

Weak Harnack inequality for fully nonlinear PDEs with superlinear growth terms in Du

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1 Introduction

In this note, we present the weak Harnack inequality for L^p -viscosity nonnegative supersolutions of fully nonlinear elliptic PDEs with unbounded coefficients and inhomogeneous terms. Moreover, we discuss the case when PDEs have superlinear growth terms in Du .

Throughout this paper, we suppose at least

$$p > \frac{n}{2}.$$

For measurable sets $U \subset \mathbf{R}^n$, we use the standard L^p -norm and $W^{2,p}$ -norm, $\|\cdot\|_{L^p(U)}$ and $\|\cdot\|_{W^{2,p}(U)}$, respectively. We will write $\|\cdot\|_p$ and $\|\cdot\|_{2,p}$ for them if there is no confusion. We also use the following notation:

$$L_+^p(U) = \{u \in L^p(U) \mid u \geq 0 \text{ a.e. in } U\}.$$

Let S^n be the set of $n \times n$ symmetric matrices with the standard order. For fixed uniform ellipticity constants $0 < \lambda \leq \Lambda$, we denote by $S_{\lambda,\Lambda}^n$ the set of all $A \in S^n$ such that $\lambda I \leq A \leq \Lambda I$. We then define the Pucci operators $\mathcal{P}^\pm$: for $X \in S^n$,

$$\mathcal{P}^+(X) = \max\{-\text{trace}(AX) \mid A \in S_{\lambda,\Lambda}^n\},$$

$$\mathcal{P}^-(X) = \min\{-\text{trace}(AX) \mid A \in S_{\lambda,\Lambda}^n\}.$$

Note that $X \rightarrow \mathcal{P}^+(X)$ (resp., $\mathcal{P}^-(X)$) is convex (resp., concave).

Let us consider the most general PDEs of second-order:

$$F(x, u, Du, D^2u) = f(x) \tag{1}$$

in Ω , where $\Omega \subset \mathbf{R}^n$ is an open set. Here, we suppose that $F : \Omega \times \mathbf{R} \times \mathbf{R}^n \times S^n \rightarrow \mathbf{R}$ and $f : \Omega \rightarrow \mathbf{R}$ are given measurable functions, and that F is continuous in the last three variables.

Definition 1.1 We call $u \in C(\Omega)$ an L^p -viscosity subsolution (resp., supersolution) of (1) in Ω if

$$\operatorname{ess\,lim\,inf}_{y \rightarrow x} \{F(y, u(y), D\phi(y), D^2\phi(y)) - f(y)\} \leq 0$$

$$\left(\text{resp., } \operatorname{ess\,lim\,sup}_{y \rightarrow x} \{F(y, u(y), D\phi(y), D^2\phi(y)) - f(y)\} \geq 0 \right)$$

whenever $\phi \in W_{\text{loc}}^{2,p}(\Omega)$ and $x \in \Omega$ is a local maximum (resp., minimum) point of $u - \phi$. Finally, we call $u \in C(\Omega)$ an L^p -viscosity solution of (1) in Ω if it is an L^p -viscosity subsolution and an L^p -viscosity supersolution of (1) in Ω .

In order to memorize the right inequality, we will often say that u is an L^p -viscosity (sub)solution of

$$F(x, u, Du, D^2u) \leq f(x)$$

when it is an L^p -viscosity subsolution of (1) for instance.

We also recall the notion of strong solutions.

Definition 1.2 We call $u \in W_{\text{loc}}^{2,p}(\Omega)$ an L^p -strong subsolution (resp., supersolution) of (1) in Ω if u satisfies

$$F(x, u(x), Du(x), D^2u(x)) - f(x) \leq 0 \quad (\text{resp., } \geq 0) \quad \text{a.e. in } \Omega.$$

Finally, we call $u \in W_{\text{loc}}^{2,p}(\Omega)$ an L^p -strong solution of (1) in Ω if the equality holds in the above.

Remark 1.3 Suppose that $p > p' > n/2$. It is trivial to see that u is an L^p -strong subsolution (resp., supersolution) of (1) in Ω , then it is an $L^{p'}$ -strong subsolution (resp., supersolution) of (1) in Ω . However, for L^p -viscosity solutions, the opposite implication holds true; if u is an $L^{p'}$ -viscosity subsolution (resp., supersolution) of (1) in Ω , then it is also an L^p -viscosity subsolution (resp., supersolution) of (1) a.e. in Ω .

2 Known results

Since the weak Harnack inequality is derived from the Aleksandrov-Bakelman-Pucci (ABP for short) maximum principle, we recall it from [8]. Thus, in what follows, we only consider the case when F is independent of u -variable.

