

THE EXISTENCE OF A NON SPECIAL ARONSZAJN TREE AND TODORČEVIĆ ORDERINGS

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ABSTRACT. It is proved that it is consistent that every forcing notions with $R_{1, \aleph_1}$ has precaliber $\aleph_1$, every Todorčević ordering for any second countable Hausdorff space also has precaliber $\aleph_1$, and there exists a non-special Aronszajn tree. This slightly extends the previous work [16, 18].

1. INTRODUCTION

Martin's Axiom was introduced by Martin and Solovay to solve Suslin's problem in [5]. In 1980's, Todorčević investigated Martin's Axiom from the view point of Ramsey theory, and introduced the countable chain condition for partitions on the set $[\omega_1]^{<\aleph_0}$. In [13], Todorčević and Veličković proved that $MA_{\aleph_1}$, which is Martin's Axiom for $\aleph_1$ many dense sets, is equivalent to the statement $\mathcal{K}'_{<\omega}$ that every ccc partition $K_0 \cup K_1$ on $[\omega_1]^{<\aleph_0}$ has an uncountable K_0 -homogeneous set. Todorčević also introduced many fragments of $MA_{\aleph_1}$ in his many papers e.g. [9, 13]. Some of them are as follows⁽¹⁾: $\mathcal{K}_{<\omega}$ is the statement that every ccc forcing notion has precaliber $\aleph_1$. For each $n \in \omega$, $\mathcal{K}_n$ is the statement that every uncountable subset of a ccc forcing notion has an uncountable n -linked subset, and $\mathcal{K}'_n$ is the statement that every ccc partition $K_0 \cup K_1 = [\omega_1]^n$ has an uncountable K_0 -homogeneous set. $\mathcal{C}^2$ is the statement that every product of ccc forcing notions has the countable chain condition. We note that they have many applications. For example, $\mathcal{C}^2$ implies Suslin's Hypothesis, every (ω_1, ω_1) -gap is indestructible, and the bounding number $\mathfrak{b}$ is greater than $\aleph_1$, and $\mathcal{K}'_2$ implies that every Aronszajn tree is special. (For other applications, see e.g. [3].) We also note the following diagram of implications

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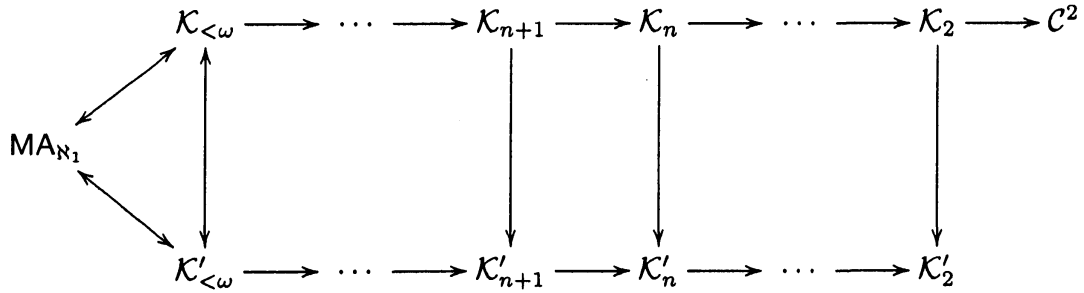
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⁽¹⁾They are defined by Todorčević in several papers. In [3, Definition 4.9] and [13, §2], $\mathcal{K}_n$'s are defined as statements for ccc forcing notions, however in [4, §4] and [9, §7], $\mathcal{K}_n$'s are defined as statements for ccc partitions. To separate them, we use notation as above. In [13], $\mathcal{K}'_{<\omega}$ above is denoted by $\mathcal{H}$.

