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Functional equations of polylogarithms

Herbert Gangl

0. Introduction

The polylogarithm functions (for short "polylogarithms"), defined by the power series

$$Li_m(z) = \sum_{n\geq 1} \frac{z^n}{n^m}, \qquad |z| < 1, \tag{1}$$

were long considered to be "just another class" of special functions with some interesting properties but they lived rather a life of mathematical outlaws. They were investigated, though, by several prominent mathematicians (Leibniz, Abel, Kummer), mainly in the 19th century.

Since the last 15 years, they have begun to conquer much mathematical ground by providing connections between formerly unrelated fields of mathematical research, often due to their strange "internal structure" which is encoded in the functional equations they satisfy.

One of the first unexpected occurrences was encountered by S. Bloch in his fundamental paper [Bl] where he introduced a certain abelian group $\mathcal{B}(F)$ which should give a constructive description of a group coming from the algebraic K-theory of a number field F. Algebraic K-theory gives a sequence of important invariants for a field (in fact, for any ring) F, which are defined non-constructively and which are very difficult to compute. Bloch showed that the dilogarithm function Li_2 (or rather some modification of it) is a map on this explicitly given group $\mathcal{B}(F)$. Suslin [Su] later proved that Bloch's map connecting the two groups is in fact a quasiisomorphism. As a consequence one can interpret the functional equations fulfilled by the dilogarithm as reflecting the arithmetic of this K-group.

The analogous picture for each m-logarithm $Li_m(z)$ of order $m > 2$ (i.e. a constructively given candidate $\mathcal{B}_m(F)$ for some higher K-group on which the—suitably modified—m-logarithm is defined) was established by Zagier [Z1], and one expects that every functional equation fulfilled by some polylogarithm mirrors some structure of the corresponding higher K-group.

A proof of the corresponding result in the case $m = 3$, namely that Zagier's candidate $\mathcal{B}_3(F)$ is quasiisomorphic to a certain K-group, was given by Goncharov [G1], [G2].

1. Properties of polylogarithms

In this paragraph we want to give some of the properties of the (classical) polylogarithms, beginning with the dilogarithm $Li_2(z)$ as defined above.

1-1. The dilogarithm

The *dilogarithm* $Li_2(z)$ as defined in (1) can be analytically continued to the cut complex plane via the integral representation

$$Li_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt, \quad z \in \mathbf{C} - [1, \infty),$$

and one can think of it as a multivalued function on $\mathbf{C} - \{1\}$ or rather as a function on the universal cover of $\mathbf{C} - \{1\}$.

There are only a few special values known explicitly, we give a typical non-trivial

Example: $Li_2(\frac{\sqrt{5}-1}{2}) = \frac{\pi^2}{10} - \log^2(\frac{\sqrt{5}+1}{2})$,

there are also several very special combinations of dilogarithm values called *ladders* involving only the powers of some algebraic number α.

Example: For $\alpha = -2\cos\frac{4\pi}{9}$ we have $Li_2(\alpha^3) - 3Li_2(\alpha^2) - 3Li_2(\alpha) = \frac{\pi^2}{18}\left(= \frac{1}{3}Li_2(\alpha^0)\right)$.

On the other hand there are lots of functional equations—so-called "trivial" ones: the "inversion relation"

$$Li_2(z) + Li_2\left(\frac{1}{z}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2(-z), \quad z \in \mathbf{C} - [0, \infty),$$

and the "distribution relation"

$$Li_2(z^n) = n \sum_{\zeta^n = 1} Li_2(\zeta z), \quad |z| < 1,$$

but also "non-trivial" ones like

$$Li_2(z) + Li_2(1-z) = \frac{\pi^2}{6} - \log(z)\log(1-z), \quad z \in \mathbf{C} - [1, \infty) - (\infty, 0],$$

the most important (because in a way basic) one being the "five term relation" discovered by Spence, Abel and others (we give only one of many possible forms):

$$Li_2\left(\frac{x}{1-x}\frac{y}{1-y}\right) - Li_2\left(\frac{y}{1-x}\right) - Li_2\left(\frac{x}{1-y}\right) + Li_2(x) + Li_2(y) = -\log(1-x)\log(1-y),$$
$$|x| + |y| < 1.$$

This functional equation is also true (after appropriate substitutions of the arguments like in the inversion relation which involve new logarithmic terms and constants) on the other regions in $\mathbf{C}$. It is known to give all functional equations where the arguments are rational functions of independent parameters (Wojtkowiak).

