

An Example of the Cauchy Problem with Infinitely Branching Solutions

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§1. Result

We consider in this note the following nonlinear first order Cauchy problem in $\mathbb{C}^2$

$$(1.1) \quad \begin{cases} u_{x_1}^p + x_1^q + a(u_{x_2}^p + x_2^q) = 0 \\ u(0, x_2) = \phi(x_2) \end{cases}$$

with the following assumptions [A.1]–[A.3]:

[A.1] $p, q \in \mathbb{N}$ satisfying $p \geq 2$, $q \geq 2$, and $p + q \geq 5$.

[A.2] ϕ is an entire function with the following property:

$\phi'(y)^p/y^q$ is not identically constant.

Notation 1.1. Let d be the greatest common divisor of p and q , and we put $p' := p/d$, $q' := q/d$. We denote the least common multiple by $m = dp'q'$. We put $\zeta_m := \exp(2\pi i/m)$ and denote the smallest field containing $\mathbb{Q} \cup \{\zeta_m\}$ by $\mathbb{Q}(\zeta_m)$.

Under this notation our third assumption is as follows:

[A.3] a is a complex constant satisfying

$$\{\alpha \in \mathbb{C}; \alpha^m = (-a)^{p'+q'}\} \cap \mathbb{Q}(\zeta_m) = \emptyset.$$

Note that [A.3] can be written as $(-a)^{1/p+1/q} \notin \mathbb{Q}(\zeta_m)$.

To state our result we give a remark on local holomorphic solutions of (1.1). We put

$$(1.2) \quad \begin{aligned} F(x; \xi) &:= \xi_1^p + x_1^q + a(\xi_2^p + x_2^q) \\ g(x; \xi_2) &:= -x_1^q - a(\xi_2^p + x_2^q) \\ \psi(y) &:= \phi'(y)^p + y^q. \end{aligned}$$

We fix a simply connected domain Ω satisfying

$$(1.3) \quad \Omega \subset \mathbb{C} \setminus \psi^{-1}(0).$$

Note that $\psi(y)/y^a \not\equiv \text{const.}$ by [A.2], in particular, $\psi(y) \not\equiv 0$, so there exists such a Ω satisfying (1.3). Since

$$g(0, y; \phi'(y)) = -a \psi(y) \neq 0 \text{ on } \Omega,$$

we can find a simply connected domain W in $\mathbb{C}^2$ containing the set $\{(0, y; \phi'(y)); y \in \Omega\}$ so that $g(W) \cap \{0\} = \emptyset$. Then the equation $f^p = g$ has p holomorphic roots $f_k \in \mathcal{O}(W)$, $1 \leq k \leq p$, so we get the following decomposition of F :

$$(1.4) \quad F(x; \xi) = \prod_{k=1}^p (\xi_1 - f_k(x; \xi_2)) \quad (\xi_1 \in \mathbb{C}, (x; \xi_2) \in W).$$

By virtue of (1.4) the Cauchy problem (1.1) can be reduced on a neighborhood $\Omega^\sim$ of $\{0\} \times \Omega$ in $\mathbb{C}^2$ to

$$(1.5)_k \quad \begin{cases} u_{x_1} = f_k(x; u_{x_2}) & x \in \Omega^\sim \\ u(0, x_2) = \phi(x_2). \end{cases} \quad (1 \leq k \leq p)$$

Notation 1.2. Since (1.5)_k has a uniquely determined holomorphic solution on a neighborhood of $\{0\} \times \Omega$ in $\Omega^\sim$, we denote it by $u_k(x; \Omega)$. We also denote the maximal analytic continuation of $u_k(x; \Omega)$ to $\mathbb{C}^2$ by $u_k^*(x; \Omega)$.

Our main result is the following theorem:

Theorem 1.3. *Under [A.1]–[A.3], for any Ω and each k , the function $u_k^*(x; \Omega)$ is an infinitely many-valued function.*

Now we explain our motivation to consider (1.1). In [2] we studied first order nonlinear Cauchy problem in a neighborhood M of x^0 in $\mathbb{C}^n$:

$$(1.6) \quad \begin{cases} G(x; u_x; u) = 0 & x = (x_1, \dots, x_n) \in M \\ u(x_1^0; x') = \phi(x') & \text{on } M \cap \{x_1 = x_1^0\}. \end{cases}$$

where $G(x; \xi; z)$ is holomorphic in the first order jet bundle $J^1(M)$ and where $\phi(x')$ is holomorphic on $M \cap \{x_1 = x_1^0\}$. We fix a point

$$e^0 = (x^0; \xi_1^0, \phi_{x'}(x^0); \phi(x^0)) \in J_{x^0}(M)$$

with $G(e^0) = 0$, and assume the following [A.4]–[A.6]:

$$[A.4] \quad \sum_j \frac{\partial G}{\partial \xi_j}(e^0) \frac{\partial}{\partial x_j} \neq 0 \quad \text{in } T_{x^0}(M).$$

[A.5] The function $\xi_1 \rightarrow G(x^0; \xi_1, \phi_{x'}(x^0); \phi(x^0))$ vanishes at ξ_1^0 with a finite vanishing order $p \geq 1$.

[A.6] There exists a holomorphic extension $\Phi(x)$ of $\phi(x')$ with $\Phi_x(x^0) = (\xi_1^0, \phi_{x'}(x^0))$ to a neighborhood of x^0 in M so that Φ has several "good" properties.

For the precise meaning of [A.6] see § 2 in [2].

The main result of [2] was the following theorem.

Theorem 1.4 ([2], Theorem 4.2). *Under [A.4]–[A.6], the Cauchy problem (1.6) has finitely many-valued analytic solutions around x^0 . Further, the ramification degrees of such solutions around x^0 can be calculable by means of the Newton polygon $N(g^*)$ of the function*

$$g^*(x'; \tau) = G(x_1^0, x'; \tau dx_1 + \Phi_x(x_1^0, x'); \phi(x')).$$

It is our motivation to consider (1.1) as an example of the Cauchy problem (1.6) which do *not* satisfy the condition [A.4]. Indeed, if we choose $\phi(x_2)$ in (1.1) as $\phi'(0) = 0$ then [A.4] at the point $e^0 = (0; 0; \phi(0)) \in J^{-1}(\mathbb{C}^2)$ is not satisfied. We also note that Theorem 1.3 concerns a *global* ramification degrees of the solutions of (1.1).

§2. Characteristic Strips

Our proof of Theorem 1.3 is based on the following three tools:

- (I) Classical theory of characteristic strips for first order nonlinear Cauchy problems.
- (II) Representation of the Hamilton flows associated with (1.1) by means of a special function $s_{pq}(\tau)$.
- (III) Automorphic property of s_{pq} or its uniformization σ_{pq} .

In this section we give a quick review of (I). For the tools (II) and (III) see §§ 3–5.

Let us consider the following Cauchy problem in $\mathbb{C}^n$:

$$(2.1) \quad \begin{cases} G(x; u_x; u) = 0 & x \in \mathbb{C}^n \\ u(0, x') = \phi(x') & x' \in S = \{x_1 = 0\} \end{cases}$$

and assume that $G(x; \xi; z)$ and $\phi(x')$ are holomorphic. We also assume that there exists a domain Ω in S so that the equation

$$(2.2) \quad G(0, y; \xi_1, \phi_{x'}(y); \phi(y)) = 0$$

has a holomorphic root $\xi_1 = f(y)$ on Ω . We put

$$\rho(y) = (0, y; f(y), \phi_{x'}(y); \phi(y))$$

and denote the characteristic strip associated with G issuing from $\{\rho(y); y \in \Omega\}$ by $\Phi(t, y) = (X; \Xi; Z)(t, y)$:

$$(2.3) \quad \partial_i X_j = [(\partial/\partial \xi_j)G](\Phi) \quad (1 \leq j \leq n)$$

$$\partial_i \Xi_j = -[(\partial/\partial x_j)G](\Phi) - \Xi_j [(\partial/\partial z)G](\Phi) \quad (1 \leq j \leq n)$$

$$\partial_i Z = \sum_{j=1}^n \Xi_j [(\partial/\partial \xi_j)G](\Phi)$$

$$(2.4) \quad \Phi(0, y) = \rho(y).$$

Proposition 2.1. *If there exists a neighborhood V of $(0, y_0)$ in $\mathbb{C}^n$ so that the restriction $X|_V$ is biholomorphic, then the function $u(x) = Z((X|_V)^{-1}(x))$, $x \in X(V)$, is a holomorphic solution of (2.1) satisfying*

$$u_{x_1}(0, y) = f(y).$$

Further, the derivatives of u are given by

$$(2.5) \quad u_{x_j}(x) = \Xi_j((X|_V)^{-1}(x)), \quad 1 \leq j \leq n$$

Proposition 2.1 follows from $G(\Phi(t, y)) \equiv 0$ and from $\Phi^* \alpha = 0$ where $\alpha = dz - \sum_{j=1}^n \xi_j dx_j$ is the fundamental 1-form on $J^1(\mathbb{C}^n)$.

