^k, R^q)) = \tilde{R}_{2B^k \times R^{m-k}}(B^k, R^q). \text{ By homotopy}$$

$$\text{equivalences of } d : I_{2B^k \times R^{m-k}, i}^{TOP}(B^k, R^q) \longrightarrow R_{2B^k \times R^{m-k}, i}^{TOP}(B^k, R^q)$$

$$\text{and } d : I_{2B^k \times R^{m-k}}^{PL}(2B^k - \frac{1}{2}B^k, R^q) \longrightarrow R_{2B^k \times R^{m-k}}^{PL}(2B^k - \frac{1}{2}B^k, R^q),$$

it follows that $d : \tilde{I}_{B^k \times R^{m-k}}(B^k, R^q) \longrightarrow \tilde{R}_{B^k \times R^{m-k}}(B^k, R^q)$ is a

homotopy equivalence.

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Lemma 6.

The natural inclusion $i : B^k \times R^{n-k} \hookrightarrow B^k \times R^{m-k}$ ($n < m$) induces the following commutative diagrams.

$$\begin{array}{ccc}
 \Pi_0(\tilde{I}_{2B^k \times R^{m-k}}(B^k, R^q)) & \xrightarrow{i^*} & \Pi_0(\tilde{I}_{2B^k \times R^{n-k}}(B^k, R^q)) \\
 \downarrow \cong & & \downarrow \cong \\
 \Pi_0(\tilde{R}_{2B^k \times R^{m-k}}(B^k, R^q)) & \xrightarrow{i^*} & \Pi_0(\tilde{R}_{2B^k \times R^{n-k}}(B^k, R^q)) \\
 \downarrow \cong & & \downarrow \cong \\
 \Pi_k(V_{q,m}^{TOP}, V_{q,m}^{PL}) & \xrightarrow{i^*} & \Pi_k(V_{q,n}^{TOP}, V_{q,n}^{PL}),
 \end{array}$$

$$\begin{array}{ccc}
 \Pi_1(\tilde{I}_{2B^k \times R^{m-k}}(B^k, R^q), I_{2B^k \times R^{m-k}}^{PL}(B^k, R^q)) & & \\
 \downarrow \cong & \searrow i^* & \\
 \Pi_1(\tilde{I}_{2B^k \times R^{n-k}}(B^k \times R^q), I_{2B^k \times R^{n-k}}^{PL}(B^k, R^q)) & & \\
 \downarrow \cong & & \downarrow \cong \\
 \Pi_1(\tilde{R}_{2B^k \times R^{m-k}}(B^k, R^q), R_{2B^k \times R^{m-k}}^{PL}(B^k, R^q)) & \xrightarrow{i^*} & \Pi_1(\tilde{R}_{2B^k \times R^{n-k}}(B^k \times R^q), R_{2B^k \times R^{n-k}}^{PL}(B^k, R^q)) \\
 \downarrow \cong & & \downarrow \cong \\
 \Pi_{1+k}(V_{q,m}^{TOP}, V_{q,m}^{PL}) & \xrightarrow{i^*} & \Pi_1(V_{q,n}^{TOP}, V_{q,n}^{PL})
 \end{array}$$

Lemma 7.

Suppose $q-m = 1$, or $q-m = 2$ $q \geq 6$. Then

$$1) \quad i^* : \Pi_0(\tilde{I}_{2B^k \times R^{q-k}}(B^k, R^q)) \longrightarrow \Pi_0(\tilde{I}_{2B^k \times R^{m-k}}(B^k, R^q))$$

is an isomorphism for $0 \leq k \leq m$.

$$2) \quad i^* : \Pi_1(\tilde{I}_{2B^k \times R^{q-k}}(B^k, R^q), I_{2B^k \times R^{q-k}}^{PL}(B^k, R^q)) \longrightarrow \Pi_1(\tilde{I}_{2B^k \times R^{m-k}}(B^k, R^q), I_{2B^k \times R^{m-k}}^{PL}(B^k, R^q))$$

is a surjection for $0 \leq k \leq m$.

Proof of 1). Suppose that $f : 2B^k \times 2B^{m-k} \longrightarrow R^q$ represent a vertex of $\tilde{I}_{2B^k \times R^{m-k}}(B^k, R^q)$. Let $(\tilde{f}, V^q, \psi)$ be an induced neighbourhood of $2B^k \times 2B^{m-k}$ by f . (cf. [10]).

$$\begin{array}{ccc} 2B^k \times 2B^{m-k} & \xrightarrow{\tilde{f}} & V^q \\ & \searrow f \quad \swarrow \psi & \\ & R^q & \end{array}$$

We may assume that V^q is a q -dim. PL manifold with PL structure induced by ψ , ψ is PL immersion and $\tilde{f}$ is an embedding.