1-2. The "higher" polylogarithms

The *polylogarithms* $Li_m(z)$ of higher order $m > 2$ defined in (1) share many properties with the dilogarithm:

- they can be analytically continued to the cut complex plane via

$$Li_m(z) = \int_0^z \frac{Li_{m-1}(t)}{t} dt, \quad z \in \mathbf{C} - [1, \infty). \tag{2}$$

- there are very few special values explicitly known (for small m),

- there are several ladders for certain algebraic numbers known (for small m),
- there are "trivial" functional equations for all m (i.e. an inversion relation and distribution relations which are very similar to the ones for $m = 2$), and
- there are "non-trivial" functional equations known for $m \leq 7$, (and expected to exist for all m). We have the following "high score table":
  $m = 3$: Spence (1809), Kummer (1840), Goncharov (1990), Wojtkowiak (1990),
  $m = 4, 5$: Kummer (1840), Wechsung (1965), Lewin (1986),
  $m = 6, 7$: the author (1990/91).

Here we have included Goncharov because his functional equation is in a way as basic as the five term relation for the dilogarithm, Wojtkowiak because his equation is of a general type, Wechsung since he generalized Kummer's approach (and was able to find new ones) and Lewin for his skillful use of ladders to produce new equations.

By differentiating functional equations of polylogarithms of higher order m one automatically gets functional equations for polylogarithms of order $k < m$ (cf. (2)), but in our computer search we also found new equations for orders $2 \leq m \leq 6$ which are not derivable from the higher-order ones known so far.

For more details on properties of these "classical" polylogarithms cf. [L1],[L2].

2. One-valued versions of polylogarithms

2.1 The Bloch-Wigner dilogarithm

There is a variant of the dilogarithm function called the Bloch-Wigner dilogarithm which has several useful properties, namely

(i) it is one-valued and defined on the projective line $P^1(\mathbf{C})$,
(ii) it is continuous on $P^1(\mathbf{C})$ and real-analytic on $P^1(\mathbf{C}) - \{0, 1, \infty\}$,
(iii) it satisfies functional equations without "lower order terms" like products of logarithms and π.

The Bloch-Wigner dilogarithm is defined as

$$D(z) = D_2(z) = \mathrm{Im}\big(Li_2(z) + \log|z|\log(1-z)\big), \qquad z \in \mathbf{C} - \{0, 1\},$$

and $D(0) = D(1) = D(\infty) = 0$. (Im denotes the imaginary part.)

The five term relation for $D(z)$ reads

$$D\left(\frac{x}{1-x}\frac{y}{1-y}\right) - D\left(\frac{y}{1-x}\right) - D\left(\frac{x}{1-y}\right) + D(x) + D(y) = 0,$$
$$x, y \in P^1(\mathbf{C}),\ (x, y) \notin \{(0,1),(1,0)\}.$$

D is (up to a constant factor) the only measurable function on $\mathbf{C}$ which fulfills this five term relation and is therefore characterized by it (cf. [Bl]).

This function D first appeared in work of D. Wigner on the cohomology of $GL(2, \mathbf{C})$ and was then encountered by S. Bloch in his attempt to make the so-called Borel regulator map explicit (cf. sec. 3).

