

Uniform non- ℓ_1^n -ness of direct sums of Banach spaces

九州工業大学・工学部 加藤幹雄 (Mikio Kato)

Department of Mathematics, Kyushu Institute of Technology

e-mail: katom@tobata.isc.kyutech.ac.jp

新潟大学・理学部 斎藤吉助 (Kichi-Suke Saito)

Department of Mathematics, Faculty of Science, Niigata University

e-mail: saito@math.sc.niigata-u.ac.jp

千葉大学大学院・人文社会科学研究科 田村高幸 (Takayuki Tamura)

Graduate School of Humanities and Social Sciences, Chiba University

e-mail: tamura@le.chiba-u.ac.jp

Abstract. This is a résumé of some recent results on the uniform non- ℓ_1^n -ness of direct sums of Banach spaces. In particular we present those for the ℓ_1 - and ℓ_∞ -sums as well.

1. Introduction

Since it was introduced in [24], the ψ -direct sum of Banach spaces have attracted a good deal of attention ([5, 6, 7, 13, 14, 19, 20, 17, 16, etc.]; see also [22, 23]). The aim of this note is to present a sequence of recent results on the uniform non- ℓ_1^n -ness of direct sums of Banach spaces. Our starting point is Theorem 1 below concerning the uniform non-squareness by the authors ([14]). To treat the uniform non- ℓ_1^n -ness is much more complicated than expected. The results presented here is almost taken from the recent paper of the present authors [16].

Let Ψ be the family of all convex (continuous) functions ψ on $[0, 1]$ satisfying

$$\psi(0) = \psi(1) = 1 \text{ and } \max\{1-t, t\} \leq \psi(t) \leq 1 \quad (0 \leq t \leq 1). \quad (1)$$

For any $\psi \in \Psi$ define

$$\|(z, w)\|_\psi = \begin{cases} (|z| + |w|)\psi\left(\frac{|w|}{|z| + |w|}\right) & \text{if } (z, w) \neq (0, 0), \\ 0 & \text{if } (z, w) = (0, 0). \end{cases} \quad (2)$$

Then $\|\cdot\| = \|\cdot\|_\psi$ is an absolute normalized norm on $\mathbb{C}^2$ (that is, $\|(z, w)\| = \|(|z|, |w|)\|$ and $\|(1, 0)\| = \|(0, 1)\| = 1$) and satisfies

$$\psi(t) = \|(1-t, t)\| \quad (0 \leq t \leq 1). \quad (3)$$

Conversely for any absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^2$ define a convex function $\psi \in \Psi$ by (3). Then $\|\cdot\| = \|\cdot\|_\psi$.

The ℓ_p -norms $\|\cdot\|_p$ are such examples and for all absolute normalized norms $\|\cdot\|$ on $\mathbb{C}^2$ we have

$$\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1 \quad (4)$$

([2]). By (3) the convex functions corresponding to the ℓ_p -norms are given by

$$\psi_p(t) := \begin{cases} \{(1-t)^p + t^p\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{1-t, t\} & \text{if } p = \infty. \end{cases} \quad (5)$$

Let X and Y be Banach spaces and let $\psi \in \Psi$. The ψ -direct sum $X \oplus_\psi Y$ of X and Y is the direct sum $X \oplus Y$ equipped with the norm

$$\|(x, y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi, \quad (6)$$

where the $\|(\cdot, \cdot)\|_\psi$ term in the right hand side is the absolute normalized norm on $\mathbb{C}^2$ corresponding to the convex function ψ ([24, 13]; see [21] for several examples). This extends the notion of the ℓ_p -sum $X \oplus_p Y$.

A Banach space X is said to be *uniformly non- ℓ_1^n* (cf. [1, 18]) provided there exists ϵ ($0 < \epsilon < 1$) such that for any $x_1, \dots, x_n \in S_X$, the unit sphere of X , there exists an n -tuple of signs $\theta = (\theta_j)$ for which

$$\left\| \sum_{j=1}^n \theta_j x_j \right\| \leq n(1-\epsilon). \quad (7)$$

We may take $x_1, \dots, x_n$ from the unit ball B_X of X in the definition. In case of $n = 2$ X is called *uniformly non-square* ([12]; cf. [1, 18]).

