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Threshold odd solutions to NLS on the line

By

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Abstract

This is a résumé of our previous paper [10] for the global dynamics of the threshold odd solutions to the nonlinear Schrödinger equation on the line.

§ 1. Introduction

We consider the following nonlinear Schrödinger equation on the real line:

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u + \partial_x^2 u + |u|^{p-1}u = 0, & (t,x) \in I \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $u = u(t,x)\colon I \times \mathbb{R} \to \mathbb{C}$, u_0 is a given function in $H^1(\mathbb{R})$, and $p > 5$. It is well known that (NLS) is locally well-posed in $H^1(\mathbb{R})$. Moreover, the energy E and the mass M, which are defined by

$$E(u) := \frac{1}{2}\|u'\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1} \text{ and } M(u) := \|u\|_{L^2}^2,$$

are conserved. Here, $u' := \partial_x u$. A blow-up alternative holds, that is, if the forward maximal existence time $T_{\max}$ is finite, then $\lim_{t\to T_{\max}-0} \|u'(t)\|_{L^2} = \infty$. See e.g. [4].

For more general NLS

$$(1.1) \qquad i\partial_t u + \Delta u + |u|^{p-1}u = 0, \quad (t,x) \in I \times \mathbb{R}^d$$

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with $u(0,x) = u_0(x) \in H^1(\mathbb{R}^d)$, where $d \in \mathbb{N}$ and $1 + 4/d < p < 2^* - 1$ ($2^* = \infty$ if $d = 1, 2$ and $2^* = 2d/(d-2)$ if $d \geq 3$), let us recall the results of the global dynamics. Let Q be the ground state, that is, the least-energy, radial, positive solution to the nonlinear elliptic equation:

$$-\Delta Q + Q - Q^p = 0 \text{ in } \mathbb{R}^d.$$

Below the ground state Q, we have the following scattering or blow-up (grow-up) dichotomy result. Let $s_c := \frac{d}{2} - \frac{2}{p-1}$.

Theorem 1.1 ([7, 1, 8, 5, 9])**.** *Assume that* $M(u_0)^{\frac{1-s_c}{s_c}} E(u_0) < M(Q)^{\frac{1-s_c}{s_c}} E(Q)$.

(1) If $\|u_0\|_{L^2}^{1-s_c} \|\nabla u_0\|_{L^2}^{s_c} < \|Q\|_{L^2}^{1-s_c} \|\nabla Q\|_{L^2}^{s_c}$, *then the solution* u *scatters in both time directions.*

(2) If $\|u_0\|_{L^2}^{1-s_c} \|\nabla u_0\|_{L^2}^{s_c} > \|Q\|_{L^2}^{1-s_c} \|\nabla Q\|_{L^2}^{s_c}$, *then the solution* u *blows up in finite time or grows up in both time directions.*

Here, we say that u scatters in positive time direction if there exists $\varphi_+ \in H^1(\mathbb{R}^d)$ such that $\|u(t) - e^{it\partial_x^2}\varphi_+\|_{H^1} \to 0$ as $t \to +\infty$ and u grows up in positive time if u exists on $[0,\infty)$ and $\limsup_{t\to\infty} \|\nabla u(t)\|_{L^2} = \infty$. In the case of (2), if we additionally assume that u_0 is radially symmetric when $d \geq 2$ and $p \leq 5$ or that $xu_0 \in L^2(\mathbb{R}^d)$, then the solution blows up in both time directions. We note that it is not known whether grow-up actually happens or not.

On the other hand, another global behavior appears in the threshold case. That is, there exist solutions of (1.1) converging to the ground state as follows.

Theorem 1.2 ([3])**.** *There exist two radial solutions* $U^{\pm}(t)$ *in* $H^1(\mathbb{R}^d)$ *such that*

- $M(U^{\pm}) = M(Q)$, $E(U^{\pm}) = E(Q)$, $U^{\pm}$ *exist at least on* $[0,\infty)$, *and there exist* $C, c > 0$ *such that* $\|U^{\pm}(t) - e^{it}Q\|_{H^1} \leq Ce^{-ct}$ *for* $t \geq 0$,
- $\|\nabla U^+(0)\|_{L^2} > \|\nabla Q\|_{L^2}$ *and* U^+ *blows up in finite negative time,*
- $\|\nabla U^-(0)\|_{L^2} < \|\nabla Q\|_{L^2}$ *and* U^- *exists globally and scatters backward in time.*

Moreover, we know the global dynamics on the threshold.

Theorem 1.3 ([3, 11])**.** *Assume that* $M(u_0)^{\frac{1-s_c}{s_c}} E(u_0) = M(Q)^{\frac{1-s_c}{s_c}} E(Q)$.

(1) If $\|u_0\|_{L^2}^{1-s_c} \|\nabla u_0\|_{L^2}^{s_c} < \|Q\|_{L^2}^{1-s_c} \|\nabla Q\|_{L^2}^{s_c}$, *then the solution* u *exists on* $\mathbb{R}$. *Moreover, either* u *scatters in both directions or* $u = U^-$ *up to symmetries of the equation.*

(2) If $\|u_0\|_{L^2}^{1-s_c} \|\nabla u_0\|_{L^2}^{s_c} = \|Q\|_{L^2}^{1-s_c} \|\nabla Q\|_{L^2}^{s_c}$, *then* $u = e^{it}Q$ *up to symmetries of the equation.*

(3) *If* $\|u_0\|_{L^2}^{1-s_c}\|\nabla u_0\|_{L^2}^{s_c} > \|Q\|_{L^2}^{1-s_c}\|\nabla Q\|_{L^2}^{s_c}$, *then either the solution* u *blows up or grows up, or* $u = U^+$ *up to symmetries of the equation.*

In the case (3), if we additionally assume that u_0 is radially symmetric when $d \geq 2$ and $p \leq 5$ or that $xu_0 \in L^2(\mathbb{R}^d)$, then the solution u either blows up in finite time in both time directions or $u = Q^+$ up to symmetries. The blow-up result with additional assumption is proved by [3] and the above blow-up or grow-up result is proved by [11]. Similar to the below case, it is an open problem whether grow-up actually happens or not.