A forcing notion $\mathbb{P}$ has precaliber $\aleph_1$ if every uncountable subset I of $\mathbb{P}$ has an uncountable subset I' of I such that every finite subset of I' has a common extension in $\mathbb{P}$. A subset I of a forcing notion $\mathbb{P}$ is called n -linked if every member of the set $[I]^n$ has a common extension in $\mathbb{P}$. A forcing notion $\mathbb{P}$ has property K if every uncountable subset of $\mathbb{P}$ has an uncountable 2-linked subset.

between them:



The equivalence of $\text{MA}_{\aleph_1}$, $\mathcal{K}_{<\omega}$ and $\mathcal{K}'_{<\omega}$ are the theorem due to Todorćević and Velićković [13]. Other implications follows from definitions or trivial arguments. It is unknown whether any other implications hold in ZFC.

The author studied about this problem in [14, 15, 16, 17, 18]. In [16, 18], The author introduced the following property on chain conditions [16, 18, Definition 2.6]: A forcing notion $\mathbb{P}$ has the property $R_{1,\aleph_1}$ if conditions of $\mathbb{P}$ are finite sets of countable ordinals, the order $\leq_{\mathbb{P}}$ is equal to the superset relation $\supseteq$, and for any large enough regular cardinal θ , any countable elementary submodel N of $H(\theta)$, any uncountable subset I of $\mathbb{P}$ which forms a Δ -system with root ν and any $\sigma \in \mathbb{P}$ with $\sigma \cap N = \nu$, there exists an uncountable subset I' of I such that every condition in I' is compatible with σ in $\mathbb{P}$. It is proved that $\mathcal{K}_2(R_{1,\aleph_1})^{(2)}$ also implies that Suslin's Hypothesis holds, every (ω_1, ω_1) -gap is indestructible and $\mathfrak{b} > \aleph_1$. It is also proved that it is consistent that every forcing notion with the property $R_{1,\aleph_1}$ has precaliber $\aleph_1$ and there exists a non-special Aronszajn tree. This says that $\mathcal{K}_{<\omega}(R_{1,\aleph_1})$ doesn't imply $\text{MA}_{\aleph_1}(R_{1,\aleph_1})$.

In this paper, we slightly develop this result by dealing with not only forcing notions with $R_{1,\aleph_1}$ but also forcing notions defined due to Todorćević and Balcar-Pazák-Thümmel [10, 1], so called *Todorćević orderings*. Namely, it is shown that it is consistent that every forcing notion with the property $R_{1,\aleph_1}$ has precaliber $\aleph_1$, Todorćević orderings for second countable Hausdorff spaces also have precaliber $\aleph_1$, and there exists a non-special Aronszajn tree.

2. PRELIMINARIES

2.1. Todorćević orderings. As said in [1], when a topological space is applied to Todorćević ordering, it is natural to require it to be sequential and have the unique limit property. A topological space X is called sequential if for any $Z \subseteq X$, Z is closed in X iff for any $A \subseteq Z$ and $x \in X$ to which A converges, x belongs to Z . A topological space X has the unique limit property if any converging subset of X converges to the unique point. For example, Hausdorff spaces have the unique limit property. For a subset F of a topological space, let F^d denote the first Cantor-Bendixson derivative of F , that is, the set of all accumulation points of F .

Definition 2.1 (Todorćević [10], see also [1, 8]). *For a topological space X , $\mathbb{T}(X)$ is the set of all subsets of X which are unions of finitely many converging sequences*

⁽²⁾ $\mathcal{K}_2(R_{1,\aleph_1})$ is the statement that every forcing notion with the property $R_{1,\aleph_1}$ has property $\mathcal{K}$.

including their limit points, and for each p and q in $\mathbb{T}(X)$, $q \leq_{\mathbb{T}(X)} p$ iff $q \supseteq p$ and $q^d \cap p = p^d$.⁽³⁾

For $p, q \in \mathbb{T}(X)$, the statement $q \leq_{\mathbb{T}(X)} p$ means that q is an extension of p (as the subset relation) and the isolated points in p are still isolated in q . $\mathbb{T}(X)$ is called *Todorćević ordering for the space X* in [1, 8] (and [19]).