As is well known ([3, 11]), if X is uniformly non- ℓ_1^n , then X is uniformly non- ℓ_1^{n+1} for every $n \in \mathbb{N}$.

2. Uniform non- ℓ_1^n -ness of $X \oplus_\psi Y$, $\psi \neq \psi_1, \psi_\infty$

The following result by the authors [14] is our starting point.

Theorem 1 (Kato-Saito-Tamura [14]). *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the following are equivalent.*

- (i) $X \oplus_\psi Y$ is uniformly non-square.
- (ii) X and Y are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$.

To treat the uniform non- ℓ_1^n -ness is much more complicated than expected. Indeed we need to prepare several lemmas, though we skip to mention them.

Theorem 2. *Let X and Y be Banach spaces and let $\psi \in \Psi, \psi \neq \psi_1, \psi_\infty$. Then the following are equivalent.*

- (i) $X \oplus_\psi Y$ is uniformly non- ℓ_1^n .
- (ii) X and Y are uniformly non- ℓ_1^n .

Theorem 2 does not answer the following question: Let X and Y be uniformly non- ℓ_1^n . Is it possible for $X \oplus_\psi Y$ to be uniformly non- ℓ_1^n with $\psi = \psi_1$ or $\psi = \psi_\infty$? The next theorem will give an answer.

Theorem 3. *Let X and Y be Banach spaces and let $\psi \in \Psi$. Assume that neither X nor Y is uniformly non- ℓ_1^{n-1} . Then the following are equivalent.*

- (i) $X \oplus_\psi Y$ is uniformly non- ℓ_1^n .
- (ii) X and Y are uniformly non- ℓ_1^n and $\psi \neq \psi_1, \psi_\infty$.

Theorem 3 includes Theorem 1 as the case $n = 2$.

Remark 1. In Theorem 3 we can not remove the condition that neither X nor Y is uniformly non- ℓ_1^{n-1} ([16, Section 6]).

3. The ℓ_1 - and ℓ_∞ -sums

Theorem 4. *Let X and Y be Banach spaces. The following are equivalent.*

- (i) $X \oplus_1 Y$ is uniformly non- ℓ_1^n .
- (ii) There exist positive integers n_1 and n_2 with $n_1 + n_2 = n - 1$ such that X is uniformly non- $\ell_1^{n_1+1}$ and Y is uniformly non- $\ell_1^{n_2+1}$.

According to Theorem 1 the uniform non-squareness of X and Y is not inherited to the ℓ_1 -sum $X \oplus_1 Y$, whereas we have the following result as the case $n = 3$ of Theorem 4.

Theorem 5. *Let X and Y be Banach spaces. Then the following are equivalent.*

- (i) $X \oplus_1 Y$ is uniformly non- ℓ_1^3 .
- (ii) X and Y are uniformly non-square.

For the ℓ_∞ -sum we obtain the following.

Theorem 6. *Let $X_1, \dots, X_m$ be uniformly non-square Banach spaces. Then $(X_1 \oplus \dots \oplus X_m)_\infty$ is uniformly non- ℓ_1^n if and only if $m < 2^{n-1}$.*

According to Theorem 5 the ℓ_1 -sum $X \oplus_1 Y$ is uniformly non- ℓ_1^3 if and only if X and Y are uniformly non-square. On the other hand for the ℓ_∞ -sum, by Theorem 6, if X and Y are uniformly non-square, then $X \oplus_\infty Y$ is uniformly non- ℓ_1^3 , whereas the converse is not true ([16, Remark 5.5]). Instead we obtain the following result which is interesting in contrast with the ℓ_1 -sum case.

Theorem 7. *Let X , Y and Z be Banach spaces. Then the following are equivalent.*

- (i) $(X \oplus Y \oplus Z)_\infty$ is uniformly non- ℓ_1^3 .
- (ii) X , Y and Z are uniformly non-square.

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