Go back to our 1d equation (NLS), which is (1.1) with $d = 1$. This is, of course, included in the previous result when we consider the general solutions in $H^1(\mathbb{R})$. However, the restriction to odd solutions gives the other threshold $2^{1/s_c}E(Q)M(Q)^{(1-s_c)/s_c}$ since Q is even and thus is excluded. For odd solutions, we also know the scattering or blow-up (grow-up) dichotomy result below the threshold. Let $H^1_{\text{odd}}(\mathbb{R})$ be the set of all odd functions in $H^1(\mathbb{R})$.

Theorem 1.4 ([15]). *Assume that* $u_0 \in H^1_{\text{odd}}(\mathbb{R})$ *satisfies* $M(u_0)^{\frac{1-s_c}{s_c}}E(u_0) < 2^{\frac{1}{s_c}}M(Q)^{\frac{1-s_c}{s_c}}E(Q)$. *Then the following hold.*

(1) *If* $\|u_0\|_{L^2}^{1-s_c}\|u_0'\|_{L^2}^{s_c} < 2\|Q\|_{L^2}^{1-s_c}\|Q'\|_{L^2}^{s_c}$, *then the solution* u *scatters in both directions.*

(2) *If* $\|u_0\|_{L^2}^{1-s_c}\|u_0'\|_{L^2}^{s_c} > 2\|Q\|_{L^2}^{1-s_c}\|Q'\|_{L^2}^{s_c}$, *then the solution* u *blows up or grows up.*

Next, let us consider the threshold case. Unlike Theorem 1.3, we do not have the ground state under the odd restriction. Thus, we have the following scattering or blow-up dichotomy result by [10].

Theorem 1.5 ([10]). *Assume that* $u_0 \in H^1_{\text{odd}}(\mathbb{R})$ *satisfies* $M(u_0)^{\frac{1-s_c}{s_c}}E(u_0) = 2^{\frac{1}{s_c}}M(Q)^{\frac{1-s_c}{s_c}}E(Q)$. *Then the following hold.*

(1) *If* $\|u_0\|_{L^2}^{1-s_c}\|u_0'\|_{L^2}^{s_c} < 2\|Q\|_{L^2}^{1-s_c}\|Q'\|_{L^2}^{s_c}$, *then the solution* u *scatters in both directions.*

(2) *We additionally assume that* $xu_0 \in L^2(\mathbb{R})$. *If* $\|u_0\|_{L^2}^{1-s_c}\|\nabla u_0\|_{L^2}^{s_c} > 2\|Q\|_{L^2}^{1-s_c}\|\nabla Q\|_{L^2}^{s_c}$, *then the solution* u *blows up.*

In the present paper, we give a résumé of the proof of Theorem 1.5 (1). See [10] for the precise proofs. It is enough to show the following proposition in order to Theorem 1.5 (1).

Proposition 1.6. *Let* $u_0 \in H^1_{\text{odd}}(\mathbb{R})$ *satisfy* $M(u_0) = 2M(Q)$ *and* $E(u_0) = 2E(Q)$. *If* $K(u_0) > 0$, *then the solution scatters in both time directions.*

The proof is based on Duyckaerts, Landoulsi, and Roudenko [6]. They proved the scattering result for the threshold solutions to 3d cubic NLS outside an obstacle. We use contradiction argument to show Theorem 1.5 (1). If we suppose that scattering result does not hold, combining a concentration compactness argument and a modulation argument, we construct a critical element u, which is a non-scattering global solution on the threshold. The critical element u has the following compactness property: There exists a translation parameter $X(t)$ satisfying for any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that

$$\int_{\{|x-X(t)|>R\}\cap\{|x+X(t)|>R\}} |u'(t,x)|^2 + |u(t,x)|^2 dx < \varepsilon$$

for all $t \in [0, \infty)$. By using this compactness property, modulation argument, and repulsion coming from oddness, we get a contradiction as follows. The modulation argument implies that $\frac{d}{dt}X(t) \to 0$ and thus $X(t)$ should converge to some constant. Such a convergence is used in [3] to get Theorem 1.3 above. On the other hand, the oddness gives the key estimate, which is considered as repulsion coming from oddness (see Remark below Lemma 2.10),

$$S(Q(\cdot - y) - Q(- \cdot -y)) - 2S(Q) \approx e^{-2y},$$

where $S = E + M/2$, for sufficiently large $y > 0$. This estimate implies that $e^{-cX(t)} \to 0$ and thus $X(t) \to \infty$. These convergence and divergence give contradiction.

We remark that we can conjecture the following by comparing Theorem 1.3 (3).

Conjecture 1.7. *Let* $u_0 \in H^1_{\text{odd}}(\mathbb{R})$ *satisfy*

$$M(u_0)^{\frac{1-s_c}{s_c}} E(u_0) = 2^{\frac{1}{s_c}} M(Q)^{\frac{1-s_c}{s_c}} E(Q)$$

and

$$\|u_0\|_{L^2}^{1-s_c} \|u_0'\|_{L^2}^{s_c} > 2\|Q\|_{L^2}^{1-s_c} \|Q'\|_{L^2}^{s_c}.$$

Then the solution blows up or grows up in both time directions.

The conjecture is natural. However, we have not yet known whether the above conjecture is valid. Moreover, even if the conjecture is true, it is difficult to show the existence or non-existence of grow-up solutions. We will give similar conjectures for NLS with repulsive delta potential in the appendix, Section A.