§3. The Function s_{pq}

In this section we define and study a special function $s_{pq}(\tau)$ which represents the Hamilton flow associated with (1.1).

We first define a function $\tau_{pq}(s)$, and next define s_{pq} as the inverse function of it.

Definition 3.1. For $p, q \in \mathbb{N}$ satisfying [A.1] we define an open sector S_q and a function $\tau_{pq}(s)$ by

$$(3.1) \quad S_q := \{s \in \mathbb{C}; 0 < \arg(s) < \pi/q\},$$

$$(3.2) \quad \tau_{pq}(s) = \int_{\Gamma(s)} (1 - z^q)^{-(p-1)/p} dz,$$

where $\Gamma(s):[0,1] \rightarrow S_q \cup \{0\}$ be a path joining 0 to $s \in S_q$. Choosing the branch of $(1-z^q)^{-1/p}$ at $z=0$ as $1^{-1/p}=1$, (3.2) determines a function $\tau_{pq} \in \mathcal{O}(S_q) \cap C(S_q^\wedge)$ independently of the choice of a path $\Gamma(s)$, where $S_q^\wedge$ denotes the closure of S_q in the extended complex plane $\mathbb{C} := \mathbb{C} \cup \{\infty\}$.

Notation 3.2. We put $\omega_{pq} := \tau_{pq}(1) \in (0, \infty)$, and define an open triangle T_{pq} by

$$(3.3) \quad T_{pq} := \left\{ \tau \in \mathbb{C} ; \begin{array}{l} 0 < \arg(\tau) < \pi/q, \text{ and} \\ \pi(p-1)/p < \arg(\tau - \omega_{pq}) < \pi \end{array} \right\}.$$

We denote the closure of T_{pq} in $\mathbb{C}$ by T_{pq}^- , and the vertex of T_{pq} distinct from 0 and ω_{pq} by λ_{pq} .

Proposition 3.3. *The function τ_{pq} maps S_q conformally onto T_{pq} [resp. maps $S_q^\wedge$ homeomorphically onto T_{pq}^-], with*

$$(3.4) \quad (\tau_{pq}(0), \tau_{pq}(1), \tau_{pq}(\infty)) = (0, \omega_{pq}, \lambda_{pq}).$$

Proof. We use the formula of Schwarz-Christoffel, which asserts that the conformal map Ψ from the upper half plane H onto T_{pq} with the property $(\Psi(0), \Psi(1), \Psi(\infty)) = (0, \omega_{pq}, \lambda_{pq})$ can be written as the following form:

$$(3.5) \quad \Psi(z) = A \int_{z_0}^z \zeta^{-(q-1)/q} (\zeta-1)^{-(p-1)/p} d\zeta + B$$

where $z_0 \in H$ and where A and B are constants. Substituting $z = \psi(s) = s^q$ into (3.5), we have the composition $\Psi_1 = \Psi \circ \psi$ given by

$$(3.6) \quad \Psi_1(s) = qA \int_{s_0}^s (\sigma^q - 1)^{-(p-1)/p} d\sigma + B$$

which maps S_q conformally onto T_{pq} with

$$(3.7) \quad (\Psi_1(0), \Psi_1(1), \Psi_1(\infty)) = (0, \omega_{pq}, \lambda_{pq}).$$

Evaluating the constants A and B by (3.7), we can deduce $\Psi_1 = \tau_{pq}$ so τ_{pq} has the desired property. $\square$

Definition 3.4. We define $s_{pq}: T_{pq} \rightarrow S_q$ by $s_{pq} := \tau_{pq}^{-1}$. By Proposition 3.3 and (3.2), s_{pq} maps T_{pq} conformally onto S_q , with

$$(3.8) \quad \begin{cases} s_{pq}'(\tau) = [1 - s_{pq}(\tau)^q]^{(p-1)/p} \\ s_{pq}(0) = 0 \end{cases}$$

The following proposition is the key result in this section.

Proposition 3.5. For $p, q \in \mathbb{N}$ satisfying [A.1], let m be the least common multiple of them (Notation 1.1). Then it follows that $p^{-1} + q^{-1} + m^{-1} \leq 1$. Moreover, the following conditions (a)–(d) are equivalent:

- (a) s_{pq} is an elliptic function on $\mathbb{C}$.
- (b) s_{pq} is single-valued around the vertex λ_{pq} of T_{pq} .
- (c) The equality $p^{-1} + q^{-1} + m^{-1} = 1$ holds.
- (d) $(p, q) \in \{(2, 3), (3, 2), (2, 4), (3, 3), (4, 2)\}$.

Proof. By [A.1], we have $1 - p^{-1} - q^{-1} > 0$, so $1 - p^{-1} - q^{-1} \in m^{-1}\mathbb{N}$. Thus we get $p^{-1} + q^{-1} + m^{-1} \leq 1$. Since (a) $\Rightarrow$ (b) is trivial, we only have to show (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

To show (b) $\Rightarrow$ (c) we apply the Schwarz's reflection principle to s_{pq} around $\tau = \lambda_{pq}$. Let T_{pq}^* be the reflection of T_{pq} with the segment $[\omega_{pq}, \lambda_{pq}]$, and let S_q^* be the reflection of S_q with the half line $(1, \infty)$. Since $s_{pq}(T_{pq}^*) = S_q^*$ by the principle, if τ rotates $2(1 - p^{-1} - q^{-1})\pi$ around λ_{pq} then the value $s_{pq}(\tau)$ rotates $2q^{-1}\pi$ around ∞ . Thus the single-valuedness of s_{pq} yields that there exists a $n' \in \mathbb{N}$ so that

$$2(1 - p^{-1} - q^{-1})\pi n' = 2\pi.$$

So, if τ rotates 2π around λ_{pq} then $s_{pq}(\tau)$ rotates $2q^{-1}\pi n'$ around ∞ , which deduces $q^{-1}n' \in \mathbb{N}$. Thus, there exists a $n \in \mathbb{N}$ satisfying $(1 - p^{-1} - q^{-1})qn = 1$. Then we deduce $qn/p \in \mathbb{N}$, which implies $n \in p'\mathbb{Z}$. Thus we get $qn \in m\mathbb{Z}$, so we conclude that $(1 - p^{-1} - q^{-1})m$ is a divisor of 1, which shows (c).

To show (c) $\Rightarrow$ (d) we remark that (c) is equivalent to

$$(3.9) \quad dp'q' = m = m(p^{-1} + q^{-1} + m^{-1}) = q' + p' + 1.$$

If $p'q' = 1$ then $p' = q' = 1$, so (3.9) implies $d = 3$, thus we get $(p, q) = (3, 3)$. If $p'q' = 2$ then $(p', q') = (1, 2)$ or $(2, 1)$, so (3.9) implies $d = 2$, thus we get $(p, q) = (2, 4)$ or $(4, 2)$. In the case $p'q' \geq 3$, we use the inequality

$$(3.10) \quad dp'q' = q' + p' + 1 \leq 2 + p'q'$$

which is a consequence of $(p' - 1)(q' - 1) \geq 0$. By (3.10) we have $0 \leq d - 1 \leq 2/(p'q') \leq 2/3$, so we get $d = 1$. Then (3.9) means $pq = q + p + 1$, which is equivalent to $(p - 1)(q - 1) = 2$. Thus we deduce $(p, q) = (2, 3)$ or $(3, 2)$.

To show (d) $\Rightarrow$ (a), we need the following notation.