2.2 The Bloch-Wigner-Ramakrishnan-Zagier-Wojtkowiak polylogarithm

There are also one-valued variants P_m of each m-logarithm function; their name "Bloch-Wigner-Ramakrishnan-Zagier-Wojtkowiak polylogarithm" stems from the fact that Ramakrishnan first defined a higher analog of the Bloch-Wigner dilogarithm implicitly (i.e. without giving actual formulae) and then Zagier and Wojtkowiak independently found an explicit form. Zagier actually gave two different versions which can be used interchangeably in our context (they agree as functions on a certain group on which we only want to consider them). One of them is defined as follows:

$$P_m(z) = \begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\Big(\sum_{k=0}^{m-1} \frac{2^k B_k}{k!} \log^k |z|\ Li_{m-k}(z)\Big), \quad \begin{matrix}\mathrm{Re\ if}\ m\ \mathrm{odd},\\ \mathrm{Im\ if}\ m\ \mathrm{even},\end{matrix} \qquad |z| \leq 1,\ z \in \mathbf{C}.$$

Here B_k denotes the k-th Bernoulli number ($B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, \ldots$) and Re the real part.
For $|z| > 1$ define it as $P_m(z) = (-1)^{m-1} P_m(\frac{1}{z})$.

These functions P_m for $m > 2$ have the same properties (i),(ii) and (iii) as listed for the Bloch-Wigner dilogarithm, the "lower order terms" in (iii) being products of logarithms, constants and $Li_k(z)$ for $k < m$. For more details cf. [Z1].

3. The polylogarithms in algebraic number theory

In this section we want to give—as an "application"—a rough picture of the context in which the functional equations of polylogarithms appear in a crucial way (it is not needed in the next section apart from the definition for $\Lambda^2(A)$ and the notation for elements in this group).

Algebraic K-theory gives a sequence of abelian groups $K_n(F)$ for each $n \geq 0$ and each ring F. These groups are defined in a highly inconstructive way (cf. [Q]) but in some cases one has a certain "explicit" amount of information about them.

For number fields F one knows about the rank of these groups by the work of Borel [Bo], namely: let r_1 be the number of real places and r_2 the number of complex places up to conjugacy, i.e. $[F : \mathbf{Q}] = r_1 + 2r_2$, put $r_+ := r_1 + r_2$ and $r_- := r_2$, then

$$\mathrm{rank}(K_n(F)) = \begin{cases} 0, & \text{if } n \text{ is even} \\ r_+, & \text{if } n \equiv 1 \pmod 4 \\ r_-, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Borel constructed a "regulator map" reg_m from $K_{2m-1}(F)$ to $\mathbf{R}^{r_\pm}$ (the definition of which we don't want to give here, cf. [Bo]) and showed that the image of this map is a lattice with covolume $q \cdot \zeta_F(m) \cdot \sqrt{|D_F|}/\pi^{mr_\mp}$ for some $q \in \mathbf{Q}$. (Here D_F denotes the discriminant of F.)

Bloch tried to make this Borel regulator map explicit by defining (in a uniform way for all number fields) a certain abelian group $\mathcal{B}(F)$ which he expected to capture the structure of the algebraic K-group $K_3(F)$, namely:
Define for an abelian group A the "second exterior power" as the quotient

$$\Lambda^2(A) = A \otimes A \,/\, \langle a \otimes b + b \otimes a \mid a, b \in A \rangle, \tag{2}$$

and write $a \wedge b$ for the image of $a \otimes b$ in $\Lambda^2(A)$.
Then define the following subgroup of $\mathbf{Z}[P^1(F)]$ (=the free abelian group on $P^1(F)$) reflecting the five term relation (and its "degenerations") for the dilogarithm: for $x \in F$ we write $[x]$ for the corresponding generator in $\mathbf{Z}[P^1(F)]$,

$$\mathcal{R}(F) = \left\{ [x]+[y]-\left[\frac{x}{1-y}\right]-\left[\frac{y}{1-x}\right]+\left[\frac{x}{1-x}\frac{y}{1-y}\right],\ [x]+\left[\frac{1}{x}\right],\ [x]+[1-x],\ [0],[1],[\infty] \right| \\ x,y \in P^1(F),\ (x,y) \notin \{(0,1),(1,0)\} \Big\}.$$