Remark. In [15], dichotomy result is shown for more general group invariant solutions to (1.1) below the corresponding threshold. The global dynamics of such solutions on the threshold is also an interesting problem.

Notation. We set $\|v\|_{L_t^p X(I)} := \|\|v\|_{X(\mathbb{R})}\|_{L_t^p(I)}$ for a space-time function $v(t,x)$, a spatial Banach space X, and a time interval I. For $y > 0$, we set $\mathcal{R}_y f(x) := f(x-y) - f(-x-y)$. We define $\mathcal{R}_0 :=$ id. We define $\mathcal{T}_y f(x) := f(x-y)$ for $y \geq 0$. Let χ_R be a smooth and even cut-off function satisfying $\chi_R(x) = 1$ on $|x| > R$ and $\chi_R(x) = 0$ on $|x| < R/2$. We set $\chi_R^{\pm} = \mathbb{1}_{\mathbb{R}_\pm}\chi_R$ and $\mathcal{G}_{R,y} f = \chi_R^+ f(x-y) - \chi_R^- f(-x-y)$. $A \approx B$ means that there exist $c, C > 0$ such that $cB \leq A \leq CB$.

§ 2. Idea of proof of Theorem 1.5 (1)

§ 2.1. Variational structure and localized virial identity

In this section, we recall the variational results for odd functions and localized virial identity. See e.g. [10] for the proofs. Let K be the virial functional defined by

$$K(f) := \|f'\|_{L^2}^2 - \frac{p-1}{2(p+1)}\|f\|_{L^{p+1}}^{p+1}.$$

We also set

$$\mu(f) := 2\|Q'\|_{L^2}^2 - \|f'\|_{L^2}^2$$

and let $\mu(t) := \mu(u(t))$ for the solution u.

2.1.1. Variational structure

Lemma 2.1. *Let $f \in H^1_{\mathrm{odd}}(\mathbb{R})$. Assume $M(f) = 2M(Q)$ and $E(f) = 2E(Q)$. Then the following are equivalent:*

(1) $K(f) > 0$.

(2) $\|f\|_{L^2}^{1-s_c}\|f'\|_{L^2}^{s_c} < 2\|Q\|_{L^2}^{1-s_c}\|Q'\|_{L^2}^{s_c}$.

(3) $\mu(f) > 0$.

In fact, we can show that $K(f) = \frac{p-5}{4}\mu(f)$ if $E(f) = 2E(Q)$.

Proposition 2.2. *Under the assumption of Proposition 1.6, the solution u exists globally and $K(u(t)) > 0$ (equivalently $\mu(t) > 0$) holds for all $t \in \mathbb{R}$.*

2.1.2. Localized virial identity

For a solution $u(t)$, we define

$$J(t) = J(u(t)) = J_\infty(u(t)) := \int_{\mathbb{R}} |x|^2 |u(t,x)|^2 dx.$$

Then we have

$$\dot{J}(t) = \frac{d}{dt}J(u(t)) = 2\,\mathrm{Im}\int_{\mathbb{R}} x\overline{u(t,x)}u'(t,x)dx,$$

$$\ddot{J}(t) = \frac{d^2}{dt^2}J(u(t)) = 8K(u(t)).$$

Let φ be an even function in $C_0^\infty(\mathbb{R})$ satisfying

$$\varphi(x) := \begin{cases} x^2, & (|x| < 1), \\ 0, & (|x| > 2). \end{cases}$$

For a solution $u(t)$, we set

$$J_R(t) = J_R(u(t)) := \int_{\mathbb{R}} R^2 \varphi\left(\frac{x}{R}\right) |u(t,x)|^2 dx.$$

Then, we have

$$\dot{J}_R(t) = \frac{d}{dt}J_R(u(t)) = 2\,\mathrm{Im} \int_{\mathbb{R}} R\varphi'\left(\frac{x}{R}\right) \overline{u(t,x)} u'(t,x) dx$$

and

$$\begin{aligned}\ddot{J}_R(t) = \frac{d^2}{dt^2}J_R(u(t)) &= 4\int_{\mathbb{R}} \varphi''\left(\frac{x}{R}\right) \left\{ |u'(t,x)|^2 - \frac{p-1}{2(p+1)}|u(t,x)|^{p+1} \right\} dx \\ &\quad - \int_{\mathbb{R}} \frac{1}{R^2} \varphi''''\left(\frac{x}{R}\right) |u(t,x)|^2 dx \\ &= 8K(u(t)) + A_R(u(t)),\end{aligned}$$

where we set

$$\begin{aligned}A_R(u(t)) := &-4\int_{|x|>R} \left\{ 2 - \varphi''\left(\frac{x}{R}\right) \right\} \left\{ |\partial_x u(t,x)|^2 - \frac{p-1}{2(p+1)}|u(t,x)|^{p+1} \right\} dx \\ &- \int_{R<|x|<2R} \frac{1}{R^2} \varphi''''\left(\frac{x}{R}\right) |u(t,x)|^2 dx.\end{aligned}$$

§ 2.2. Strichartz estimates and long time perturbation

Definition 2.3 (admissible pair)**.** We say that $(\widetilde{q}, \widetilde{r})$ is an admissible pair if $(\widetilde{q}, \widetilde{r})$ satisfies $2 \leq \widetilde{r} \leq \infty$ and $2/\widetilde{q} = 1/2 - 1/\widetilde{r}$.

We set

$$q := \frac{4(p+1)}{p-1},\ r := p+1,\ a := \frac{2(p^2-1)}{p+3},\ \text{and } b := \frac{2(p^2-1)}{(p-1)^2 - (p-1) - 4}.$$

Here (q, r) is an admissible pair and (a, r) is a non-admissible pair.