Notation 3.6. Let Q be the closure of the union of T_{pq} and its reflection with the real axis. Let r_k [resp. ρ] be the $2\pi/p$ [$2\pi/q$] rotation in $\mathbb{C}$ with center $\zeta_q^k \omega_{pq}$ [the origin]. We put

$$(3.11) \quad F_{pq} := \text{the interior of } \left[\bigcup_{j=0}^{p-1} \bigcup_{k=0}^{q-1} r_k^j(\rho^k(Q)) \right],$$

which is an open $2(p-1)q$ -gon in $\mathbb{C}$. For each $n \in \mathbb{Z}$ satisfying $0 \leq n \leq (p-1)q-1$, dividing n by $(p-1)$, we write n as

$$(3.12) \quad n = (p-1)k + j \quad (0 \leq k, 0 \leq j \leq p-2),$$

and we define *sides* s_n and s'_n of F_{pq} , and *side-pairing maps*

$$\{g_s \in \text{Aut}(\mathbb{C}); s \in \{s_n, s'_n; 0 \leq n \leq (p-1)q-1\}\}$$

as follows:

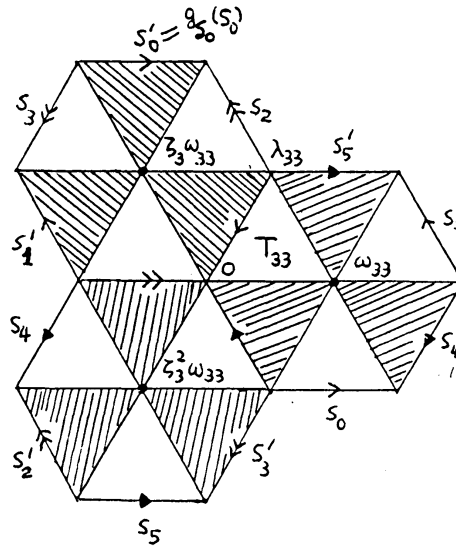
$$(3.13) \quad s_n = r_k^{j+1}(\rho^k([0, \lambda_{pq}))), \quad s'_n = r_{k+1}^{j+1}(\rho^k([0, \lambda_{pq})))$$

$$(3.14) \quad g_s = r_{k+1}^{j+1} \circ r_k^{-j-1} \text{ if } s = s_n, \text{ and}$$

$$g_s = r_k^{j+1} \circ r_{k+1}^{-j-1} \text{ if } s = s'_n.$$

We illustrate the polygon F_{pq} in the case $(p, q) = (3, 3)$:

Figure 3.7. F_{33} with the side-pairing maps.



Let G_{pq} be the group generated by the side-pairing maps (3.14). Since any $g \in G_{pq}$ is a translation in $\mathbb{C}$, there exists a constant $c(g) \in \mathbb{C}$ so that $g(z) = z + c(g)$ for $z \in \mathbb{C}$. For all (p, q)

satisfying the condition (d), it can be verified that G_{pq} is generated by g_0 and g_1 , which are independent over $\mathbb{R}$. Further, F_{pq} is a *fundamental domain* for the group G_{pq} , that is, the following conditions hold:

$$(3.15) \quad \mathbb{C} = \bigcup_{g \in G_{pq}} g(F_{pq}^-)$$

$$(3.16) \quad g \in G_{pq} \setminus \{1\} \text{ implies } g(F_{pq}) \cap F_{pq} = \emptyset.$$

Since s_{pq} is invariant under the action of G_{pq} and is single-valued on $\mathbb{C}$, s_{pq} is an elliptic function. $\square$

§4. Representation of Hamilton Flows

In this section we give the representation of the Hamilton flows associated with (1.1) by means of $s_{pq}(\tau)$. Let us recall the simply connected domain Ω in $\mathbb{C} \setminus \psi^{-1}(0)$ and the decomposition (1.4) of $F(x; \xi)$. We put

$$(4.1) \quad \rho_k(y) = (0, y; f_k(0, y; \phi'(y)), \phi'(y))$$

which lies in $T^*(\mathbb{C}^2) \cap F^{-1}(0)$.

Lemma 4.1. *Let Ω_1 be a simply connected subdomain of Ω so that $\{y^q/\psi(y); y \in \Omega_1\} \subset \mathbb{C} \setminus [1, \infty)$. Then there exist $A_k, B, E \in \mathcal{O}(\Omega_1)$ such that*

$$A_k(y)^{q'} = f_k(0, y; \phi'(y))$$

$$B(y)^{p'} s_{pq}(E(y)) = y$$

$$B(y)^{q'} [1 - s_{pq}(E(y))^q]^{1/p} = \phi'(y).$$

Moreover, the function $\alpha_k(y) = -a(B(y)/A_k(y))^{m-p'-q'}$ is constant on Ω_1 so that $\alpha_k^m = (-a)^{p'+q'}$.

Proof. Since $f_k(0, y; \phi'(y))^p = -a\psi(y) \neq 0$ on Ω , the existence of A_k is obvious. To show the existence of B , we put

$$(4.2) \quad V_q = \mathbb{C} \setminus \bigcup_{k=0}^{q-1} \zeta_q^k [1, \infty).$$

Let $B(y)$ be a holomorphic root of $B^m = \psi$ on Ω_1 . Then, $y \in \Omega_1$ implies $(y/B^{p'})^q = y^q/\psi(y) \in \mathbb{C} \setminus [1, \infty)$, so $y/B^{p'} \in V_q$. Since $\tau_{pq}|V_q$ is single-valued, we can define $E_B \in \mathcal{O}(\Omega_1)$ by

$$E_B(y) = (\tau_{pq}|V_q)(y/B(y)^{p'}).$$

By $s_{pq} \circ \tau_{pq} = \text{id}$, we get $B(y)^{p'} s_{pq}(E(y)) = y$ on Ω_1 . Let us put $B^*(y) := \zeta_m B(y)$. Then

$$s_{pq}(E_{B^*}(y)) = y / (\zeta_m B(y))^{p'} = s_{pq}(E_B(y)) / \zeta_q$$

so we get $s_{pq}(E_{B^*})^q = s_{pq}(E_B)^q = y^q / \psi(y) \in \mathbb{C} \setminus [1, \infty)$. Since $(1-z)^{-1/p}$ is single-valued on $\mathbb{C} \setminus [1, \infty)$, we deduce that

$$[1 - s_{pq}(E_{B^*})^q]^{1/p} = [1 - s_{pq}(E_B)^q]^{1/p}.$$

Since $B^m = \psi$ implies

$$\begin{aligned} (B^{q'} [1 - s_{pq}(E_B)^q]^{1/p})^p &= B^m (1 - s_{pq}(E_B)^q) \\ &= \psi [1 - (y^q / \psi)] = \phi'(y)^p, \end{aligned}$$

we can find some $j \in \{0, \dots, m-1\}$ so that $B^{*(j)} = \zeta_m^j B$ satisfies

$$(B^{*(j)})^{q'} [1 - s_{pq}(E_{B^{*(j)}})^q]^{1/p} = \phi'(y).$$

The last assertion on α_k easily follows from $(A_k/B)^m = -a$. $\square$

The representation of the Hamilton flows is as follows:

Proposition 4.2. *Let $\Phi_k = (X; \Xi)(t, y)$ be the Hamilton flows associated with $F(x; \xi) := \xi_1^p + x_1^q + a(\xi_2^p + x_2^q)$, issuing from $\rho_k(y)$ at $t = 0$. Let Ω_1 and $A_k, B, E \in \mathcal{O}(\Omega_1)$ and $\alpha_k \in \mathbb{C}$ be chosen as in Lemma 4.1. Then Φ_k can be represented as*

$$\begin{aligned} X_1 &= A_k(y)^{p'} s_{pq}(\tau) \\ X_2 &= B(y)^{p'} s_{pq}(-\alpha_k \tau + E(y)) \\ \Xi_1 &= A_k(y)^{q'} [1 - s_{pq}(\tau)^q]^{1/p} \\ \Xi_2 &= B(y)^{q'} [1 - s_{pq}(-\alpha_k \tau + E(y))^q]^{1/p} \end{aligned}$$

where we put $\tau = \tau(t, y) = pA_k(y)^{m-p'-q'} t$.