Now we suppose the uniform ellipticity for F :

$$\mathcal{P}^-(X - Y) \leq F(x, \xi, X) - F(x, \xi, Y) \leq \mathcal{P}^+(X - Y)$$

for $x \in \Omega$, $\xi \in \mathbf{R}^n$, and $X, Y \in S^n$. A typical example of F is given by

$$F(x, \xi, X) := \max_{1 \leq i \leq M} \min_{1 \leq j \leq N} \{-\text{trace}(A(x; i, j)X) + \langle b(x; i, j), \xi \rangle\},$$

where for $M, N > 1$, functions $x \in \Omega \rightarrow A(x; i, j) \in S_{\lambda, \Lambda}^n$ and $x \in \Omega \rightarrow b(x; i, j) \in \mathbf{R}^n$ are measurable ($1 \leq i \leq M$, $1 \leq j \leq N$). Notice that the above F is non-convex and non-concave in general.

Under the uniform ellipticity assumption, if u is an L^p -viscosity solution of (1) in Ω , then it is also an L^p -viscosity solution of

$$\mathcal{P}^-(D^2u) + F(x, Du, O) \leq f(x), \text{ and } \mathcal{P}^+(D^2u) + F(x, Du, O) \geq f(x)$$

in Ω . Therefore, for the sake of simplicity, instead of (1), we shall study the following extremal PDEs: for $m \geq 1$,

$$\mathcal{P}^\pm(D^2u) \pm \mu(x)|Du|^m = \mp f(x), \quad (2)_{m, \pm}$$

where μ, f are often supposed to be nonnegative.

We recall the ABP maximum principle for L^n -strong solutions of $(2)_{1, -}$.

Proposition 2.1 (cf. [6]) There exist $C_k = C_k(n, \lambda, \Lambda) > 0$ ($k = 1, 2$) such that if $f, \mu \in L_+^n(\Omega)$, and $u \in C(\overline{\Omega}) \cap W_{\text{loc}}^{2, n}(\Omega)$ is an L^n -strong subsolution of $(2)_{1, -}$ in Ω , then it follows that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C_1 \exp(C_2 \|\mu\|_n^n) \|f\|_{L^n(\{u>0\})},$$

where $\{u > 0\} := \{x \in \Omega \mid u(x) > 0\}$.

Remark 2.2 In the above statement, we can replace $\|f\|_{L^n(\{u>0\})}$ by $\|f\|_{L^n(\Gamma[u])}$, where $\Gamma[u]$ is the upper contact set of u in Ω . See Gilbarg-Trudinger's book for the definition of $\Gamma[u]$.

From Proposition 2.1, it is trivial to obtain the corresponding result for L^p -strong supersolutions of $(2)_{1,+}$ by taking $v = -u$.

Now, we recall an L^p -viscosity version of the ABP maximum principle. We will use a constant $p_0 = p_0(n, \lambda, \Lambda) \in [\frac{n}{2}, n)$, which was introduced in [4]. We note that p_0 does not depend on Ω because we only need to solve extremal PDEs in balls. See [8] (also [5]) for the details.

Theorem 2.3 (cf. Proposition 2.8 and Theorem 2.9 in [8]) Assume that

$$q \geq p > p_0 \quad \text{and} \quad q > n \quad \text{hold.} \quad (3)$$

For $\mu \in L_+^q(\Omega)$, there exists $C_3 = C_3(n, \lambda, \Lambda, \|\mu\|_q) > 0$ such that if $f \in L_+^p(\Omega)$, and $u \in C(\overline{\Omega})$ is an L^p -viscosity subsolution of $(2)_{1,-}$ in Ω , then it follows that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C_3 \|f\|_{L^p(\{u>0\})}.$$

Remark 2.4 For more precise dependence of C_3 with respect to $\|\mu\|_q$, we refer to [8].

We next consider the case when $m > 1$ for $(2)_{m,-}$. In general, when $m > 1$, the ABP maximum principle for $(2)_{m,-}$ fails even for classical solutions (see [7, 8]).