Lemma 2.4 (Strichartz estimates)**.** *We have the following estimates. Let $I \ni 0$ be a time interval.*

$$\|e^{it\partial_x^2} f\|_{L_t^q L_x^r(I)} + \|e^{it\partial_x^2} f\|_{L_t^a L_x^r(I)} \lesssim \|f\|_{H^1},$$

$$\left\| \int_I e^{i(t-s)\partial_x^2} F ds \right\|_{L_t^q L_x^r(I)} \lesssim \|F\|_{L_t^{q'} L_x^{r'}(I)},$$
$$\left\| \int_I e^{i(t-s)\partial_x^2} F ds \right\|_{L_t^a L_x^r(I)} \lesssim \|F\|_{L_t^{b'} L_x^{r'}(I)}.$$

We have the long time perturbation lemma. See [7, Proposition 4.7] for the proof.

Lemma 2.5 (Long time perturbation). *Let I be a time interval with $0 \in I$. Given $A \geq 0$, there exist $\varepsilon(A) > 0$ and $C(A) > 0$ with the following property. If $u \in C(I : H^1(\mathbb{R}))$ is a solution of* (NLS), *if $v \in C(I : H^1(\mathbb{R}))$ and $e \in L^1_{\mathrm{loc}}(I : H^{-1}(\mathbb{R}))$ satisfy $i\partial_t v + \partial_x^2 v + |v|^{p-1}v = e$, for a.e. $t \in I$, and if*

$$\|v\|_{L^a(I:L^r)} \leq A,$$
$$\|e\|_{L^{b'}(I:L^{r'})} \leq \varepsilon \leq \varepsilon(A),$$
$$\|e^{it\partial_x^2}(u(0) - v(0))\|_{L^a(I:L^r)} \leq \varepsilon \leq \varepsilon(A),$$

then $u \in L^a(I : L^r(\mathbb{R}))$ and $\|u - v\|_{L_t^a L_x^r(I)} \leq C\varepsilon$.

We note that if u exists globally forward in time and satisfies $\|u\|_{L^a([0,\infty):L^r)} < \infty$ then u scatters in positive time direction (see [7, Proposition 4.2]).

§ 2.3. Profile decomposition

We have the following linear profile decomposition. See [10, Proposition 22].

Lemma 2.6 (Linear profile decomposition). *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^1_{\mathrm{odd}}(\mathbb{R})$. Then there exist a subsequence of $\{\varphi_n\}_n$ (which is also denoted by $\{\varphi_n\}_n$), a sequence $\{\widetilde{\psi}^j\}_j \subset H^1(\mathbb{R})$, a sequence $\{\widetilde{W}_n^j\}_{n,j} \subset H^1_{\mathrm{odd}}(\mathbb{R})$, a time sequence $\{t_n^j\}_{n,j}$, and a spatial sequence $\{x_n^j\}_{n,j} \subset \mathbb{R}_{\geq 0}$ such that*

$$\varphi_n = \sum_{j=1}^{J} \mathcal{R}_{x_n^j} e^{-it_n^j \partial_x^2} \widetilde{\psi}^j + \widetilde{W}_n^J \text{ for all } J \in \mathbb{N}$$

satisfying the following.

(1) For any fixed j, we have:

$$\text{either } t_n^j = 0 \text{ for any } n \in \mathbb{N}, \text{ or } t_n^j \to \pm\infty \text{ as } n \to \infty,$$
$$\text{either } x_n^j = 0 \text{ for any } n \in \mathbb{N}, \text{ or } x_n^j \to \infty \text{ as } n \to \infty.$$

If $x_n^j = 0$, the profile $\widetilde{\psi}^j$ is an odd function.

(2) Orthogonality of the parameters:

$$|t_n^j - t_n^k| + |x_n^j - x_n^k| \to \infty \text{ as } n \to \infty, \quad \forall j \neq k.$$

(3) Smallness of the remainder:

$$\forall \varepsilon > 0, \exists J = J(\varepsilon) \in \mathbb{N} \text{ such that } \limsup_{n\to\infty} \|e^{it\partial_x^2}\widetilde{W}_n^J\|_{L^\infty_{t,x}} < \varepsilon.$$

(4) Orthogonality in norms: for any $J \in \mathbb{N}$

$$\|\varphi_n\|_{L^2}^2 = \sum_{j=1}^J \|\mathcal{R}_{x_n^j}\widetilde{\psi}^j\|_{L^2}^2 + \|\widetilde{W}_n^J\|_{L^2}^2 + o_n(1),$$
$$\|\partial_x\varphi_n\|_{L^2}^2 = \sum_{j=1}^J \|\partial_x\mathcal{R}_{x_n^j}\widetilde{\psi}^j\|_{L^2}^2 + \|\partial_x\widetilde{W}_n^J\|_{L^2}^2 + o_n(1).$$

Moreover, we have

$$\|\varphi_n\|_{L^q}^q = \sum_{j=1}^J \|\mathcal{R}_{x_n^j}e^{-it_n^j\partial_x^2}\widetilde{\psi}^j\|_{L^q}^q + \|\widetilde{W}_n^J\|_{L^q}^q + o_n(1), \quad q \in (2,\infty), \quad \forall J \in \mathbb{N}.$$

§ 2.4. Modulation

We have the following modulation lemma. See [10, Lemma 26].