We show that $\tilde{f}$ can be extended to an embedding $\tilde{F} : 2B^k \times 2B^{q-k} \longrightarrow V^q$ such that $\tilde{F}|((2B^k - \frac{1}{2}B^k) \times 2B^{q-k})$ is PL, moreover such extension is unique up to isotopy. An extension $\tilde{F}$ of $\tilde{f}$ can be constructed as follows.

In the case where $q-m = 1$. By existence of codimension 1 topological normal bundle, there is a topological embedding $g : 2B^k \times 2B^{m-k} \times R \longrightarrow V^q$ such that $\tilde{f} \cdot \text{pr}_1 = g$, where $\text{pr}_1 : 2B^k \times 2B^{m-k} \times R \longrightarrow 2B^k \times 2B^{m-k}$ is the projection on $2B^k \times 2B^{m-k}$. By the existence of codimension 1 PL normal bundle, there is a PL bundle $p : E \longrightarrow (2B^k - \frac{1}{2}B^k) \times 2B^{m-k}$ with fibre being R , and PL embedding $g' : E \longrightarrow V^q$ such that $(\tilde{f}|((2B^k - \frac{1}{2}B^k) \times 2B^{m-k})) \cdot p = g'$. Because codimension 1 topological normal bundle is unique up to isotopy (cf. Brown [2]), g can be chosen such that $p \cdot g'^{-1} \cdot g = \text{pr}_1$. Therefore $p : E \longrightarrow (2B^k - \frac{1}{2}B^k) \times 2B^{m-k}$ is topological trivial bundle. The homotopy equivalence between TOP_1 and PL_1 implies that $p : E \longrightarrow (2B^k - \frac{1}{2}B^k) \times 2B^{m-k}$

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is PL trivial bundle. Then we may assume that $E = (2B^k - \frac{1}{2}B^k) \times 2B^{m-k} \times R$. $p = pr_1$. By the argument of Brown [2], PL embedding $g' : (2B^k - \frac{1}{2}B^k) \times 2B^{m-k} \times R \longrightarrow V^q$ can be extended to a topological embedding $\tilde{F} : 2B^k \times 2B^{m-k} \times R \longrightarrow V^q$.

Let $\tilde{F}_\alpha : 2B^k \times 2B^{m-k} \times R \longrightarrow V^q$ ($\alpha=0,1$) be extensions of $\tilde{f}$ such that $\tilde{F}_\alpha|((2B^k - \frac{1}{2}B^k) \times 2B^{m-k} \times R)$ is PL. By the uniqueness up to isotopy of codimension 1 PL normal bundle and the isotopy extension theorem of topological manifolds (Edward-Kirby [3]), there is a topological isotopy $H_t : 2B^k \times 2B^{m-k} \times R \longrightarrow 2B^k \times 2B^{m-k} \times R$ ($0 \leq t \leq 1$) fixing $2B^k \times 2B^{m-k} \times 0$, such that $H_0 = id$.

$H_t|((2B^k - \frac{1}{2}B^k) \times 2B^{m-k} \times R)$ is PL, and $H_1 \cdot \tilde{F}_1^{-1} \cdot \tilde{F}_0|((2B^k - \frac{1}{2}B^k) \times 2B^{m-k} \times R)$ commutes with the projection on $(2B^k - \frac{1}{2}B^k) \times 2B^{m-k}$.

By the uniqueness up to isotopy of codimension 1 topological normal bundle, there is a topological isotopy H_t ($1 \leq t \leq 2$) fixing $((2B^k - \frac{1}{2}B^k) \times 2B^{m-k} \times R) \cup (2B^k \times 2B^{m-k} \times 0)$ such that $H_2 \cdot \tilde{F}_1^{-1} \cdot \tilde{F}_0$ commutes with the projection on $2B^k \times 2B^{m-k}$. Because

$\Pi_1(TOP_1, PL_1) = 0$ for $i \geq 0$, there is a topological isotopy H_t ($2 \leq t \leq 3$) such that $H_t|((2B^k - \frac{1}{2}B^k) \times 2B^{m-k} \times R)$ ($2 \leq t \leq 3$) is PL, $H_t \cdot \tilde{F}_1^{-1} \cdot \tilde{F}_0$ ($2 \leq t \leq 3$) commutes with the projection on $2B^k \times 2B^{m-k}$, $H_3 \cdot \tilde{F}_1^{-1} \cdot \tilde{F}_0$ is PL. The contractibility of $2B^k \times 2B^{m-k}$ implies that there is a PL isotopy H_t ($3 \leq t \leq 4$) which commutes with the projection on the $2B^k \times 2B^{m-k}$ such that $H_4 \cdot \tilde{F}_1^{-1} \cdot \tilde{F}_0 = id$. $\tilde{F}_1 \cdot H_t^{-1}$ ($0 \leq t \leq 4$) is the required isotopy from $\tilde{F}_1$ to $\tilde{F}_0$.

