

Linear operators and c.o.n.s. in Hilbert spaces associated with chaotic dynamical systems

(カオス力学系に付随する線形作用素および完全正規直交系)

Shinzo Kawamura: Yamagata University
(山形大学理学部 河村 新蔵)

In this note, we will show a non-chaotic property in chaotic dynamical system (X, φ) , where X is a compact set and φ is a continuous map of X onto X . One of the most important property in the chaotic dynamical theory is to be sensitive dependence on initial conditions, which is a chaotic property in the sense that there exists $\delta > 0$ such that, for any x in X and neighbourhood $U(x)$ of x , there exists y in $U(x)$ and $n \geq 0$ such that

$$d(\varphi^n(x), \varphi^n(y)) > \delta.$$

On the other hand, some chaotic dynamical systems have the property of topological-mixing on a measure space (X, m) , that is,

$$\lim_{n \rightarrow \infty} \int_X f(\varphi^n(x))g(x)dm = \int_X f(x)g(x)dm$$

for any continuous function f on a metric space X and L^1 -function g on the measurable space (X, m) with $\int_X g(x)dm = 1$.

We study this non-chaotic property by representing chaotic dynamical systems (X, φ) on Hilbert spaces $\mathfrak{H}$.

1. Covariant representation of dynamical systems

Let $C(X)$ be the C^* -algebra of all continuous functions on X . Then a continuous map φ from X onto itself induces a $*$ -endomorphism α_φ of $C(X)$, which is defined by

$$\alpha_\varphi(f)(x) = f(\varphi(x)), \quad x \in X.$$

Let π be a covariant representation of $(C(X), \alpha_\varphi)$ on a Hilbert space $\mathfrak{H}$ in the following sense:

$$\pi(\alpha_\varphi(f)) = V_1\pi(f)V_1^* + V_2\pi(f)V_2^*$$

for all f in $C(X)$, where (V_1, V_2) is a couple of isometries on $\mathfrak{H}$ with the property

$$V_1V_1^* + V_2V_2^* = I.$$

In this case, (V_1, V_2) induces a $*$ -endomorphism of $\mathfrak{L}(\mathfrak{H})$ as follows:

$$\alpha_V(a) = V_1 a V_1^* + V_2 a V_2^*$$

for all a in $\mathfrak{L}(\mathfrak{H})$.

For some couples (V_1, V_2) , we can find a c.o.n.s. $\{e_n\}_{n=1}^\infty$ satisfying following condition:

$$V_1 e_n = e_{2n-1} \quad \text{and} \quad V_2 e_n = e_{2n} \quad \text{for all } n \geq 1,$$

which are called a c.o.n.s. of Walsh type with respect to (V_1, V_2) .

Related to these systems, we have already had a theorem which shows a non-chaotic property.

Theorem 1.1. [2:Theorem 2.2.3] *Let π be a covariant representation of $(C(X), \alpha_\varphi)$ implemented by (V_1, V_2) . If (V_1, V_2) has a c.o.n.s. $\{e_n\}_{n=1}^\infty$ of Walsh type, then we have*

$$\lim_{n \rightarrow \infty} (\alpha_V^n(a)\xi, \xi) = (ae_1, e_1)$$

for all a in A and ξ in $\mathfrak{H}$ with $\|\xi\| = 1$.

Here we give some examples. Let φ be a unimodal map of $[0, 1]$ onto itself in the following sense.

- (1) φ is a continuous map of $[0, 1]$ onto $[0, 1]$.
- (2) There exists a point c in $(0, 1)$ such that
 - (i) $\varphi(0) = \varphi(1) = 0$ and $\varphi(c) = 1$,
 - (ii) φ is strictly monotone increasing on $[0, c]$ and strictly monotone decreasing on $[c, 1]$,
 - (iii) φ and the two inverse maps β, γ of φ are absolutely continuous functions on $[0, 1]$, where $\beta([0, 1]) = [0, c]$ and $\gamma([0, 1]) = [c, 1]$.

Given a unimodal map φ , we define a couple (V_1, V_2) of isometries associated with φ as follows:

$$V_1 = V_1(\varphi) = M_{\sqrt{\varphi}} M_{\chi_{[0, c]}} T_\varphi \quad \text{and} \quad V_2 = V_2(\varphi) = -M_{\sqrt{-\varphi}} M_{\chi_{[c, 1]}} T_\varphi,$$

where M_f means the multiplication operator on $L^2[0, 1]$, χ_E the characteristic function of E and $(T_\varphi \xi)(x) = \xi(\varphi(x))$.