The Bloch group for a number field F and also for $F = \overline{\mathbf{Q}}$ and $F = \mathbf{C}$ is given by

$$\mathcal{B}(F) = \frac{\{\sum_i n_i[x_i] \in \mathbf{Z}[P^1(F)] \mid \sum_i n_i\ (x_i \wedge (1-x_i)) = 0 \mod \text{torsion} \in \Lambda^2(F^\times)\}}{\mathcal{R}(F)}$$

One immediately computes that $\mathcal{R}(F)$ is in fact a subgroup of the group given in the "numerator " of the definition for $\mathcal{B}(F)$. Moreover, extending the (Bloch-Wigner) dilogarithm linearly to a function on $\mathbf{Z}[P^1(F)]$ and using the fact that it fulfills the five term relation (and its "degenerations") and it is clear that the (Bloch-Wigner) dilogarithm can be regarded as a function on the quotient group $\mathcal{B}(F)$.

The complex embeddings $\sigma : F \hookrightarrow \mathbf{C}$ induce maps $\mathcal{B}(F) \hookrightarrow \mathcal{B}(\mathbf{C})$, and by composition with the above we get a map

$$\mathcal{B}(F) \hookrightarrow \oplus_{i=1}^{r_2} \mathcal{B}(\mathbf{C}) \xrightarrow{(D,\ldots,D)} \mathbf{R}^{r_2}.$$

Theorem (Suslin [Su]+Bloch [Bl]):
There is an isomorphism ψ (up to tensoring with $\mathbf{Q}$) between $K_3(F)$ and $\mathcal{B}(F)$ for all number fields F which makes the following diagram commute:

$$\begin{array}{ccc} K_3(F) & \xrightarrow{\psi} & \mathcal{B}(F) \\ \downarrow reg_2 & & \downarrow (D,\ldots,D) \\ \mathbf{R}^{r_2} & = & \mathbf{R}^{r_2}. \end{array}$$

As a corollary (combining the results of Borel, Bloch and Suslin), one has the following **"Rigidity" fact:**
For each number field F one can express $\zeta_F(2)$ (up to a known factor) as a determinant with entries $\sum n_i D(x_i)$ where $x_i \in F$ and $n_i \in \mathbf{Q}$.

A slightly weaker statement had been proved earlier by Zagier [Z2].

Essentially the same picture has been established (conjecturally) by Zagier [Z1] for $K_{2m-1}(F)$, reg_m, P_m and the following generalized Bloch group $\mathcal{B}_m(F)$. The idea is to take again the free abelian group on $P^1(F)$, to impose some suitable algebraic condition which

should be fulfilled by functional equations for the m-logarithm and to divide by "the subgroup generated by specializing the functional equations defined over F (or, alternatively, over $\mathbf{Q}$) for the m-logarithm to arguments in F."

More precisely, we set

$$\mathcal{R}_2(F) = \mathcal{R}(F), \quad \mathcal{B}_2(F) = \mathcal{B}(F),$$

and for $m > 2$ define the following map

$$\begin{aligned} \delta_m^F : \mathbf{Z}[P^1(F)] &\longrightarrow \mathcal{B}_{m-1}(F) \otimes F^\times \\ [x] &\longmapsto [x]_{m-1} \otimes x \end{aligned}$$

and $\delta_m^F([0]) = \delta_m^F([1]) = \delta_m^F([\infty]) = 0$, where $[x]_{m-1}$ is the projection of $[x]$ onto $\mathcal{B}_{m-1}(F)$.

We write $F(t)$ for the function field over F in one variable, for which we can define $\delta_m^{F(t)}$ in the same way.