In the case where $q-m = 2$, $q \geq 6$. There is a topological extension $F : 2B^k \times 2B^{m-k} \times R^2 \longrightarrow V^q$ as f . By Kirby-Siebenmann [8], $F|((2B^k - \frac{1}{2}B^k) \times 2B^{m-k} \times R^2)$ is isotopic to a PL embedding.

Extending this isotopy by isotopy extension theorem ([3]), we obtain the required embedding $\tilde{F}$. Similar argument to the proof in the case where $q-m = 1$ implies the uniqueness of the extension of $\hat{f}$, by the existence and uniqueness of codimension 2 PL normal bundle ([12]) and the homotopy equivalence between TOP_2 and PL_2 (Akiba [1], Kneser [9]).

Proof of 2). Let $f : 2B^k \times 2B^{m-k} \times I \longrightarrow R^q \times I$ be an 1-simplex of $\tilde{I}_{2B^k \times R^{m-k}}(B^k, R^q)$, such that $f|_{(2B^k \times 2B^{m-k} \times \dot{I})}$ is a PL immersion.

There is a topological immersion $g : 2B^k \times 2B^{m-k} \times R^{q-m} \times I \longrightarrow R^q \times I$ such that g is an extension of f , g commutes with the projection on I and $g|_{((2B^k - \frac{1}{2}B^k) \times 2B^{m-k} \times R^{q-m} \times I)}$ is PL. Similarly to the proof of 1), there is regular homotopies $rel.$

$2B^k \times 2B^{m-k} \times \{\alpha\} \xrightarrow{g_t} 2B^k \times 2B^{m-k} \times R^{q-m} \times \{\alpha\} \longrightarrow R^q \times \{\alpha\} \quad (\alpha=0,1),$

keeping PL on $(2B^k - \frac{1}{2}B^k) \times 2B^{m-k} \times R^{q-m} \times \{\alpha\}$, which take

$g|_{(2B^k \times 2B^{m-k} \times R^{q-m} \times \{\alpha\})}$ to PL immersions. Extending g_t to a regular homotopy over $2B^k \times 2B^{m-k} \times R^{q-m} \times I$. We obtain an immersion $g' : 2B^k \times 2B^{m-k} \times R^{q-m} \times I \longrightarrow R^q \times I$, which satisfies the following. g' is an 1-simplix of $\tilde{I}_{2B^k \times R^{q-k}}(B^k, R^q)$, an extension

of $\tilde{f}$ and $g'|_{(2B^k \times 2B^{m-k} \times R^{q-m} \times \dot{I})}$ is PL immersion. This shows

that $i^* : \Pi_1(\tilde{I}_{2B^k \times R^{q-k}}(B^k, R^q), I^{PL}_{2B^k \times R^{q-k}}(B^k, R^q))$

$$\longrightarrow \Pi_1(\tilde{I}_{2B^k \times R^{m-k}}(B^k, R^q), I^{PL}_{2B^k \times R^{m-k}}(B^k, R^q))$$

is surjective. The proof is complete.

Lemma 8.

Suppose $q-m \geq 3$, $f : 2B^k \times 2B^{m-k} \longrightarrow R^q$ be a vertex of

$\tilde{I}_{2B^k \times 2B^{m-k}}(B^k, R^q)$. Then there is a regular homotopy g_t which takes f to a PL immersion with $g_t \in \tilde{I}_{2B^k \times 2B^{m-k}}(B^k, R^q)$.

Proof.

Let $(\hat{f}, V^q, \psi)$ be an induced neighbourhood by f of $2B^k \times 2B^{m-k}$, where ψ is PL immersion, V^q is PL manifold and $\hat{f}$ is embedding with $\hat{f}|((2B^k - \frac{1}{2}B^k) \times 2B^{m-k})$ being PL. By Kirby-Siebenmann [8] and Miller [11], there is an isotopy $\hat{f}_t$ which takes $\hat{f}$ to a PL embedding keeping PL on $(2B^k - \frac{1}{2}B^k) \times 2B^{m-k}$. $g_t = \psi \circ \hat{f}_t$ is a required regular homotopy.

§3. Proof of theorems.

Theorem 1 is obtained by lemma 5~8.

Corollary 2 follows immediately from Theorem 1 by Kirby-Siebenmann [7].

Proof of Theorem 3.

Let $f : M^m \longrightarrow Q^q$ be a topological immersion. $df : TM \longrightarrow TQ$ is its differential. There is a homotopy $f_t : M \longrightarrow Q$ which takes f to a PL map f_1 . There is a homotopy of locally flat bundle monomorphism $\psi_t : TM \longrightarrow TQ$ such that $\psi_0 = df$ and ψ_t covers f_t . When $q-m > 3$, ψ_1 can deform to a PL bundle map and when $q-m = 1$ $q \geq 5$ or $q-m = 2$ $q \geq 6$ the obstruction to deform ψ_1 to a PL bundle map lies in $H^3(M, Z_2)$ by Corollary 2. By topological and PL immersion theorems ([5], [10]), it follows from theorem 3.1). Proof of 2) is similar to those of 1).

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