Proposition 4.2 follows from a direct computation with Lemma 4.1 and with the differential equation (3.8).

§5. Uniformization of s_{pq}

We will treat the case $p+q \geq 7$ and construct a uniformization of the multi-valued function s_{pq} on $\mathbb{C}$. In the case $p+q \geq 7$, note that Proposition 3.5 implies

$$(5.1) \quad p^{-1} + q^{-1} + m^{-1} < 1.$$

Notation 5.1 Let $\mathbb{D}$ be the unit open disk in $\mathbb{C}$. We regard $\mathbb{D}$

as a hyperbolic plane with the *Poincaré metric*

$$ds^2 = (1 - |z|^2)^{-2} (dx^2 + dy^2).$$

It is well-known that geodesics in $\mathbb{D}$ with respect to this metric consist of all (Euclidean) circles which are orthogonal to the circle at infinity $\{|z|=1\}$.

Definition 5.2. By (5.1) there exists a hyperbolic triangle with inner angles $p^{-1}\pi$, $q^{-1}\pi$ and $m^{-1}\pi$. We denote by T_{pq}^* the uniquely determined triangle with vertices 0 , ω_{pq}^* , λ_{pq}^* so that

$$(5.2) \quad \omega_{pq}^* \in (0, 1) \text{ and } \lambda_{pq}^* \in \mathbb{D} \cap \{\operatorname{Im}(z) > 0\}$$

$$(5.3) \quad \angle 0 = q^{-1}\pi, \quad \angle \omega_{pq}^* = p^{-1}\pi, \quad \text{and} \quad \angle \lambda_{pq}^* = m^{-1}\pi.$$

By Riemann's mapping theorem, there exists a map π which maps T_{pq}^* conformally onto the triangle T_{pq} defined by (3.3) [resp. maps $(T_{pq}^*)^\sim$ homeomorphically onto T_{pq}^-], where $\sim$ denotes the closure in $\mathbb{D}$.

Notation 5.3. We denote by π_{pq} the conformal map $\pi: T_{pq}^* \rightarrow T_{pq}$ with $(\pi(0), \pi(\omega_{pq}^*), \pi(\lambda_{pq}^*)) = (0, \omega_{pq}, \lambda_{pq})$. We denote the composition $s_{pq} \circ \pi_{pq}$ by σ_{pq} .

We consider the analytic [resp. meromorphic] continuation of π_{pq} [σ_{pq}] to $\mathbb{D}$. To do this we need to construct a polygon F_{pq} in $\mathbb{D}$, which is obtained by similar way of the construction (3.11) of F_{pq} in the case $p+q \leq 6$.

Definition 5.4. Let $(T_{pq}^*)'$ be the reflection of T_{pq}^* with the geodesic $(-1, 1)$, and we put $Q := [T_{pq}^* \cup (T_{pq}^*)']^\sim$. We denote by r_k [resp. ρ] the elliptic $2\pi/p$ [$2\pi/q$] rotation in $\mathbb{D}$ with center $\zeta_q^{-k} \omega_{pq}^*$ [the origin]. We define F_{pq} by

$$(5.4) \quad F_{pq} := \text{the interior of } \left[\bigcup_{j=0}^{p-1} \bigcup_{k=0}^{q-1} r_k^j(\rho^k(Q)) \right].$$

Note that F_{pq} is a $2(p-1)q$ -gon in $\mathbb{D}$. For each $n \in \mathbb{Z}$ satisfying $0 \leq n \leq (p-1)q-1$, dividing n by $(p-1)$, we write

$$(5.5) \quad n = (p-1)k + j \quad (0 \leq k, 0 \leq j \leq p-2).$$

Definition 5.5. We define *sides* s_n and s_n' of F_{pq} and *side-pairing maps*

$$\{g_s \in \operatorname{Aut}(\mathbb{D}); s \in \{s_n, s_n'; 0 \leq n \leq (p-1)q-1\}\}$$

as follows:

$$(5.6) \quad s_n = r_{k+1}^{j+1}(\rho^k([0, \lambda_{pq}^*])), \text{ and}$$

$$s_n' = r_{k+1}^{j+1}(\rho^k([0, \lambda_{pq}^*]))$$

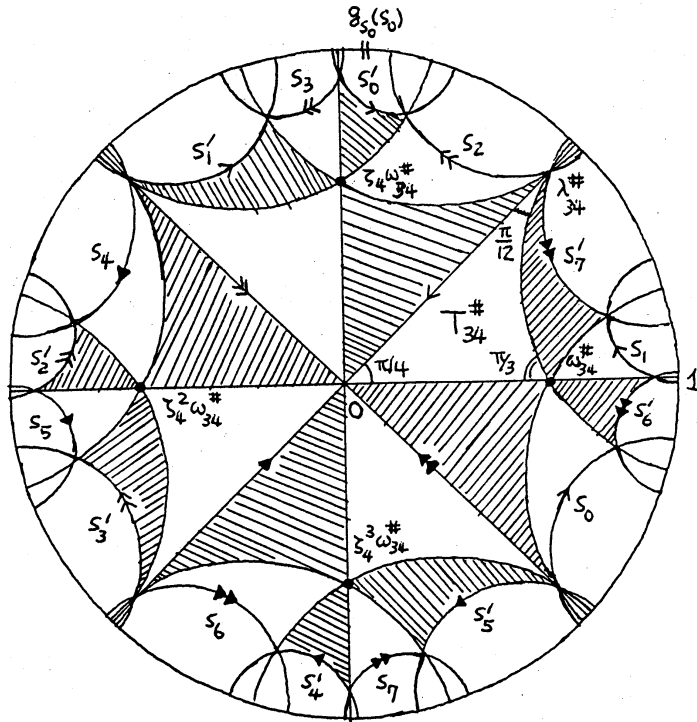
$$(5.7) \quad g_S = r_{k+1}^{j+1} \circ r_k^{-j-1} \text{ if } s = s_n, \text{ and}$$

$$g_S = r_k^{j+1} \circ r_{k+1}^{-j-1} \text{ if } s = s_n',$$

where $[0, \lambda_{pq}^*]$ denotes the geodesic segment joining 0 to λ_{pq}^* .

We illustrate the polygon F_{pq} in the case $(p, q) = (3, 4)$:

Figure 5.6. F_{34} with the side-pairing maps.



We recall the following general definition.

Definition 5.7. Let G be a subgroup of $\text{Aut}(\mathbb{D})$. A subset Ω is called a *fundamental domain* for G if

(i) Ω is a domain.

(ii) The hyperbolic area of $\partial\Omega$ is 0.

(iii) $\mathbb{D} = \bigcup_{g \in G} g(\Omega^{\sim})$ (where $\sim$ denotes the closure in $\mathbb{D}$).

(iv) $g \in G \setminus \{1\}$ implies $g(\Omega) \cap \Omega = \emptyset$.

Definition 5.8. A fundamental domain Ω for G is called *locally finite* if for any compact K in $\mathbb{D}$ only finitely many g in G satisfy $g(\Omega) \cap K \neq \emptyset$.

We now use the Poincaré's theorem originated by [3], which gives a sufficient condition for a polygon P with side-pairing maps

$$(5.8) \quad \{g_s; s \in \{\text{sides of } P\}\}$$

to be a locally finite fundamental domain for the group generated by the side-pairing maps. To state the Poincaré's theorem we need a notion of a cycle of a vertex of P .

Definition 5.9. Let x be a vertex of P . We call a finite sequence $\{x_0, x_1, \dots, x_{n-1}\}$ a *cycle* of x if

$$(i) \quad x_0 = x_n = x, \quad \text{and} \quad x_i \neq x_j \quad \text{for} \quad 0 \leq i < j \leq n-1.$$

(ii) For any $1 \leq j \leq n$ there exist sides s_j and t_j of P so that

$$(5.9) \quad \{x_j\} = s_j \cap t_j, \quad g_{s_j}(s_j) = t_{j-1} \quad \text{and} \quad g_{s_j}(x_j) = x_{j-1}.$$

We note that if $C(x) = \{x_0, x_1, \dots, x_{n-1}\}$ is a cycle of x then every x_j is a vertex of P , and that $C^{-1}(x) = \{x_n, x_{n-1}, \dots, x_1\}$ is also a cycle of x . Further, there is no cycle of x different from $C(x)$ and $C^{-1}(x)$.