Lemma 2.7 (Modulation)**.** *Let $R > 0$ be sufficiently large. There exist $\mu_0 > 0$ and a function $\varepsilon : (0,\mu_0) \to (0,\infty)$ with $\varepsilon(\mu) \to 0$ as $\mu \to 0$ such that the following holds. For any $0 < \mu < \mu_0$ and for all $f \in H^1_{\mathrm{odd}}(\mathbb{R})$ satisfying $E(f) = 2E(Q)$, $M(f) = 2M(Q)$ and $0 < \mu(f) < \mu$, there exist $(\tilde{\theta}, y) \in \mathbb{R} \times (R,\infty)$ such that*

$$\|e^{-i\tilde{\theta}}f - \mathcal{R}_yQ\|_{H^1} < \varepsilon(\mu)$$

and, letting $g = e^{-i\tilde{\theta}}f - \mathcal{R}_yQ$,

$$\mathrm{Im}\int_{\mathbb{R}} g\chi_R^+\mathcal{T}_yQdx = 0, \quad \mathrm{Re}\int_{\mathbb{R}} g(\chi_R^+\mathcal{T}_yQ)'dx = 0. \tag{2.1}$$

The parameter $(\tilde{\theta}, y)$ is unique in $\mathbb{R}/2\pi\mathbb{Z} \times (R,\infty)$ and the mapping $f \mapsto (\tilde{\theta}, y)$ is C^1.

Let u be an odd solution satisfying $M(u(t)) = 2M(Q)$ and $E(u(t)) = 2E(Q)$. We set $I_{\mu_0} := \{t \in I_{\max} \ ; \ \mu(t) < \mu_0\}$, where $I_{\max}$ denotes the maximal existence time interval of the solution. By Lemma 2.7, we have C^1 functions $\tilde{\theta} = \tilde{\theta}(t) := \tilde{\theta}(u(t))$ and $y = y(t) := y(u(t))$ for $t \in I_{\mu_0}$. We set $\theta := \tilde{\theta} - 1$. We also have orthogonality conditions (2.1). We set

$$\begin{aligned} u(t,x) &= e^{i\theta(t)+it}(\mathcal{R}_{y(t)}Q(x) + g(t,x)) \\ &= e^{i\theta(t)+it}(\mathcal{R}_{y(t)}Q(x) + \rho(t)\mathcal{G}_{R,y(t)}Q(x) + h(t,x)), \end{aligned} \tag{2.2}$$

where

$$\rho(t) := \frac{\mathrm{Re} \int g\chi_R^+ (\mathcal{T}_{y(t)}Q)^p dx}{\int (\chi_R^+)^2 (\mathcal{T}_{y(t)}Q)^{p+1} dx}. \tag{2.3}$$

Then it follows from (2.1), (2.2) and (2.3) that

$$\mathrm{Im} \int_{\mathbb{R}} h\chi_R^+ \mathcal{T}_{y(t)} Q dx = \mathrm{Re} \int_{\mathbb{R}} h(\chi_R^+ \mathcal{T}_{y(t)}Q)' dx = \mathrm{Re} \int_{\mathbb{R}} h\chi_R^+ (\mathcal{T}_{y(t)}Q)^p dx = 0. \tag{2.4}$$

2.4.1. Estimate of the parameters We will give estimates of the parameters.

Lemma 2.8. *We have the estimates*

$$|\rho| \lesssim \|g\|_{L^2}, \quad \|g\|_{H^1} \lesssim |\rho| + \|h\|_{H^1}, \quad \|h\|_{H^1} \lesssim |\rho| + \|g\|_{H^1} \lesssim \|g\|_{H^1}.$$

The above lemma is easily showed by the definitions of ρ and h.

Lemma 2.9. *We have* $|\rho| \approx |\mu(t)| + O(e^{-2y} + \|h\|_{H^1}^2)$.

Proof. By the definition of μ and decomposition of $u = e^{i\theta + it}(\mathcal{R}_y Q + \rho \mathcal{G}_{R,y} Q + h)$, we get the estimate, where we also use the fact that Q is the solution of the elliptic equation. □

The following lemma plays a key role.

Lemma 2.10. *For a sufficiently large number* $y > 0$, *it holds that*

$$S(\mathcal{R}_y Q) - 2S(Q) \approx e^{-2y}.$$

In the one dimensional case, the function Q is explicitly given by

$$Q(x) = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \left\{\cosh\left(\frac{p-1}{2}x\right)\right\}^{-\frac{2}{p-1}}.$$

This means that Q decays like $e^{-|x|}$. This decay gives the above exponential estimate.

Remark. Oddness of $\mathcal{R}_y Q = Q(\cdot - y) - Q(- \cdot - y)$ is used to show this key lemma. If we consider the even case, we have

$$S(Q(\cdot - y) + Q(- \cdot - y)) - 2S(Q) \approx -e^{-2y}.$$

This does not give the following lemma. Intuitively, these difference means that oddness, which we may consider as Dirichlet zero boundary condition at the origin, gives repulsion and, on the other hand, evenness gives attraction. The attraction by evenness causes an even logarithmic two-soliton with the same action as $2S(Q)$ (see [17]). Such a solution does not appear in the odd case as shown in Theorem 1.5.

By using the above key estimate and coercivity of the corresponding linearized operator, we get the following.

Lemma 2.11. *We have* $e^{-2y} + \|h\|_{H^1}^2 \lesssim \mu(t)^2$.

Rough sketch of the proof. Since u is decomposed as the main term $\mathcal{R}_y Q$ and the residue term g, we add $S(\mathcal{R}_y Q)$ with the assumption $S(u) = 2S(Q)$. We get

$$0 = S(u) - 2S(Q) = S(u) - S(\mathcal{R}_y Q) + S(\mathcal{R}_y Q) - 2S(Q).$$

We estimate the first term in the right hand side from below using the coercivity of the linearized operator. (Here, $\mathcal{R}_y Q$ is not the precise solution and thus an error term arises. However, since the error is smaller than e^{-2y}, it can be absorbed by the second term.) The second term is estimated from below by e^{-2y} as shown in Lemma 2.10. These implies the desired estimate. $\square$

Combining the above estimates, we get the following.

Corollary 2.12. *We have* $|\rho| + \|h\|_{H^1} + \|g\|_{H^1} + e^{-2y} \lesssim |\mu(t)|$.

2.4.2. Estimate of the derivatives of the parameters

We have the following estimate of the derivatives of the parameters.