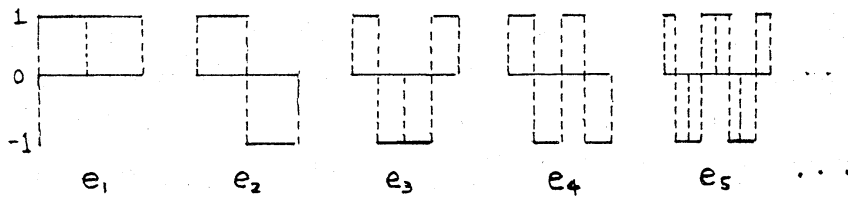
Let $\pi(f) = M_f$ for f in $C(X)$. Then π is a covariant representation of $(C(X), \alpha_\varphi)$ with respect to this couple (V_1, V_2) , but it has no c.o.n.s. of Walsh type. However, in

the following example, we can find a couple (W_1, W_2) of isometris which implements the $*$ -endomorphism α_V and has a c.o.n.s of Walsh type.

Example 1.2. Let $X = [0, 1]$ and $\varphi = 1 - |1 - 2x|$: the tent map.

$$W_1 = \frac{1}{\sqrt{2}}V_1 - \frac{1}{\sqrt{2}}V_2 (= T_\tau) \quad \text{and} \quad W_2 = W_2(\tau) = \frac{1}{\sqrt{2}}V_1 + \frac{1}{\sqrt{2}}V_2.$$

Then π is also a covariant representation of $(C([0, 1]), \alpha_\varphi)$ implemented by (W_1, W_2) having a c.o.n.s. $\{e_n\}_{n=1}^\infty$ of Walsh type, which is just the following Walsh series (cf.[3]).



Putting $a = M_f$ for f in $C(X)$ in the theorem above, we have

$$\lim_{n \rightarrow \infty} (\alpha_\varphi^n(f)\xi, \xi) = \lim_{n \rightarrow \infty} (\alpha_V^n(f)\xi, \xi) = (fe_1, e_1) = (f, e_1)$$

for all ξ in $\mathfrak{H}$ with $\|\xi\| = 1$.

Remark.1.3. The Walsh series becomes a group with respect to product of fuctoins on $[0, 1]$, which satisfies the following relation.

Group $G = \{e_n\}_{n=1}^\infty$

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	..
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	
e_2	e_2	e_1	e_4	e_3	e_6	e_5	e_8	e_7	
e_3	e_3	e_4	e_1	e_2	e_7	e_8	e_5	e_6	
e_4	e_4	e_3	e_2	e_1	e_8	e_7	e_6	e_5	
e_5	e_5	e_6	e_7	e_8	e_1	e_2	e_3	e_4	
e_6	e_6	e_5	e_8	e_7	e_2	e_1	e_3	e_4	
e_7	e_7	e_8	e_5	e_6	e_3	e_4	e_1	e_2	
e_8	e_8	e_7	e_6	e_5	e_4	e_3	e_2	e_1	
...									

Let φ be topologically conjugate to the tent map τ , that is, $\varphi = h \circ \tau \circ h^{-1}$ for some homeomorphism h of $[0,1]$ onto itself. In our case, the maps h and h^{-1} are assumed to be absolutely continuous functions on $[0,1]$.

Then $(C(X), \alpha_\varphi)$ has a covariant representation implemented by (W_1, W_2) defined as in Example 1.2. The couple (W_1, W_2) has a c.o.n.s. $\{e_n\}_{n=1}^\infty$ of Walsh type with $e_1 = \sqrt{(h^{-1})'}$, where $(h^{-1})'$ is the derivative of h^{-1} .

Example 1.4. Let $X = [0,1]$ and $\varphi = 4x(1-x)$: the logistic map. Then φ is topologically conjugate to the tent map with conjugacy $h(x) = \sin^2(\pi x/2)$. Thus the couple (W_1, W_2) has a c.o.n.s. $\{e_n\}_{n=1}^\infty$ of Walsh type with $e_1(x) = 1/(\pi(x(1-x))^{1/2})^{1/2}$.

2. Convergence of sequences $\{(\alpha_V^n(\cdot)\xi, \xi)\}_{n=1}^\infty$ in σ -weak topology

We consider the convergence in Theorem 1.1 in the context of duality between $\mathfrak{L}(\mathfrak{H})$ and the predual space $\mathfrak{L}(\mathfrak{H})_*$. Let M be an α_V -invariant von Neuman subalgebra of $\mathfrak{L}(\mathfrak{H})$ and $A = A_V^M$ the adjoint operator of the restriction of α_V to M . Then Theorem 1.1 implies the following. If (V_1, V_2) has a c.o.n.s. $\{e_n\}_{n=1}^\infty$ of Walsh type, then we have

$$\lim_{n \rightarrow \infty} A^n(\omega_{\xi, \xi}) = \omega_{e_1, e_1}$$

for all ξ in $\mathfrak{H}$ with $\|\xi\| = 1$, where $\omega_{\xi, \xi}$ is a vector state on $\mathfrak{L}(\mathfrak{H})$.