Theorem 5.10 (Poincaré's theorem of a restricted type). *Let P be a relatively compact polygon in $\mathbb{D}$ with side-pairing maps (5.8). Assume the following condition (angle condition) (5.10) for any vertex x : let $C(x) = \{x_0, x_1, \dots, x_{n-1}\}$ be a cycle of x , and let θ_j be the inner angle of P at the vertex x_j . In this situation, there exists a $N = N(C(x)) \in \mathbb{N}$ so that*

$$(5.10) \quad \sum_{j=0}^{n-1} \theta_j = 2\pi / N.$$

Then P is a locally finite fundamental domain for the group which is generated by the collection (5.8).

For the proof of Theorem 5.10 see [1] or [3].

We want now to apply Theorem 5.10 to our F_{pq} with the side-pairing maps (5.7). To do this we must verify the following lemma.

Lemma 5.11. *The polygon F_{pq} with the side-pairing maps (5.7) satisfies the angle condition (5.10) with $N(C(x)) = 1$ for all vertices x of F_{pq} .*

Proof. Let x be a vertex of F_{pq} . Then there exists a uniquely determined $n \in \{0, 1, \dots, (p-1)q-1\}$ such that x lies in s_n . There are two cases which can occur:

$$\text{Case 1.} \quad x = r_k^{j+1}(\rho^k(\lambda_{pq}^\#)),$$

$$\text{Case 2.} \quad x = r_k^{j+1}(\rho^k(0)) = r_k^{j+1}(0),$$

where j and k are related with n by (5.5).

We will first show the following claim:

Claim 5.12. *Let n' be the uniquely determined integer by*

$$(5.11) \quad x' := g_{s_n}(x) \in s_{n'}.$$

Then it follows that

$$(5.12) \quad n' \equiv n + p \pmod{(p-1)q} \text{ in the case 1,}$$

$$(5.13) \quad n' \equiv n + p - 1 \pmod{(p-1)q} \text{ in the case 2.}$$

Proof of Claim 5.12. We first show (5.12). By the definition (5.7) of the side-pairing maps, we have

$$x' = r_{k+1}^{j+1} r_k^{-j-1}(r_k^{j+1}(\rho^k(\lambda_{pq}^\#))) = r_{k+1}^{j+1}(\rho^k(\lambda_{pq}^\#)).$$

Since $\rho^k(\lambda_{pq}^\#) = r_{k+1}(\rho^{k+1}(\lambda_{pq}^\#))$, we get

$$x' = r_{k+1}^{j+2}(\rho^{k+1}(\lambda_{pq}^\#)).$$

Thus, by (5.6), $x' \in s_{n'}$ is equivalent to

$$n' \equiv (p-1)(k+1) + j + 1 \pmod{(p-1)q},$$

which implies

$$n' - n \equiv (p-1)(k+1) + j + 1 - [(p-1)k + j] = p.$$

Next we show (5.13). By (5.7) we also have

$$x' = r_{k+1}^{j+1} r_k^{-j-1}(r_k^{j+1}(0)) = r_{k+1}^{j+1}(\rho^{k+1}(0)).$$

So we get by (5.6) that $n' \equiv (p-1)(k+1) + j \pmod{(p-1)q}$, which implies

$$n' - n \equiv (p-1)(k+1) + j - [(p-1)k + j] = p - 1.$$

This completes the proof of Claim 5.12. $\square$

We continue the proof of Lemma 5.11. Let $C(x) = \{x_0, \dots, x_{\nu-1}\}$ be the cycle of $x = x_0$. In the case 1, Claim 5.12 shows that $\nu = \#C(x)$ is the minimum of $\mu \in \mathbb{N}$ satisfying

$$(5.14) \quad n + \mu p \equiv n \pmod{(p-1)q}.$$

Since (5.14) is equivalent to $\mu p' \in (p-1)q' \mathbb{Z}$, the coprimeness of p' and $(p-1)q'$ implies $\mu \in (p-1)q' \mathbb{Z}$. Thus we get

$$\nu = \min[(p-1)q' \mathbb{N}] = (p-1)q'$$

Further, a vertex x_h ($1 \leq h \leq \nu = (p-1)q'$) in $C(x)$ coincides with $\rho^k(\lambda_{pq})$ for some $k \in \{0, 1, \dots, q-1\}$ if and only if h satisfies $n + hp \in (p-1)\mathbb{Z}$, which is equivalent to

$$(5.15) \quad j + h \in (p-1)\mathbb{Z}$$

where $n = (p-1)k + j$. Since (5.15) has q' -solutions h , we have

$$\#\{x_h; \theta_h = 4m^{-1}\pi\} = q' \text{ and}$$

$$\#\{x_h; \theta_h = 2m^{-1}\pi\} = \nu - q' = (p-2)q'.$$

Thus we deduce in the case 1 that

$$\begin{aligned} \sum_{j=0}^{\nu-1} \theta_j &= (4m^{-1}\pi)q' + (2m^{-1}\pi)(p-2)q' \\ &= 2\pi m^{-1}pq' = 2\pi. \end{aligned}$$

In the case 2, Claim 5.12 shows that $\nu = \#C(x)$ is the minimum of $\mu \in \mathbb{N}$ satisfying

$$(5.16) \quad n + \mu(p-1) \equiv n \pmod{(p-1)q}.$$

Then it is trivial that $\nu = q$, so we deduce in the case 2 that

$$\sum_{j=0}^{\nu-1} \theta_j = (2q^{-1}\pi)q = 2\pi.$$

It completes the proof of Lemma 5.11. $\square$

By virtue of Lemma 5.11 we can apply Theorem 5.10 (Poincaré's theorem) to the polygon F_{pq} with the side-pairing maps (5.7), and have the following proposition.

Proposition 5.13. *The polygon F_{pq} defined by (5.4) is a locally finite fundamental domain for the group generated by the side-pairing maps (5.7).*

Notation 5.14. We denote by G_{pq} the group generated by the side-pairing maps (5.7).

As a consequence of Proposition 5.13 we have the following uniformization of S_{pq} .

Corollary 5.15. *The following (i)~(iv) hold:*

(i) σ_{pq} is meromorphic on $\mathbb{D}$, and is G_{pq} -invariant.

- (ii) π_{pq} is holomorphic on $\mathbb{D}$, and for any g in G_{pq} the function $c(g)(z) := \pi_{pq}(gz) - \pi_{pq}(z)$ is constant on $\mathbb{D}$.
- (iii) The map $g \rightarrow c(g)$ is a group homomorphism from G_{pq} into the additive group $(\mathbb{C}, +)$.
- (iv) $\sigma_{pq}(z) = s_{pq}(\pi_{pq}(z))$ on $\mathbb{D}$, that is, the following diagram is commutative:

$$(5.17) \quad \begin{array}{ccc} \mathbb{D} & \xrightarrow{\sigma_{pq}} & \mathbb{C} \\ \pi_{pq} \downarrow & \nearrow s_{pq} & \\ \mathbb{C} & & \end{array}$$

§6. A Picard Type Theorem

In this section we give a Picard type theorem for a uniformization of the equation (6.2) below.

Let $\Phi_k = (X; \Xi)$ be the Hamilton flow, and we consider the following equation in $(t, y) \in \mathbb{C} \times \Omega_1$:

$$(6.1) \quad X_j(t, y) = x_j \quad (j = 1, 2)$$

where Ω_1 is the domain chosen as in Lemma 4.1. By Proposition 4.2, putting $\tau = pA_k(y)^{m-p'-q'}t$, (6.1) can be written as

$$(6.2) \quad \begin{aligned} X_1 \sim(\tau, y) &= A_k(y)^{p'} s_{pq}(\tau) = x_1 \\ X_2 \sim(\tau, y) &= B(y)^{p'} s_{pq}(-\alpha_k \tau + E(y)) = x_2 \end{aligned}$$

Since in the case $p+q \geq 7$ the function s_{pq} is multi-valued, we introduce the following uniformization of the map $X \sim$.