Lemma 2.13. *We have* $|\dot{\theta}(t)| + |\dot{\rho}(t)| + |\dot{y}(t)| \lesssim |\mu(t)|$.

Proof. The previous calculation in [10] is a little bit incorrect. So, we give a modification of the proof. By a direct calculation, h satisifies

$$\begin{aligned}
i\dot{h} + \partial_x^2 h = {} & \dot{\theta}(t)(\mathcal{R}_{y(t)}Q + g) + h \\
& - |\mathcal{R}_{y(t)}Q + g|^{p-1}(\mathcal{R}_{y(t)}Q + g) + |\mathcal{R}_{y(t)}Q|^p \\
& - |\mathcal{R}_{y(t)}Q|^p + |\mathcal{T}_y Q|^p - |\mathcal{T}_{-y}Q|^p \\
& - i(-\dot{y}(t)\mathcal{R}_{y(t)}^{+}\partial_x Q + \dot{\rho}(t)\mathcal{G}_{R,y(t)}Q - \rho(t)\dot{y}(t)\mathcal{G}_{R,y(t)}^{+}\partial_x Q) \\
& - \rho(t)\{(\partial_x^2\chi_R^+)\mathcal{T}_y Q - (\partial_x^2\chi_R^-)\mathcal{T}_{-y}Q\} \\
& - \rho(t)\{(\partial_x\chi_R^+)\partial_x\mathcal{T}_y Q - (\partial_x\chi_R^-)\partial_x\mathcal{T}_{-y}Q\} \\
& + \rho(t)\mathcal{G}_{R,y(t)}(Q^p)
\end{aligned}$$

where we use the fact that Q is a solution to $-Q'' + Q - Q^p = 0$.

Since by the orthogonality condition (2.4) we have

$$\frac{d}{dt}\operatorname{Im}\int h\chi_R^+\mathcal{T}_y Q dx = 0,$$

we get

$$\operatorname{Im}\int \dot{h}\chi_R^+\mathcal{T}_yQdx - \dot{y}\operatorname{Im}\int h\chi_R^+\mathcal{T}_yQ'dx = 0.$$

Multiplying $\chi_R^+\mathcal{T}_yQ$ into the equation of h and taking the integral and the real part, we obtain

$$\dot{\theta}(t) = O(\|h\|_{H^1} + \|h\|_{L^2}|\dot{y}| + \|g\|_{H^1} + |\rho| + e^{-2y}),$$

where we used $|\dot{y}\operatorname{Im}\int h\chi_R^+\mathcal{T}_yQ'dx| \lesssim \|h\|_{L^2}|\dot{y}|$.

Similarly, since we have

$$\operatorname{Re}\int \dot{h}\chi_R^+(\mathcal{T}_yQ)^p dx - \dot{y}\operatorname{Re}\int h\chi_R^+ p(\mathcal{T}_yQ)^{p-1}\mathcal{T}_yQ'dx = 0$$

by the orthogonality condition (2.4), multiplying $\chi_R^+(\mathcal{T}_yQ_\omega)^p$ into this equation and taking the integral and the imaginary part, we obtain

$$\dot{\rho}(t) = O(\|h\|_{H^1} + \|h\|_{L^2}|\dot{y}| + |\dot{\theta}(t)|\|g\|_{H^1} + \|g\|_{H^1} + e^{-2y} + e^{-2y}|\dot{y}(t)|).$$

In the same way, since we have

$$\operatorname{Re}\int \dot{h}(\chi_R^+\mathcal{T}_yQ)'dx - \dot{y}\operatorname{Re}\int h(\chi_R^+\mathcal{T}_yQ')'dx = 0$$

by the orthogonality condition (2.4), multiplying $(\chi_R^+\mathcal{T}_yQ)'$ into this equation and taking the integral and the imaginary part, we obtain

$$\dot{y}(t) = O(\|h\|_{H^1} + \|h\|_{L^2}|\dot{y}| + |\dot{\theta}(t)|\|g\|_{H^1} + e^{-2y}|\dot{\rho}(t)| + \|g\|_{H^1} + |\rho| + e^{-2y}).$$

Thus, we get

$$\begin{aligned}&|\dot{\theta}(t)| + |\dot{\rho}(t)| + |\dot{y}(t)| \\ &\lesssim \|h\|_{H^1} + \|h\|_{L^2}|\dot{y}| + |\dot{\theta}(t)|\|g\|_{H^1} + \|g\|_{H^1} + e^{-2y} + e^{-2y}|\dot{y}(t)| + e^{-2y}|\dot{\rho}(t)| + |\rho|.\end{aligned}$$

By taking $R > 0$ sufficiently large and $\|h\|_{L^2}$ and $\|g\|_{H^1}$ small enough, we get

$$\|h\|_{L^2}|\dot{y}| + |\dot{\theta}(t)|\|g\|_{H^1} + e^{-2y}|\dot{y}(t)| + e^{-2y}|\dot{\rho}(t)| \le \frac{1}{2C}(|\dot{y}| + |\dot{\theta}(t)| + |\dot{\rho}(t)|).$$

Therefore, it holds that

$$|\dot{\theta}(t)| + |\dot{\rho}(t)| + |\dot{y}(t)| \lesssim \|h\|_{H^1} + \|g\|_{H^1} + e^{-2y} + |\rho|$$

and thus $|\dot{\theta}(t)| + |\dot{\rho}(t)| + |\dot{y}(t)| \lesssim |\mu(t)|$. □

§ 2.5. Construction of a critical element

Proposition 2.14. *We suppose that Proposition 1.6 fails, that is, there exists a solution* $u \in H^1_{\mathrm{odd}}(\mathbb{R})$ *satisfying* $M(u_0) = 2M(Q)$, $E(u_0) = 2E(Q)$, $K(u_0) > 0$, *and* u *does not scatter in positive time direction. Then, there exists a function* $x\colon [0,\infty) \to [0,\infty)$ *such that for any* $\varepsilon > 0$ *there exists* $R = R(\varepsilon) > 0$ *such that*

$$\int_{\{|x-x(t)|>R\}\cap\{|x+x(t)|>R\}} |u'(t,x)|^2 + |u(t,x)|^2 dx < \varepsilon \text{ for all } t \geq 0.$$

This can be shown by the profile decomposition and long time perturbation. Since u is odd, it seems to be difficult to rewrite this into a terminology of set theory such that $\{u(t, \cdot - x(t))\ ;\ t > 0\}$ is precompact in $H^1(\mathbb{R})$, which appears in the proofs for general solutions.

