

# The Nonequivariant Coherent-Constructible Correspondence for Toric Surfaces

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## Motivation -Homological Mirror Symmetry-

Mirror symmetry: A mysterious relationship between symplectic geometry and complex geometry from string theory. (e.g. period integrals and Gromov-Witten invariants)

Homological mirror symmetry: An explanation of mirror symmetry from categorical viewpoint.

derived category of coherent sheaves on  $X$   
 $\cong$  Fukaya-type category of the mirror of  $X$

Ex) HMS for  $\mathbb{P}^1 \leftrightarrow (\mathbb{C}^\times, W := z + \frac{1}{z})$

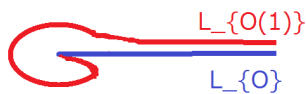
① Beilinson's theorem

$$D^b(\text{coh}\mathbb{P}^1) \cong \langle \mathcal{O}, \mathcal{O}(1) \rangle$$

② HMS for  $\mathbb{P}^1$

$$D^b(\text{coh}\mathbb{P}^1) \cong DFS(\mathbb{C}^\times, W)$$

Lefschetz thimbles in  $\mathbb{C}^\times$



## Nadler-Zaslow microlocalization

$X$ : real analytic manifold

$$D_c^b(X) \cong DFuk(T^*X).$$

LHS: the bdd derived cat of constructible sheaves on  $X$

How to see this equivalence?

## Constructible sheaf

"Locally constant sheaf on each stratum."

Ex) constant sheaf  $\mathbb{C}_{S^1}$ , skyscraper sheaf  $\mathbb{C}_x, p_*\mathbb{C}_{[0,1]}$   
where  $p: \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \cong S^1$ .

## Microsupport $\subset T^*X$

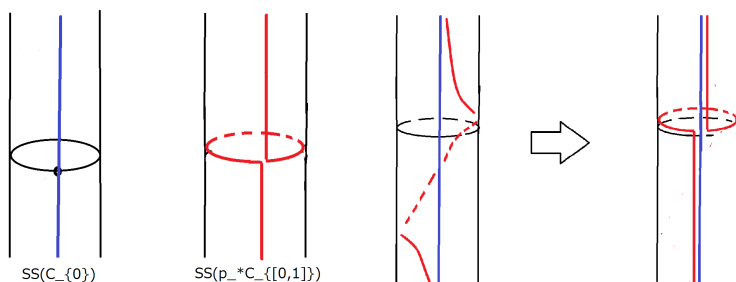
The set of codirections of which "the cohomology of the sheaf does not extend isomorphically".

$$(\text{Sh}(S^1) \ni \mathcal{E} \mapsto \text{SS}(\mathcal{E}) \subset T^*S^1).$$

Ex)  $T^*S^1 \cong \mathbb{C}^\times$

NZ-microlocalization:

$$\begin{aligned} L_{\mathcal{O}} &\mapsto \mathbb{C}_0 \\ L_{\mathcal{O}(1)} &\mapsto \mathbb{C}_{S^1} \end{aligned}$$



$$D^b(\text{coh}\mathbb{P}^1) \xrightarrow{\cong} \langle \mathbb{C}_0, p_*\mathbb{C}_{(0,1)} \rangle \subset D_c^b(S^1) \cong DFuk(T^*S^1).$$

After the works of Bondal and Nadler-Zaslow, Fang-Liu-Treumann-Zaslow proposed the (nonequivariant) coherent-constructible correspondence.

## Notations

- $M, N$ : free abelian groups of finite rank which are dual each other.
- $\Sigma$ : a smooth complete fan defined in  $N_{\mathbb{R}}$ .
- $X_{\Sigma}$ : the toric variety associated with  $\Sigma$ .
- $\overline{\Lambda}_{\Sigma} := \bigcup_{\sigma \in \Sigma} p(\sigma^\perp) \times (-\sigma)$ : the subset of  $T \times N_{\mathbb{R}}$  where  $p: M_{\mathbb{R}} \rightarrow M_{\mathbb{R}}/M := T$  is the quotient map.
- $D_c^b(T, \overline{\Lambda}_{\Sigma})$ : the full subcategory of  $D_c^b(T)$  spanned by objects whose microsupports contained in  $\overline{\Lambda}_{\Sigma}$ .

## Theorem[Fang-Liu-Treumann-Zaslow, Treumann]

$$\exists \kappa_{\Sigma}: D^b(\text{coh}X_{\Sigma}) \hookrightarrow D_c^b(T, \overline{\Lambda}_{\Sigma}).$$

Conjecture (the nonequivariant coherent-constructible correspondence) [Fang-Liu-Treumann-Zaslow, Treumann]

$$\kappa_{\Sigma}: D^b(\text{coh}X_{\Sigma}) \xrightarrow{\cong} D_c^b(T, \overline{\Lambda}_{\Sigma}).$$

Known results[Treumann, Scherotzke-Sibilla]

- Zonotopally unimodular fans (e.g.  $(\mathbb{P}^1)^n$ )
- Cragged fans (e.g. toric Fano surfaces)

## Main Theorem[K]

If  $\dim \Sigma = 2$ , the above conjecture holds.

Key of the proof: Blow-up formula

$\hat{\Sigma}$ : a toric point blow-up of  $\Sigma$

## Theorem[Orlov]

$\dim \Sigma = n + 1 \Rightarrow$

$$\begin{aligned} D^b(\text{coh}X_{\hat{\Sigma}}) \\ \cong \langle \mathcal{O}_E(nE), \dots, \mathcal{O}_E(E), D^b(\text{coh}X_{\Sigma}) \rangle \end{aligned}$$

## Theorem[K]

$\dim \Sigma = n + 1 \Rightarrow$

$$\begin{aligned} D_c^b(T, \Lambda_{\hat{\Sigma}}) \\ \cong \langle \kappa_{\hat{\Sigma}}(\mathcal{O}_E(nE)), \dots, \kappa_{\hat{\Sigma}}(\mathcal{O}_E(E)), D_c^b(T, \overline{\Lambda}_{\Sigma}) \rangle \end{aligned}$$

This formula, Toric MMP and the functoriality of  $\kappa_{\Sigma}$  together reduces the proof to the known case if  $\dim \Sigma = 2$ .

The proof of the formula is relied on microlocal sheaf theory of Kashiwara-Schapira.

Thank you for your attention!