Each element $t_0 \in F$ defines a "specialization map"

$$\begin{aligned} \mathbf{Z}[P^1(F(t))] &\longrightarrow \mathbf{Z}[P^1(F)] \\ \xi(t) = \sum n_i[x_i(t)] &\longmapsto \xi(t_0) = \sum n_i[x_i(t_0)] \end{aligned}$$

Now define

$$\mathcal{R}_m(F) = \{\xi(0) - \xi(1) \mid \xi(t) \in \ker\left(\delta_m^{F(t)}\right)\}$$

and finally

$$\mathcal{B}_m(F) = \ker(\delta_m^F)/\mathcal{R}_m(F).$$

With these notations, we can state a version of

Zagier's conjecture:

There is a quasiisomorphism ψ making the following diagram commute

$$\begin{array}{ccc} K_{2m-1}(F) & \xrightarrow{\psi} & \mathcal{B}_m(F) \\ \downarrow \mathrm{reg}_m & & \downarrow (P_m, ..., P_m) \\ \mathbf{R}^{r_\mp} & = & \mathbf{R}^{r_\mp}. \end{array}$$

As a corollary, one would have:

Rigidity (conjecture for $m > 3$):

For each number field F one can express $\zeta_F(m)$ (up to a known factor) as a determinant with entries $\sum n_i P_m(x_i)$ where $x_i \in F$ and $n_i \in \mathbf{Q}$.

Goncharov [G1] has proved Zagier's conjecture in the case $m = 3$ and has also given an approach for handling the general case. Beilinson and Deligne ([B-D]) have given a map $\mathcal{B}_m(F) \longrightarrow K_{2m-1}(F)$ such that the (mirrored) diagram above commutes.

4. An algebraic criterion for functional equations

We have seen that it is desirable to find functional equations for higher polylogarithms, and we now state a criterion for their existence which was used to actually find new ones (using a computer).

Recall the definition of $\Lambda^2(A)$ for an abelian group A (sec. 3, (2)) and the notation $a \wedge b$ for the image of $a \otimes b$ under the natural projection $A \otimes A \longrightarrow \Lambda^2(A)$.

Let $\beta_m : \mathbf{Z}[\mathbf{C}(t)^\times] \longrightarrow \mathrm{Sym}^{m-2}(\mathbf{C}(t)^\times) \otimes \Lambda^2(\mathbf{C}(t)^\times)$ be the homomorphism given on generators by

$$\beta_m([x]) = [x]^{m-2} \otimes x \wedge (1-x),$$

where $[x]^k$ denotes the image of $x^{\otimes k}$ under the natural map $\bigotimes_{j=1}^{k} \mathbf{C}(t)^\times \longrightarrow \mathrm{Sym}^k(\mathbf{C}(t)^\times)$ (i.e. sum over the action of the symmetric group Σ_k which permutes the factors in the tensor product).

Let P_m be extended linearly to a map on $\mathbf{Z}[\mathbf{C}(t)^\times]$. Then we have the following

Criterion (Zagier [Z1]): For $\xi(t) \in \mathbf{Z}[\mathbf{C}(t)^\times]$ the following holds:

$$\beta_m(\xi(t)) = 0 \quad \Longrightarrow \quad P_m(\xi(t)) = \text{constant}.$$

Thus in order to find functional equations it is sufficient to take a (finite) set $\{x_i(t)\} \subset \mathbf{C}(t)^\times$, determine a basis of a finite dimensional subspace of the vector space $\left(\mathrm{Sym}^{m-2}(\mathbf{C}(t)^\times) \otimes \Lambda^2(\mathbf{C}(t)^\times)\right) \otimes \mathbf{Q}$ (in order to use vector space argumentation we tensor with $\mathbf{Q}$), namely a subspace in which all the $\beta_m([x_i(t)])$ lie, and then look for linear dependencies among these images.

Example 1. $m = 3$,

$$2\beta_3\left([t] + [\tfrac{1}{1-t}] + [1-\tfrac{1}{t}]\right) = 0 \quad \text{in } \mathbf{C}(t)^\times \otimes \Lambda^2(\mathbf{C}(t)^\times).$$