Notation 6.1. Let us put

$$(6.3) \quad \Sigma_{pq} := \begin{cases} \mathbb{D} & \text{if } p+q \geq 7 \\ \mathbb{C} & \text{if } 5 \leq p+q \leq 6. \end{cases}$$

In the case $\Sigma_{pq} = \mathbb{D}$, the maps π_{pq} and σ_{pq} are already defined in Notation 5.3. In the case $\Sigma_{pq} = \mathbb{C}$, we denote the identity map on $\mathbb{C}$ by π_{pq} , and we put $\sigma_{pq} = s_{pq}$.

Using this notation, we have $\pi_{pq} \in \mathcal{O}(\Sigma_{pq})$ and $\sigma_{pq} \in \text{Mero}(\Sigma_{pq})$ so that

$$(6.4) \quad \sigma_{pq}(z) = s_{pq}(\pi_{pq}(z)) \quad \text{on } \Sigma_{pq}.$$

Definition 6.2. For the constant α_k and $E \in \mathcal{O}(\Omega_1)$ in (6.2), we define a surface $M_{pq}(\alpha_k)$ by

$$(6.5) \quad M_{pq}(\alpha_k) := \{(z_1, z_2; y) \in \Sigma_{pq}^2 \times \Omega_1 : \\ -\alpha_k \pi_{pq}(z_1) + E(y) = \pi_{pq}(z_2)\}.$$

We also define a map $X^*: M_{pq}(\alpha_k) \rightarrow \mathbb{C}^2$ by

$$(6.6) \quad X^*(z_1, z_2; y) = (A_k(y)^{p'} \sigma_{pq}(z_1), B(y)^{p'} \sigma_{pq}(z_2)).$$

Finally we define a map $P: M_{pq}(\alpha_k) \rightarrow \mathbb{C} \times \Omega_1$ by

$$(6.7) \quad P(z_1, z_2; y) = (\pi_{pq}(z_1), y).$$

Remark 6.3. The identity (6.4) implies the following identity:

$$(6.8) \quad X^*(z_1, z_2; y) = X^\sim(P(z_1, z_2; y)).$$

Proof. Indeed, it follows from (6.4) and (6.5) that

$$\begin{aligned} X_1^\sim(P(z_1, z_2; y)) &= A_k(y)^{p'} s_{pq}(\pi_{pq}(z_1)) \\ &= A_k(y)^{p'} \sigma_{pq}(z_1) = X_1^*(z_1, z_2; y), \\ X_2^\sim(P(z_1, z_2; y)) &= B(y)^{p'} s_{pq}(-\alpha_k \pi_{pq}(z_1) + E(y)) \\ &= B(y)^{p'} s_{pq}(\pi_{pq}(z_2)) = B(y)^{p'} \sigma_{pq}(z_2) \\ &= X_2^*(z_1, z_2; y) \quad \square \end{aligned}$$

By virtue of Remark 6.3, the equation (6.2) also has the following uniformization

$$(6.9) \quad \begin{aligned} X_1^*(z_1, z_2; y) &= A_k(y)^{p'} \sigma_{pq}(z_1) = x_1 \\ X_2^*(z_1, z_2; y) &= B(y)^{p'} \sigma_{pq}(z_2) = x_2. \end{aligned}$$

Notation 6.4. We put $(G_{pq})_* = \{c(g) \in \mathbb{C}; g \in G_{pq}\}$, where $c(g)$ is the constant defined by $c(g) = \pi_{pq}(gz) - \pi_{pq}(z)$, $z \in \Sigma_{pq}$.

By Corollary 5.15 (iii) $(G_{pq})_*$ forms an additive subgroup of $\mathbb{C}$.

The following lemma is fundamental to solve (6.9).

Lemma 6.5. *The vector sum*

$$\alpha_k (G_{pq})_* + (G_{pq})_* = \{\alpha_k x + y; x, y \in (G_{pq})_*\}$$

is dense in $\mathbb{C}$.

We omit the proof of Lemma 6.5, which is obtained by the fact that α_k lies in $\mathbb{C} \setminus \mathbb{Q}(\zeta_m)$ (see Lemma 4.1 and [A.3]), and that $c(g)/\omega_{pq}$ lies in $\mathbb{Z}[\zeta_m]$.

Using Lemma 6.5, we can show the following Picard type theorem for the map $X^*: M_{pq}(\alpha_k) \rightarrow \mathbb{C}^2$.

Proposition 6.6. *There exists a relatively compact subdomain Ω_2 of Ω_1 such that the following (i) and (ii) hold:*

(i) *There exists an open neighborhood V of the origin in $\mathbb{C}^2$ such that for any $x \in V$ we can find a distinct sequence $\{(z_{1\nu}, z_{2\nu}; y_\nu); \nu \in \mathbb{N}\}$ in $M_{pq}(\alpha_k) \cap [\Sigma_{pq}^2 \times \Omega_2]$ satisfying*

$$(6.10) \quad X^*(z_{1\nu}, z_{2\nu}; y_\nu) = x \quad \text{for any } \nu \in \mathbb{N}.$$

(ii) *Moreover, if $x \in V \setminus \{0\}$ then $\{(z_{1\nu}, z_{2\nu}; y_\nu)\}$ has the following property: for any $\mu \neq \nu$,*

$$(6.11) \quad z_{1\mu} \in G_{pq}(z_{1\nu}) \quad \text{and} \quad z_{2\mu} \in G_{pq}(z_{2\nu})$$

are not compatible, where $G_{pq}(z)$ denotes the G_{pq} -orbit containing $z \in \Sigma_{pq}$.

Proof of (i). Let us recall the facts shown in § 4:

$$(6.12) \quad A_k(y)^m = -a \psi(y) \neq 0, \quad B(y)^m = \psi(y) \neq 0 \quad \text{on } \Omega_1,$$

$$(6.13) \quad E(y) = (\tau_{pq}|V_q)(y/B(y)^p),$$

where V_q is the simply connected domain given by (4.2). Since $\{y/B(y)^p\}^q = y^q/\psi(y)$ is not constant by [A.2], we get

$$(6.14) \quad E(y) \text{ is not constant on } \Omega_1.$$

We also remark that

$$(6.15) \quad \psi(y) \text{ is not constant on } \Omega_1.$$

Indeed, if we assume $\psi(y) \equiv c$ then $c \neq 0$, because $\psi(y) \equiv 0$ implies $\phi'(y)^p/y^q \equiv -1$ which violates [A.2]. On the other hand, $\psi(y) \equiv c \neq 0$ implies $\phi'(y) \equiv [c - y^q]^{1/p}$ which contradicts $\phi'(y) \in \mathcal{O}(\mathbb{C})$. Thus we get (6.15).

By (6.14) and (6.15), there exists a relatively compact subdomain Ω_2 of Ω_1 so that

$$(6.16) \quad \min_{y \in \Omega_2^-} |E'(y)| > 0 \quad \text{and} \quad \min_{y \in \Omega_2^-} |\psi'(y)| > 0.$$

Since $\psi(y) \neq 0$ on Ω_1 , we also have

$$\min_{y \in \Omega_2^-} |\psi(y)| > 0.$$

Thus (6.12) yields

$$(6.17) \quad \min_{y \in \Omega_2^-} |A_k(y)| > 0 \quad \text{and} \quad \min_{y \in \Omega_2^-} |B(y)| > 0.$$

Then, by (6.17) and the former of (6.16), we get

$$(6.18) \quad \min\{ \min_{y \in \Omega_2^-} |A_k(y)|, \min_{y \in \Omega_2^-} |B(y)|, \min_{y \in \Omega_2^-} |E'(y)| \} > 0.$$

We denote the left hand side of (6.18) by ε . We consider the following function H :

$$(6.19) \quad H(y; x_1, x_2) := -\alpha_k(\tau_{pq}|V_q)(x_1/A_k(y)^{p'}) + E(y) \\ + (\tau_{pq}|V_q)(x_2/B(y)^{p'})$$

$$\text{for } y \in \Omega_2 \quad \text{and} \quad |x_j| < \varepsilon^{p'} \quad (j=1,2).$$

Since $y \in \Omega_2$ and $|x_j| < \varepsilon^{p'}$ yield $|x_1/A_k(y)^{p'}|, |x_2/B(y)^{p'}| < 1$, we have $x_1/A_k(y)^{p'}, x_2/B(y)^{p'} \in V_q$, so $H(y; x_1, x_2)$ is well-defined. Moreover, since

$$(\partial/\partial y)H = -\alpha_k(\tau_{pq}|V_q)'(x_1/A_k(y)^{p'})x_1(d/dy)[A_k^{-p'}] \\ + E'(y) + (\tau_{pq}|V_q)'(x_2/B(y)^{p'})x_2(d/dy)[B^{-p'}],$$

we can find a small $\delta \in (0, \varepsilon^{p'})$ such that $|x_j| < \delta$ yield

$$(6.20) \quad |(\partial/\partial y)H(y; x_1, x_2)| \\ \geq \min_{y \in \Omega_2^-} |E'(y)| - \delta \left(\max_{|z| \leq \delta/\varepsilon^{p'}} |(\tau_{pq}|V_q)'(z)| \right) \\ \times \max_{y \in \Omega_2^-} (|\alpha_k| |(d/dy)[A_k^{-p'}]| + |(d/dy)[B^{-p'}]|) \\ \geq \varepsilon/2.$$

We put $V := \{x \in \mathbb{C}^2; |x_j| < \delta\}$.

