

Spitzer's Markov chains and non-linear integral
equations of the Hammerstein type

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Spitzer [5] has introduced Markov chains, whose space of "time parameters" is an infinite tree T , and whose state space is a set $\{-1, +1\}$. He investigates Gibbs distributions on T that are Markov chains of such construction. We generalize Spitzer's results to a case when the state space is a compact set. If the state space consists of two points as in a case of Spitzer, all Markov chains are reversible. So, in that case, the "time parameter" space T need not be equipped with a direction. But, since Markov chains may not be reversible in our case, we must introduce a direction into T .

Let T be an infinite directed tree, in which s branches emanate from every vertex and n branches flow into every vertex. Generalizing Spitzer's construction [5], we define Markov chains whose space of "time parameters" is the tree T , and whose state space is a compact metric measure space $(X, \mathcal{B}, \mu)$. Let $F(x, y)$ be a measurable function on $X \times X$. We assume neither boundedness nor symmetry $F(x, y) = F(y, x)$ of F . If F satisfies

$$(A,1) \quad \iint e^{-(n+s)F(x,y)} \mu(dx) \mu(dy) < +\infty$$

or

$$(A,2) \quad \sup_x \{ \int e^{-F(x,y)} \mu(dy), \int e^{-F(y,x)} \mu(dy) \} < +\infty,$$

then we can define Gibbs distributions on T with the potential F ([1],[3]). Let $\mathcal{M}(F)$ be the set of Markov chains that are Gibbs distributions with the potential F .

Putting $X(x,M) = \{y \in X; F(x,y) \leq M\}$ and $X^*(x,M) = \{y \in X; F(y,x) \leq M\}$,

we assume that there exist M and an integer k such that

$$(A,3) \quad \begin{cases} \mu^k\{(x_1, x_2, \dots, x_k); \mu(X \setminus \bigcup_{i=1}^k X(x_i, M)) = 0\} > 0, \\ \mu^k\{(x_1, x_2, \dots, x_k); \mu(X \setminus \bigcup_{i=1}^k X^*(x_i, M)) = 0\} > 0. \end{cases}$$

Theorem 1. Under the above assumptions, a Markov chain with the transition density $p(x,y)$ belongs to $\mathcal{M}(F)$, if and only if $p(x,y)$ has the expression;

$$p(x,y) = \lambda(s,n) u(x)^{-1} u(y)^s v(y)^{n-1} e^{-F(x,y)},$$

where $\lambda(s,n)$ is the Perron-Frobenius eigenvalue of the kernel $e^{-F(x,y)}$ if $s=n=1$, and $\lambda(s,n)=1$ if otherwise, and u and v are positive measurable functions satisfying the following integral equations of the Hammerstein type;

$$(*) \quad \begin{cases} u(x) = \lambda(s,n) \int e^{-F(x,y)} u(y)^s v(y)^{n-1} \mu(dy), \\ v(x) = \lambda(s,n) \int e^{-F(y,x)} u(y)^{s-1} v(y)^n \mu(dy), \\ \int u^s v^{n-1} d\mu = \int u^{s-1} v^n d\mu, \\ \int u d\mu = \int v d\mu = 1, \text{ if } s=n=1, \\ \int u^s v^n d\mu < +\infty. \end{cases}$$

The expression is unique. The invariant density $h(x)$ of $p(x,y)$ has the form

$$h(x) = c u(x)^s v(x)^n,$$

where c is a normalizing constant.

Theorem 2. The set $\mathcal{M}(F)$ is not empty, either if

(A,4) $\int e^{-F(x,y)} \mu(dy)$ and $\int e^{-F(y,x)} \mu(dy)$ do not depend on x ,

or if

(A,5) $\sup_x \{ \int e^{-(n+s)F(x,y)} \mu(dy), \int e^{-(n+s)F(y,x)} \mu(dy) \} < +\infty$

and

(A,6) $\sup_x \{ \int e^{(n+s)(n+s-2)F(x,y)} \mu(dy), \int e^{(n+s)(n+s-2)F(y,x)} \mu(dy) \} < +\infty$

We say that $p(x,y)$ is reversible, if $h(x)p(x,y) = h(y)p(y,x)$.

We say that a potential F is uniformly symmetrizable, if there exists a symmetric potential $\hat{F}$ such that $\sup_{x,y} |F(x,y) - \hat{F}(x,y)| < +\infty$

and such that $\hat{F}$ determines the Hamiltonian which is equivalent to that determined by F .

Theorem 3. We assume (A,3), (A,5) and

(A,6)' $\sup_x \{ \int e^{(n+s)(n+s-2)'F(x,y)} \mu(dy), \int e^{(n+s)(n+s-2)'F(y,x)} \mu(dy) \} < +\infty$,

where $(n+s)(n+s-2)' = \max\{(n+s)(n+s-2), 1\}$. Then, the following three statements are equivalent to each other.

- 1) A potential F is uniformly symmetrizable.
- 2) There exists a reversible Markov chain in $\mathcal{M}(F)$.
- 3) All Markov chains in $\mathcal{M}(F)$ are reversible.

If F is symmetric, u and v in Theorem 1 can be regarded as positive eigenfunctions of the kernel $e^{-F(x,y)} u(y)^{s-1} v(y)^{n-1}$, which implies $u = c v$. Therefore, (*) reduces to

$$(**) \begin{cases} u(x) = \lambda(s,n) \int e^{-F(x,y)} u(y)^{s+n-1} \mu(dy), \\ \int u d\mu = 1, \text{ if } s=n=1, \\ \int u^{s+n} d\mu < +\infty. \end{cases}$$

In case the state space is the unit circle S^1 which we identify with $[0,1)$, we can construct an example

$$u(x) = \int_0^1 \Gamma(x-y) u(y)^2 dy,$$

where Γ is positive, even and of C^∞ -class and u is positive and non-constant. The Markov chain in $\mathcal{M}(-\log \Gamma)$ determined by u is not rotation-invariant. On the contrary, all Gibbs distributions in Z^2 with the state space S^1 , whose potential is of finite range, of C^2 -class and rotation-invariant, are also rotation-invariant [2].

In the following we consider potentials with the form βF , where $\beta > 0$ is called the reciprocal temperature.

Theorem 4. Assume (A,3) and

$$(A,7) \quad \sup_x \{ \int e^{|F(x,y)|} \mu(dy), \int e^{|F(y,x)|} \mu(dy) \} < +\infty.$$

If β is sufficiently small, then $\mathcal{M}(\beta F)$ consists of a unique Markov chain.

Theorem 5. Let X be a finite set and let $\mu(\{i\}) > 0$ for all $i \in X$. Let F be a symmetric potential on X satisfying

$$(A,8) \quad F(i,j) > F(j,j) + \frac{1}{n+s-1} |F(i,i) - F(j,j)|$$

for all $i \neq j \in X$. Then, the number of Markov chains in $\mathcal{M}(\beta F)$ is equal to $2^{\#X} - 1$ for sufficiently large β , if $s+n > 2$.

Details are found in [4].

References

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