We define X by

$$X(t) \coloneqq \begin{cases} x(t), & (t \notin I_{\mu_0}), \\ y(t), & (t \in I_{\mu_0}). \end{cases}$$

Then we see that for any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that

$$\int_{\{|x-X(t)|>R\}\cap\{|x+X(t)|>R\}} |u'(t,x)|^2 + |u(t,x)|^2 dx < \varepsilon \text{ for all } t \geq 0.$$

§ 2.6. Extinction of the critical element

We show that there is no critical element by contradiction by showing $X(t)$ is bounded and unbounded.

2.6.1. $X(t)$ is bounded

Proposition 2.15. *Let* u *be an odd solution with* $M(u) = 2M(Q)$, $E(u) = 2E(Q)$, *and* $K(u(t)) > 0$ *such that for any* $\varepsilon > 0$ *there exists* $R = R(\varepsilon) > 0$ *such that*

$$\int_{\{|x-X(t)|>R\}\cap\{|x+X(t)|>R\}} |u'(t,x)|^2 + |u(t,x)|^2 dx < \varepsilon$$

for all $t \in [0,\infty)$. *Then* $X(t)$ *is bounded.*

By using the scattering result for general solutions on the threshold [3], long time perturbation lemma, and Lemma 2.11, we can show the following.

Lemma 2.16. $X(t_n) \to \infty$ *is equivalent to* $\mu(t_n) \to 0$ *when* $t_n \to \infty$.

Thus, roughly, if we suppose that $X(t_n) \to \infty$, then $\mu(t_n) \to 0$ and this implies $X(t_n)$ converges since $|\frac{d}{dt}X(t)| \lesssim \mu(t) \to 0$ (see Lemma 2.13). This is a contradiction and thus X should be bounded. See [10, Section 5.2.1] for precise proofs.

2.6.2. $X(t)$ is unbounded

As shown in Proposition 2.15, $X(t)$ is bounded. Then we get the following: For any $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{\{|x|>R\}} |u'(t,x)|^2 + |u(t,x)|^2 dx < \varepsilon.$$

By using the localized virial identity, the above compactness, and $K \approx \mu$, it is shown that there exists time sequence $\{t_n\}$ such that $\mu(t_n) \to 0$. This implies $X(t_n) \to \infty$ by Lemma 2.16. This contradicts that X is bounded.

§ A. Conjectures for NLS with repulsive delta potential

Let us consider the following NLS with repulsive delta potential:

$$(\delta\text{NLS}) \qquad \begin{cases} i\partial_t u + \partial_x^2 u + \gamma\delta u + |u|^{p-1}u = 0, & (t,x) \in I \times \mathbb{R}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}), \end{cases}$$

where $\gamma < 0$, δ is the Dirac delta with the mass at the origin, and $p > 5$. Here, $\Delta_\gamma = \partial_x^2 + \gamma\delta$ is defined as follows:

$$-\Delta_\gamma f := -f'' \text{ for } f \in \mathcal{D}(-\Delta_\gamma),$$
$$\mathcal{D}(-\Delta_\gamma) := \{f \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) : f'(0+) - f'(0-) = -\gamma f(0)\}.$$

The energy E_γ and the mass M, which are defined by

$$E_\gamma(u) := \frac{1}{2}\|u'(t)\|_{L^2}^2 - \frac{\gamma}{2}|u(t,0)|^2 - \frac{1}{p+1}\|u(t)\|_{L^{p+1}}^{p+1} \text{ and } M(u) := \|u\|_{L^2}^2,$$

are conserved. For (δNLS), we have the global dynamics results as follows. Let $\omega > 0$ and we define

$$K_\gamma(f) := \|u'\|_{L^2}^2 - \frac{\gamma}{2}|u(t,0)|^2 - \frac{p-1}{2(p+1)}\|u\|_{L^{p+1}}^{p+1}$$
$$S_{\omega,\gamma}(u) := E_\gamma(u) + \frac{\omega}{2}M(u).$$

Let us consider the two minimizing problems:

$$n_{\omega,\gamma} := \inf\{S_{\omega,\gamma}(f) : f \in H^1(\mathbb{R}) \setminus \{0\}, K_\gamma(f) = 0\},$$
$$r_{\omega,\gamma} := \inf\{S_{\omega,\gamma}(f) : f \in H^1_{\text{even}}(\mathbb{R}) \setminus \{0\}, K_\gamma(f) = 0\},$$

where $H^1_{\text{even}}(\mathbb{R})$ is the set of even functions in $H^1(\mathbb{R})$. It is known that $n_{\omega,\gamma} < r_{\omega,\gamma}$ for $\omega > 0$ and $\gamma < 0$ (see e.g [14] and references therein). This difference gives different thresholds for general solutions and for even solutions, which is similar to (NLS) for general solutions and for odd solutions. First, we recall that the dichotomy result for general solutions below the threshold. Here, Q denotes the ground state of (NLS), which appears in the previous sections.

Theorem A.1 ([14]). *Assume that* $M(u_0)^{\frac{1-s_c}{s_c}}E_\gamma(u_0) < M(Q)^{\frac{1-s_c}{s_c}}E(Q)$. *Then the following hold for* (δNLS).