For any $x \in V$ we construct a sequence $\{(z_{1\nu}, z_{2\nu}; y_\nu); \nu \in \mathbb{N}\}$ of solutions of (6.10) as follows. Since (6.20) implies that, for any fixed $x \in V$, the function $y \rightarrow H(y; x)$ is not constant on Ω_2 , the image $W(x) = \{H(y; x); y \in \Omega_2\}$ is a non-empty open set in $\mathbb{C}$. Then, by Lemma 6.5, $W(x) \cap [\alpha_k(G_{pq})_* + (G_{pq})_*]$ contains infinitely many elements, so we can choose sequences $\{y_\nu\}$ in Ω_2 , and $\{g_\nu\}, \{h_\nu\}$ in G_{pq} such that

$$(6.21) \quad H(y_\nu; x) = -\alpha_k c(g_\nu) + c(h_\nu) \quad \text{with the property}$$

$$H(y_\nu; x) \neq H(y_\mu; x) \quad \text{for any } \nu \neq \mu.$$

Taking subsequences if necessary, we may assume that there is a $y_0 \in \Omega_2^-$ so that $y_\nu \rightarrow y_0$ ($\nu \rightarrow \infty$). Now we define a sequence

$\{(z_{1\nu}, z_{2\nu}); \nu \in \mathbb{N}\}$ in Σ_{pq}^2 by

$$(6.22) \quad z_{1\nu} := g_\nu((\sigma_{pq}|F_{pq}^*)^{-1}(x_1/A_k(y_\nu)^{p'}))$$

$$z_{2\nu} := h_\nu((\sigma_{pq}|F_{pq}^*)^{-1}(x_2/B(y_\nu)^{p'})),$$

where F_{pq}^* is, using Q and ρ in Definition 5.4, defined by

$$(6.23) \quad F_{pq}^* := \text{the interior of } [\bigcup_{k=0}^{q-1} \rho^k(Q)].$$

We note that $F_{pq}^* \subset F_{pq}$ and the following diagram commutes:

$$(6.24) \quad \begin{array}{ccc} F_{pq}^* & \xrightarrow[\sim]{\sigma_{pq}|F_{pq}^*} & V_q \\ \pi_{pq}|F_{pq}^* \downarrow \wr & \swarrow \wr & \tau_{pq}|V_q \\ \tau_{pq}(V_q) & & \end{array}$$

Then the G_{pq} -invariance of σ_{pq} yields

$$\begin{aligned} X_1^*(z_{1\nu}, z_{2\nu}; y_\nu) &= A_k(y_\nu)^{p'} \sigma_{pq}(z_{1\nu}) \\ &= A_k(y_\nu)^{p'} \sigma_{pq}((\sigma_{pq}|F_{pq}^*)^{-1}(x_1/A_k(y_\nu)^{p'})) = x_1, \\ X_2^*(z_{1\nu}, z_{2\nu}; y_\nu) &= B(y_\nu)^{p'} \sigma_{pq}(z_{2\nu}) \\ &= B(y_\nu)^{p'} \sigma_{pq}((\sigma_{pq}|F_{pq}^*)^{-1}(x_2/B(y_\nu)^{p'})) = x_2. \end{aligned}$$

Moreover, (6.24) and the property $\pi_{pq}(gz) = \pi_{pq}(z) + c(g)$ imply

$$\begin{aligned} & -\alpha_k \pi_{pq}(z_{1\nu}) + E(y_\nu) - \pi_{pq}(z_{2\nu}) \\ &= -\alpha_k [\pi_{pq}((\sigma_{pq}|F_{pq}^*)^{-1}(x_1/A_k(y_\nu)^{p'})) + c(g_\nu)] + E(y_\nu) \\ & \quad - [\pi_{pq}((\sigma_{pq}|F_{pq}^*)^{-1}(x_2/B(y_\nu)^{p'})) + c(h_\nu)] \\ &= H(y_\nu; x) - [\alpha_k c(g_\nu) + c(h_\nu)] \\ &= 0. \end{aligned}$$

Thus, the assertion (i) is proved.

Proof of (ii). It suffices to show that if there exist ν and μ with $\nu \neq \mu$ so that (6.11) are compatible, then $x=0$. Since $z_{1\mu} \in G_{pq}(z_{1\nu})$, there exists a $g \in G_{pq}$ so that $z_{1\mu} = g(z_{1\nu})$, so (6.22) yields

$$\begin{aligned} & g_\mu((\sigma_{pq}|F_{pq}^*)^{-1}(x_1/A_k(y_\mu)^{p'})) \\ &= g g_\nu((\sigma_{pq}|F_{pq}^*)^{-1}(x_1/A_k(y_\nu)^{p'})). \end{aligned}$$

Since the image of $(\sigma_{pq}|F_{pq}^*)^{-1}$ lies in F_{pq} which is a fundamental

domain for G_{pq} by Proposition 5.13, we get $g_\mu = gg_\nu$ and

$$(\sigma_{pq}|F_{pq}^*)^{-1}(x_1/A_k(y_\mu)^{p'}) = (\sigma_{pq}|F_{pq}^*)^{-1}(x_1/A_k(y_\nu)^{p'}).$$

Then the injectivity of $(\sigma_{pq}|F_{pq}^*)^{-1}$ implies

$$(6.25) \quad x_1/A_k(y_\mu)^{p'} = x_1/A_k(y_\nu)^{p'}.$$

If we assume $x_1 \neq 0$ then $A_k(y_\mu)^{p'} = A_k(y_\nu)^{p'}$. Since we get

$$(d/dy)[A_k(y)^{p'}] = p' A_k(y)^{p'-1} A_k'(y) \neq 0$$

on Ω_2^- , which is a consequence of (6.12), (6.17) and the latter of (6.16), there exists an open neighborhood U of $y_0 = \lim y_\nu$ so that $y \rightarrow A_k(y)^{p'}$ is injective on U . Thus, taking a subsequence if necessary, we may assume that $y_\nu \in U$ for all ν . Then we have $A_k(y_\mu)^{p'} \neq A_k(y_\nu)^{p'}$, so $x_1 \neq 0$ is impossible. Hence we get $x_1 = 0$. Since the similar argument also yields $x_2 = 0$, we get the assertion (ii). It completes the proof. $\square$

§7. Proof of Theorem 1.3

Now we give a proof of our main result (Theorem 1.3) in this last section. The Picard type theorem (Proposition 6.6) shows that the fiber $(X^*)^{-1}(x)$ is an infinite set in $M_{pq}(\alpha_k)$ for $x \in V$.

On the other hand, we can show the following fact.