(Keep in mind that we write the "group addition" multiplicatively, so since $1 \in \mathbf{C}(t)^\times$ is the neutral element we have e.g. $2\big(y \otimes z \wedge (-1)\big) = y \otimes z \wedge (-1)^2 = 0$.) Indeed:

$$\begin{aligned}
&2\beta_3\left([t] + [\tfrac{1}{1-t}] + [1-\tfrac{1}{t}]\right) \\
&= 2\Big(\big(t \otimes t \wedge (1-t)\big) + \big((1-t)^{-1} \otimes (1-t)^{-1} \wedge (-1)\cdot t \cdot (1-t)^{-1}\big) \\
&\quad + \big((-1)\cdot(1-t)\cdot t^{-1} \otimes (-1)\cdot(1-t)\cdot t^{-1} \wedge t^{-1}\big)\Big) \\
&= 2\Big(\big(t \otimes t \wedge (1-t)\big) \\
&\quad + \big((1-t)\otimes(1-t)\wedge(-1) + (1-t)\otimes(1-t)\wedge t - (1-t)\otimes(1-t)\wedge(1-t)\big) \\
&\quad - \big((-1)\otimes(-1)\wedge t - (-1)\otimes(1-t)\wedge t + (-1)\otimes t\wedge t \\
&\quad - (1-t)\otimes(-1)\wedge t \ (1-t)\otimes(1-t)\wedge t + (1-t)\otimes t\wedge t \\
&\quad + t\otimes(-1)\wedge t + t\otimes(1-t)\wedge t - t\otimes t\wedge t\big)\Big) \\
&= 0.
\end{aligned}$$

Example 2. $m = 4$. Let

$$\begin{aligned}\xi(t) = \; & 2([t(1-t)] + [-\tfrac{t}{(1-t)^2}] + [-\tfrac{1-t}{t^2}]) \\ & -3([\tfrac{1}{1-t+t^2}] + [\tfrac{(1-t)^2}{1-t+t^2}] + [\tfrac{t^2}{1-t+t^2}]) \\ & -6([-\tfrac{1-t+t^2}{t(1-t)}] + [\tfrac{1-t+t^2}{t}] + [\tfrac{1-t+t^2}{1-t}]).\end{aligned}$$

Then $2\beta_4(\xi(t)) = 0$ in $\mathrm{Sym}^2(\mathbf{C}(t)^\times) \otimes \Lambda^2(\mathbf{C}(t)^\times)$.

The verification is left to the reader.

There is a more conceptual way to prove that such a linear combination (as in Example 2) among the $\beta_4([x_i(t)])$ must hold: on the submodule

$$V = \langle \pm t^a(1-t)^b(1-t+t^2)^c \mid a, b, c \in \mathbf{Z} \rangle \subset \mathbf{C}(t)^\times$$

we have an Σ_3-action induced by the Σ_3-action on $\mathbf{C}(t)^\times$ generated by $(t \longmapsto \frac{1}{t})$ and $(t \longmapsto 1-t)$.

All the 9 arguments $x_i(t)$ in $\xi(t)$ given above have the property that $1 - x_i(t)$ also lies in V and the arguments in each row form a full orbit under the Σ_3-action.

Viewing $\left(\mathrm{Sym}^2(V) \otimes \Lambda^2(V)\right) \otimes \mathbf{Q}$ as a representation space for Σ_3 we can show via a dimension argument (namely the subspace of Σ_3-invariants has dimension 2) that the β_4-images of the three formal sums over the Σ_3-orbits must satisfy a linear dependence relation—and in fact it is (essentially) unique, given by the above.

Using Zagier's criterion, we obtain a functional equation for the 4-logarithm, and one easily finds that the constant (occurring in the formulation of the criterion) must be zero in this case.

In many of the functional equations that we've found there is some non-trivial symmetry group acting on the set of arguments, and in several cases the use of linear representation theory of these finite groups is actually sufficient to prove that a (non-trivial) functional equation must hold (regardless of coefficient considerations)—given that one has found enough $x_i(t)$ such that all the $\beta_m([x_i(t)])$ lie in a "low"-dimensional subvector space. In general, though, it is not sufficient.

For further examples cf. also [Z3] and [L2] (Ch. 16.4).

The functional equations for higher m-logarithms are of a rather complex nature, so we refrain from giving them here—there is e.g. a functional equation for the 7-logarithm in two variables with 274 terms (which are grouped into 24 orbits under the action of a group of order 18).

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Address:
812 Fukuoka, Kyushu University
Faculty of Science
Dept. of Mathematics