(1) If $K_\gamma(u_0) \geq 0$, *the solution scatters.*

(2) If $K_\gamma(u_0) < 0$, *the solution blows up or grows up.*

We also have a dichotomy result at the threshold.

Theorem A.2 ([2, 16]). *Assume that* $M(u_0)^{\frac{1-s_c}{s_c}}E_\gamma(u_0) = M(Q)^{\frac{1-s_c}{s_c}}E(Q)$. *Then the following hold for* (δNLS).

(1) If $K_\gamma(u_0) > 0$, *the solution scatters.*

(2) If $K_\gamma(u_0) < 0$ *and* $xu_0(x) \in L^2(\mathbb{R})$, *the solution blows up.*

For this result, we have the following conjecture similarly to Conjecture 1.7.

Conjecture A.3. *Assume that* $M(u_0)^{\frac{1-s_c}{s_c}}E_\gamma(u_0) = M(Q)^{\frac{1-s_c}{s_c}}E(Q)$. *If* $K_\gamma(u_0) < 0$, *then the solution blows up or grows up in both time directions.*

Next, we consider even solutions to (δNLS). Then, $r_{\omega,\gamma}$ is attained by

$$Q_{\omega,\gamma}(x) := \omega^{\frac{1}{p-1}}Q\left(\sqrt{\omega}|x| + \frac{2}{p-1}\tanh^{-1}\left(\frac{\gamma}{2\sqrt{\omega}}\right)\right)$$

if $\omega > \gamma^2/4$. The function $Q_{\omega,\gamma}$ is the solution to

$$-Q_{\omega,\gamma}'' - \gamma\delta Q_{\omega,\gamma} + \omega Q_{\omega,\gamma} - Q_{\omega,\gamma}^p = 0.$$

On the other hand, there is no minimizer if $\omega \leq \gamma^2/4$.

We have the following dichotomy result for even solutions below the threshold $r_{\omega,\gamma}$.

Theorem A.4 ([14]). *Assume that* $u_0 \in H^1_{\text{even}}(\mathbb{R})$ *satisfies* $S_{\omega,\gamma}(u_0) < r_{\omega,\gamma}$ *for some* $\omega > 0$. *Then the following hold for* (δNLS).

(1) If $K_\gamma(u_0) \geq 0$, *the solution scatters.*

(2) If $K_\gamma(u_0) < 0$, *the solution blows up or grows up.*

Next, when $\omega > \gamma^2/4$, we have the following two results on the threshold, which is similar to (NLS).

Theorem A.5 ([12]). *Let $\omega > \gamma^2/4$. There exist two even solutions $U^{\pm}_{\omega,\gamma}$ to* (δNLS) *such that the following hold:*

- $M(U^{\pm}_{\omega,\gamma}) = M(Q_{\omega,\gamma})$, $E_\gamma(U^{\pm}_{\omega,\gamma}) = E_\gamma(Q_{\omega,\gamma})$, $U^{\pm}_{\omega,\gamma}$ *exist at least on* $[0,\infty)$, *and there exists* $c > 0$ *such that*

$$\|U^{\pm}_{\omega,\gamma}(t) - e^{i\omega t}Q_{\omega,\gamma}\|_{H^1} \lesssim e^{-ct} \text{ for } t \geq 0.$$

- $K_\gamma(U^{+}_{\omega,\gamma}(0)) < 0$ *and* $U^{+}_{\omega,\gamma}$ *blows up in finite negative time.*
- $K_\gamma(U^{-}_{\omega,\gamma}(0)) > 0$ *and* $U^{-}_{\omega,\gamma}$ *scatters backward in time.*

Theorem A.6 ([12]). *Let $\omega > \gamma^2/4$. Assume that $u_0 \in H^1_{\text{even}}(\mathbb{R})$ satisfies $M(u_0) = M(Q_{\omega,\gamma})$ and $E_\gamma(u_0) = E_\gamma(Q_{\omega,\gamma})$. Then the following are true for* (δNLS).

(1) If $K_\gamma(u_0) > 0$, the solution u scatters, or else $u = U^{-}_{\omega,\gamma}$ up to symmetries.

(2) If $K_\gamma(u_0) = 0$, then $u = e^{i\omega t}Q_{\omega,\gamma}$ up to symmetries.

(3) If $K_\gamma(u_0) < 0$ and $xu_0 \in L^2(\mathbb{R})$, the solution u blows up, or else $u = U^{+}_{\omega,\gamma}$ up to symmetries.

Thus we conjecture the following.

Conjecture A.7. *Under the assumption of Theorem A.6, if $K_\gamma(u_0) < 0$, then the solution u blows up or grows up, or else $u = U^{+}_{\omega,\gamma}$ up to symmetry.*

When $0 < \omega \leq \gamma^2/4$, the dichotomy result on the threshold is valid.

Theorem A.8 ([13]). *Let $0 < \omega \leq \gamma^2/4$. Assume that $u_0 \in H^1_{\text{even}}(\mathbb{R})$ satisfies $M(u_0) = 2M(Q_{\omega,0})$ and $E_\gamma(u_0) = 2E(Q_{\omega,0})$. Then we have the following:*

(1) If $K_\gamma(u_0) > 0$, the solution u scatters.

(2) If $K_\gamma(u_0) < 0$ and $xu_0 \in L^2(\mathbb{R})$, the solution u blows up.

Conjecture A.9. *Under the assumption of Theorem A.8, if $K_\gamma(u_0) < 0$, then the solution u blows up or grows up.*

Remark. We only consider evenness as the symmetry for (δNLS), since the oddness erase the potential. Indeed, the jump condition in the definition of Δ_γ becomes

$$f'(0+) - f'(0-) = -\gamma f(0) = 0$$

for an odd function f and thus it is equivalent to consider odd solutions to (NLS).

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