Theorem 7.1. *Let Ω_2 be the relatively compact subdomain chosen as in Proposition 6.6. We put $M_{pq}^*(\alpha_k) := M_{pq}(\alpha_k) \cap (\Sigma_{pq}^2 \times \Omega_2)$. Then, for any $k = 1, 2, \dots, p$, the surface $M_{pq}^*(\alpha_k)$ is a non-singular connected surface.*

Partial proof. The non-singularity follows from the former of (6.16). The connectivity of $M_{pq}^*(\alpha_k)$ in the case $p+q \leq 6$ is easily verified, because $\pi_{pq} = \text{id}$ on $\Sigma_{pq} = \mathbb{C}$ implies that the surface $M_{pq}^*(\alpha_k) = \{(z_1, z_2; y) \in \mathbb{C}^2 \times \Omega_2; -\alpha_k z_1 + E(y) = z_2\}$ is a continuous image of the connected set $\mathbb{C} \times \Omega_2$. But in the case $p+q \geq 7$ it needs a long proof with full use of $\alpha_k \in \mathbb{C} \setminus \mathbb{Q}(\zeta_m)$ to show that $M_{pq}^*(\alpha_k)$ is connected. So, we omit it here. $\square$

To show Theorem 1.3, we recall the following diagram, which is introduced in Definition 6.2 and Remark 6.3.

$$\begin{array}{ccc}
 (7.1) & (z; y) \in M_{pq}^*(\alpha_k) & \xrightarrow{X^*} \mathbb{C}^2 \\
 & \downarrow P & \nearrow X \sim \\
 & (\pi_{pq}(z_1), y) \in \mathbb{C}_1 \times \Omega_2 & \\
 & \downarrow & \nearrow X \\
 & \left(\frac{\pi_{pq}(z_1)}{p A_k(y)^{m-p'-q'}}, y \right) \in \mathbb{C}_1 \times \Omega_2 &
 \end{array}$$

Let $(X^*)^{-1}$ be the multi-valued analytic inverse of X^* , and let V be the open neighborhood of the origin in $\mathbb{C}^2$ chosen as in Proposition 6.6. For any fixed $x \in V$ and for two points $(z_\nu; y_\nu)$ and $(z_\mu; y_\mu)$ in the fiber $(X^*)^{-1}(x)$, by Theorem 7.1, there exists a path $\Gamma_{\nu\mu}$ in $M_{pq}^*(\alpha_k)$ which joins $(z_\nu; y_\nu)$ to $(z_\mu; y_\mu)$. Since the 2-form $\beta := dX_1^* \wedge dX_2^*$ does not vanish identically on $M_{pq}^*(\alpha_k)$, the complement of $\beta^{-1}(0)$ is open dense and connected in $M_{pq}^*(\alpha_k)$. Thus we can choose $\Gamma_{\nu\mu} \setminus \{(z_\nu; y_\nu), (z_\mu; y_\mu)\}$ in the complement of $\beta^{-1}(0)$. So, the two germs ρ_ν and ρ_μ of $(X^*)^{-1}$ at x with $\rho_\nu(x) = (z_\nu; y_\nu)$, $\rho_\mu(x) = (z_\mu; y_\mu)$ can be continued analytically each other along the path $X^*(\Gamma_{\nu\mu})$ in $\mathbb{C}^2$. We put

$$\begin{aligned}
 (7.2) \quad (X^*)^{-1}(x) &= (z(x); y(x)) \\
 X^{-1}(x) &= (t(x), y(x))
 \end{aligned}$$

where $t(x)$ is given by

$$(7.3) \quad t(x) = \pi_{pq}(z_1(x)) / [p A_k(y(x))^{m-p'-q'}].$$

Note that, if we restrict analytic continuations of $(X^*)^{-1}$ to the continuations along paths of the type $X^*(\Gamma_{\nu\mu})$, then $y(x) \in \Omega_2$, so $t(x)$ is well-defined by (7.3).

Let $Z_k(t, y)$ be the solution of

$$\begin{aligned}
 (7.4) \quad (\partial/\partial t)Z &= \Xi_1[(\partial/\partial \xi_1)F](\Phi_k) + \Xi_2[(\partial/\partial \xi_2)F](\Phi_k) \\
 &= p \Xi_1^p + a p \Xi_2^p
 \end{aligned}$$

$$Z(0, y) = \phi(y)$$

where $\Phi_k(t, y) = (X(t, y); \Xi(t, y))$ is the Hamilton flow issuing from $\rho_k(y)$ at $t=0$. We put

$$(7.5) \quad w_k(x) := Z_k(X^{-1}(x)).$$

By the theory of characteristic strips in § 2, $w_k(x)$ is an

analytic continuation of the solution $u_k(x; \Omega)$ of the Cauchy problem (1.1), so all germs of w_k are contained in germs of the maximal continuation $u_k^*(x; \Omega)$.

Now we assume that the conclusion of Theorem 1.3 is false, that is, $u_k^*(x; \Omega)$ is finitely many-valued. Then w_k is also finitely many-valued, so are $(\partial/\partial x_1)w_k$ and $(\partial/\partial x_2)w_k$. By (2.5) in Proposition 2.1 and by the uniqueness of continuations, we have

$$(7.6) \quad (\partial/\partial x_j)w_k(x) = \Xi_j(X^{-1}(x)) \quad \text{for } j=1,2.$$

Since

$$\Xi_1(t, y)^p + X_1(t, y)^q = \Xi_1(0, y)^p + X_1(0, y)^q = -a\psi(y),$$

we have $[(\partial/\partial x_1)w_k(x)]^p + x_1^q = -a\psi(y(x))$, which implies that $\psi(y(x))$ is finitely many-valued. Then the relations

$$A_k(y)^m = -a\psi(y) \quad \text{and} \quad B(y)^m = \psi(y)$$

yield that both $A_k(y(x))$ and $B(y(x))$ are finitely many-valued functions. Then, by the equations

$$x_1 = X_1^*(z(x); y(x)) = A_k(y(x))^p \sigma_{pq}(z_1(x))$$

$$x_2 = X_2^*(z(x); y(x)) = B(y(x))^p \sigma_{pq}(z_2(x)),$$

we deduce

$$(7.7) \quad \sigma_{pq}(z_1(x)) \quad \text{and} \quad \sigma_{pq}(z_2(x)) \quad \text{are finitely many-valued.}$$

From now on we fix $x \in V \setminus \{0\}$, and let $\{(z_{1\nu}, z_{2\nu}; y_\nu)\}$ be the sequence in Proposition 6.6. Then (7.7) yields that the set

$$(7.8) \quad \{\sigma_{pq}(z_{j\nu}); \nu \in \mathbb{N}\} \quad \text{is finite for } j=1,2.$$

By (7.8), taking a subsequences of $\{z_{1\nu}\}$ and $\{z_{2\nu}\}$ if necessary, we may assume that there exist constants c_j ($j=1,2$) so that

$$(7.9) \quad \sigma_{pq}(z_{j\nu}) = c_j \quad (j=1,2).$$

Since the restriction $\sigma_{pq}|_{F_{pq}}: F_{pq} \rightarrow \hat{\mathbb{C}}$ is a p -to-1 map, and since σ_{pq} is G_{pq} -invariant, we deduce from (7.9) the following inequality:

$$(7.10) \quad \#\{z_{1\nu}\}/G_{pq} \leq p.$$

Then, taking a subsequences of $\{z_{1\nu}\}$ if necessary, we may assume that $\{z_{1\nu}\}$ is contained in the same G_{pq} -orbit $G_{pq}(z_{11})$.

Finally we consider the finite sequence $\{z_{2\nu}; 1 \leq \nu \leq p+1\}$. Since (7.9) implies the inequality

$$\#[\{z_{2\nu}; 1 \leq \nu \leq p+1\}/G_{pq}] \leq p$$

as similar as (7.10), there exist $\nu, \mu \in \{1, 2, \dots, p+1\}$ with $\nu \neq \mu$ such that $z_{2\mu} \in G_{pq}(z_{2\nu})$. Thus we conclude that there exist ν and μ with $\nu \neq \mu$ such that

$$z_{1\mu} \in G_{pq}(z_{1\nu}) \text{ and } z_{2\mu} \in G_{pq}(z_{2\nu})$$

are compatible. This contradicts the assertion (ii) of Proposition 6.6. Thus the maximal analytic continuation $u_*(x; \Omega)$ is an infinitely many-valued function. It completes the proof of Theorem 1.3.

References

- [1] Beardon, A.F., *The geometry of discrete groups*, Graduate Text in Math. 91, Springer-Verlag, 1983.
- [2] Kametani, M., On multi-valued analytic solutions of first order non-linear Cauchy problems, Publ. R.I.M.S. **27** (1991) 1-131.
- [3] Poincaré, H., Théorie des groupes fuchsien, Acta Math. **1** (1882), 1-62.