Theorem 2.5 (Theorems 2.11 and 2.12 in [8]) Assume that (3) and

$$mq(n - p) < n(q - p) \quad (4)$$

holds. For $m > 1$, there exists $\delta_1 = \delta_1(n, \lambda, \Lambda, m, p, q) > 0$ satisfying the following property: for $\mu \in L_+^q(\Omega)$, there is $C_4 = C_4(n, \lambda, \Lambda, m, p, q, \|\mu\|_q) > 0$ such that if $f \in L_+^p(\Omega)$ satisfies

$$\|f\|_p^{m-1} \|\mu\|_q < \delta_1,$$

and $u \in C(\overline{\Omega})$ is an L^p -viscosity subsolution of $(2)_{m,-}$ in Ω , then it follows that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C_4 \|f\|_{L^p(\{u>0\})}.$$

Remark 2.6 We note that under (3), the relation (4) holds true when $p \geq n$. Thus, when $p \geq n$, we may choose arbitrary $m > 1$.

3 Weak Harnack inequality ($m = 1$)

From now on, we consider PDEs in cubes although it is possible to replace them by balls. We denote by Q_R the open cube with its center at the origin and with its length $R > 0$; $Q_R = (-\frac{R}{2}, \frac{R}{2}) \times \cdots \times (-\frac{R}{2}, \frac{R}{2})$.

Theorem 3.1 (Theorems 4.5 and 4.7 in [9]) Assume that (3) holds. There exists $r = r(n, \lambda, \Lambda) > 0$ satisfying the following property: for $\mu \in L_+^q(Q_2)$, there is $C_5 = C_5(n, \lambda, \Lambda, p, q, \|\mu\|_q) > 0$ such that if $f \in L_+^p(Q_2)$ and $u \in C(\overline{Q_2})$ is a nonnegative L^p -viscosity supersolution of $(2)_{1,+}$ in Q_2 , then it follows that

$$\left(\int_{Q_1} u^r dx \right)^{\frac{1}{r}} \leq C_5 \left\{ \inf_{Q_1} u + \|f\|_{L^p(Q_2)} \right\}.$$

Idea of proof: We first reduce the assertion to the case when $f \equiv 0$. For this purpose, due to our strong solvability (Theorem 2.3 in [9]), we find an L^p -strong supersolution $v \in C(\overline{Q_2}) \cap W_{\text{loc}}^{2,p}(Q_2)$ of

$$\mathcal{P}^-(D^2v) - \mu(x)|Dv| \geq f(x) \quad \text{in } Q_2$$

such that $0 \leq v \leq C_6 \|f\|_p$ in Q_2 for some $C_6 = C_6(n, \lambda, \Lambda, p, \|\mu\|_q) > 0$. Setting $w := u + v$, we see that w is an L^p -viscosity supersolution of $(2)_{1,+}$ in Q_2 with $f \equiv 0$. Thus, if we verify the assertion when $f \equiv 0$, then we find $C_7 = C_7(n, \lambda, \Lambda, p, q, \|\mu\|_q) > 0$ such that

$$\left(\int_{Q_1} u^r dx \right)^{\frac{1}{r}} \leq \left(\int_{Q_1} w^r dx \right)^{\frac{1}{r}} \leq C_7 \inf_{Q_1} w \leq C_7 \inf_{Q_1} u + C_7 C_6 \|f\|_p,$$

which concludes our proof.

Next, by considering $U := u/(\inf_{Q_1} u + \varepsilon)$ ($\forall \varepsilon > 0$), it is enough to show that $\inf_{Q_1} u \leq 1$ implies that $\int_{Q_1} u^r dx \leq C_0$ for some $r, C_0 > 0$, which are independent of u and $\varepsilon > 0$. (In fact, we can prove a weaker fact that $\inf_{Q_3} u \leq 1$ implies $\int_{Q_1} u^r dx \leq C_0$. However, we skip this because we will not go into the details of “cube-decomposition lemma”.)

By the strong solvability (Theorem 2.3 in [8]) again, we then choose an L^p -strong solution $\phi \in C(\overline{Q_2}) \cap W_{\text{loc}}^{2,p}(Q_2)$ of

$$\mathcal{P}^-(D^2\phi) + \mu(x)|D\phi| = \xi(x) \quad \text{in } Q_2$$

such that $0 \geq \phi$ in Q_2 , $-2 \geq \phi$ in Q_1 , and $\xi \in C(Q_2)$ with $\text{supp } \xi \subset Q_1$. Setting $V := -u - \phi$, we see that V is an L^p -viscosity subsolution of

$$\mathcal{P}^-(D^2V) - \mu(x)|DV| \leq -\xi(x) \quad \text{in } Q_2.$$

Hence, the ABP maximum principle (Theorem 2.3) implies

$$1 \leq \sup_{Q_1} V \leq C_3 \|\xi\|_{L^n(\{V>0\})} \leq C_3 \|\xi\|_\infty |\{x \in Q_1 \mid u(x) < M_1\}|,$$

where $M_1 = \sup(-\phi) > 1$. Therefore, we have

$$|\{x \in Q_1 \mid u(x) \geq M_1\}| \leq \theta$$

for some $\theta \in (0, 1)$. It is now enough to obtain

$$|\{x \in Q_1 \mid u(x) \geq M_1^k\}| \leq \theta^k \quad (5)$$

because this yields $\int_{Q_1} u^r dx \leq C_0$ for some $r, C_0 > 0$. To prove (5), we need a “cube-decomposition” lemma (e.g. in [1, 2]) but we omit this here.