Let M be the abelian von Neumann subalgebra $M_{L^\infty[0,1]}$. Then the predual M_* is regarded as $L^1[0,1]$. Let (W_1, W_2) be as in Example 1.2 or 1.4. Then we have $A = A_V^M = A_W^M$ and the convergence mentioned above means the following.

If φ is the tent map on $[0,1]$, we have

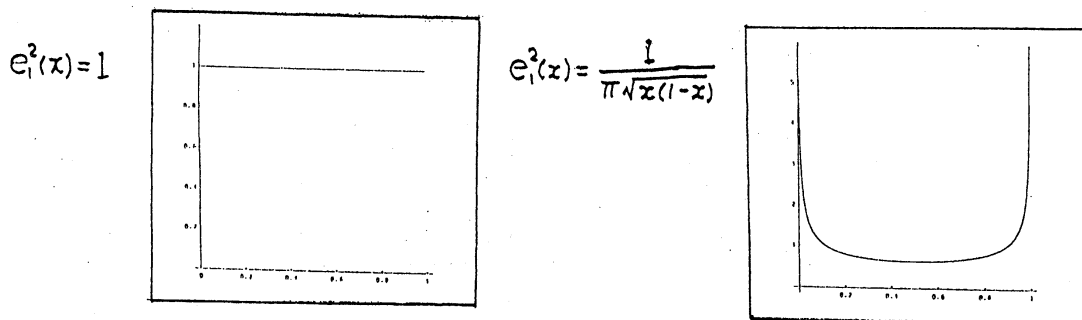
$$\lim_{n \rightarrow \infty} A^n(\eta) = e_1^2 = e_1$$

for all η in $L^1[0,1]$ with $\|\eta\|_1 = 1$, where $e_1(x) = 1$.

On the other hand, if φ is the logistic map on $[0,1]$, we have.

$$\lim_{n \rightarrow \infty} A^n(\eta) = e_1^2$$

for all η in $L^1[0,1]$ with $\|\eta\|_1 = 1$, where $e_1^2(x) = 1/\pi\sqrt{x(1-x)}$.



In the case of the tent map, we have more detailed results mentioned below.

Theorem 2.1 *Let φ be the tent map on $[0,1]$ and M the abelian von Neumann algebra $M_{L^\infty[0,1]}$. Put $A = A_V^M$. Then we have the following.*

- (1) $A(\eta)(x) = \frac{1}{2}(\eta(\frac{x}{2}) + \eta(1 - \frac{x}{2}))$ for η in $L^1[0,1]$.
- (2) $\lim_{n \rightarrow \infty} A^n(\eta) = 1$ in $\sigma(L^1[0,1], L^\infty[0,1])$ topology, for η in $L^1[0,1]$ with $\|\eta\|_1 = 1$.
- (3) $\lim_{n \rightarrow \infty} \|A^n(\eta) - 1\|_1 = 0$ for η in $L^2[0,1]$ with $\|\eta\|_1 = 1$.
- (4) $\lim_{n \rightarrow \infty} \|A^n(\eta) - 1\|_\infty = 0$ for η in $C_W[0,1]$, where $C_W[0,1]$ is the C^* -subalgebra of $L^\infty[0,1]$ generated by the Walsh series $\{e_n\}_{n=1}^\infty$.

We note that

$$C[0,1] \subset C_W[0,1] \subset L^\infty[0,1] \subset L^2[0,1] \subset L^1[0,1].$$

Remark 2.2. Let A be the map of $L^1[0,1]$ into $L^1[0,1]$ mentioned in the theorem above. Then we have the following.

- (1) $A(1) = 1$.
- (2) $A(2x) = 1$.
- (3) $A^n(3x^2) = \frac{3}{4^n} - \frac{3}{4^{n-1} \cdot 2} + \frac{4^{n-1} \cdot 2 + 1}{4^{n-1} \cdot 2}$ for each positive integer n .

For other α_V -invariant von Neuman subalgebras M , we have some results concerning the property of convergence of the sequence $\{(A_V^M)^n(\omega_{\xi,\xi})\}_{n=1}^\infty$. Moreover we are studying representations of chaotic dynamical systems in the case of other unimodal maps and Dai and Larson [1] provide an interesting theory which is deeply related to our study.

References

- [1] X. Dai and D. Larson, *Wandering vectors for unitary systems and orthogonal wavelets* to appear as Memoire of A.M.S.
- [2] S. Kawamura, Covariant representations chaotic dynamical systems, to appear in Tokyo J. Math.
- [3] F. Schipp, W.R. Wade and P. Simon, *Walsh series*, Adam Hilger, Bristol-New York, 1990.