4 Weak Harnack inequality ($m > 1$)

To follow the argument in section 3, we need to establish the existence of L^p -strong solutions of the associated extremal PDEs:

$$\mathcal{P}^+(D^2u) + \mu(x)|Du|^m = f(x).$$

In order to show the strong solvability of the above PDEs, we will apply the Schauder fixed point theorem. To this end, we use a recent result by Winter in [14] on the global $W^{2,p}$ -estimate of L^p -viscosity solutions of extremal PDEs:

$$\mathcal{P}^+(D^2u) = f(x) \quad \text{in } B_1$$

under “smooth” Dirichlet condition.

Our strong solvability result is as follows:

Theorem 4.1 (Theorem 3.1 in [10]) Assume that $\partial\Omega \in C^{1,1}$, $f \in L^p(\Omega)$, $\mu \in L^q(\Omega)$ and $\psi \in W^{2,p}(\Omega)$ hold. Assume also that one of the following conditions holds:

$$\begin{cases} (a) & q = \infty, p_0 < p, m(n-p) < n, \\ (b) & n < p \leq q < \infty, \\ (c) & p_0 < p \leq n < q < \infty, mq(n-p) < n(q-p). \end{cases} \quad (6)$$

There exists $\delta_2 = \delta_2(n, \lambda, \Lambda, p, q, m, \Omega) > 0$ such that if

$$\|\mu\|_q(\|f\|_p + \|\psi\|_{2,p})^{m-1} < \delta_2,$$

then there exists L^p -strong solutions $u \in W^{2,p}(\Omega)$ of

$$\begin{cases} \mathcal{P}^+(D^2u) + \mu(x)|Du|^m = f(x) & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega. \end{cases}$$

Moreover, there is $C_8 = C_8(n, \lambda, \Lambda, p, q, m, \Omega) > 0$ such that

$$\|u\|_{2,p} \leq C_8(\|f\|_p + \|\psi\|_{2,p}).$$

Idea of proof: It is enough to verify that we can apply the Schauder fixed point theorem to the mapping $T : v \in W^{1,r}(\Omega) \rightarrow Tv \in W^{2,p}(\Omega)$ (for some $r > 1$), where $w := Tv$ is an L^p -strong solution of

$$\begin{cases} \mathcal{P}^+(D^2w) + \mu(x)|Dv|^m = f(x) & \text{in } \Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases}$$

See [10] for the details.

Since we do not know if the weak Harnack inequality holds true even when μ is bounded, we will also consider this case. We refer to [13] for related results.

Theorem 4.2 (Theorem 4.2 in [10]) Assume that one of (6) holds. Assume also that

$$1 < m < 2 - \frac{n}{q}. \quad (7)$$

For $M > 0$, there exist $\delta_3 = \delta_3(n, \lambda, \Lambda, p, m, M) > 0$, $C_9 = C_9(n, \lambda, \Lambda, p, q, m) > 0$ and $r = r(n, \lambda, \Lambda, p, q, m) > 0$ such that if $f \in L_+^p(Q_2)$ and $\mu \in L_+^q(Q_2)$ satisfy

$$\|\mu\|_q(1 + \|f\|_p^{m-1}) < \delta_3,$$

and an L^p -viscosity supersolution $u \in C(Q_2)$ of $(2)_{m,+}$ in Q_2 satisfies $0 \leq u \leq M$ in Q_2 , then it follows that

$$\left(\int_{Q_1} u^r dx \right)^{\frac{1}{r}} \leq C_9 \left\{ \inf_{Q_1} u + \|f\|_p \right\}.$$

Remark 4.3 The hypothesis (7) is necessary when we use a scaling argument to apply the cube-decomposition lemma.

Idea of proof: In section 3, we used strong solvability of extremal PDEs $(2)_{1,\pm}$ twice in the idea of proof of Theorem 3.1. Instead, we need to utilize Theorem 4.1 here. In order to obtain (5), we have to modify the scaling argument in [1] (also [2]) as in [11].

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