Todorćević orderings were firstly introduced by Todorćević in [10]. The motivation is to demonstrate a Borel definable ccc forcing which consistently does not have property K. He defined it on a separable metric space. By generalizing it and applying it to other topological spaces, Thümmel discovered a forcing notion which has the σ -finite chain condition but does not have the σ -bounded chain condition, and so he solved the problem of Horn and Tarski [8]. (For Horn-Tarski's problem, see [2, 11].) Right after Thümmel's result, Todorćević introduced a Borel definable solution of the problem of Horn and Tarski [12].

In [12], Todorćević introduced the Borel definable version of Todorćević orderings, which consists of all countable compact subsets whose first Cantor-Bendixson derivative is finite. In [1], Balcar-Pazák-Thümmel introduced a separative version of Todorćević orderings, which consists of all functions f from members p of $\mathbb{T}(X)$ into $\{0, 1\}$ such that $f^{-1}(1)$ is a finite set including p^d as a subset, ordered by the function-extension. In this paper, as in [19], we adopt the definition of Todorćević orderings in Definition 2.1.

Some of Todorćević orderings may not be ccc [1, Theorem 2.3], but many of them are ccc. From the proof of [10], we note that for a space X , if each of finite powers of X is hereditarily separable, then Todorćević ordering for X has the ccc. In [1, Definition 2.1], Balcar-Pazák-Thümmel introduced the property of topological spaces which is a sufficient condition to introduce Todorćević orderings to have the ccc (see also [19]). In this paper, we use the following property of Todorćević orderings.

Lemma 2.2. *For a second countable Hausdorff space X , $\mathbb{T}(X)$ is powerfully ccc, that is, a finite support product of any number of copies of $\mathbb{T}(X)$ has the countable chain condition.*

Proof. It suffices to show that for any $n \in \omega$, the finite support product ${}^n\mathbb{T}(X)$ is ccc. Let I be an uncountable subset of ${}^n\mathbb{T}(X)$. By shrinking I if necessary, we may assume that for each $i < n$, the set $\{p_i^d; \langle p_j; j < n \rangle \in I\}$ forms a Δ -system with root d_i . Take a countable elementary submodel N of $H(\theta)$ (for some large enough regular cardinal θ) such that $\{X, I\} \in N$.

Take $\langle p_i; i < n \rangle$ and $\langle q_i; i < n \rangle$ in $I^{(4)}$ such that for each $i < n$,

- $(p_i^d \setminus d_i) \cap N = \emptyset$, and

⁽³⁾This definition is slightly different from the original one, in [10], which consists of all finite sets σ of convergent sequences in X including their limit points such that for any $A, B \in \sigma$,

$$\lim(A) \notin (B \setminus \{\lim(B)\}),$$

ordered by the reverse inclusion. But essentially, both are same. In fact, both are forcing-equivalent.

⁽⁴⁾Since the set $\{p_i^d; \langle p_j; j < n \rangle \in I\}$ forms an uncountable Δ -system for each $i < n$ and N is countable, we can find such a $\langle p_i; i < n \rangle \in I$. Similarly, since the set $N \cup \bigcup_{i < n} p_i$ is countable, we can find such a $\langle q_i; i < n \rangle \in I$.

- $(q_i^d \setminus d_i) \cap (N \cup p_i) = \emptyset$.

Since X is second countable Hausdorff and N is an elementary submodel, there exists a sequence $\langle U_i, V_i; i < n \rangle \in N$ of open subsets of X such that for each $i < n$,

- $U_i \cap V_i = \emptyset$,
- $p_i^d \setminus d_i \subseteq U_i$,
- $q_i^d \setminus d_i \subseteq V_i$, and
- $V_i \cap (p_i \setminus U_i) = \emptyset$.

This can be done because the sets $p_i^d \setminus d_i$, $q_i^d \setminus d_i$ and $p_i \setminus U_i$ are finite and $(q_i^d \setminus d_i) \cap p_i = \emptyset$. By the elementarity of N , there exists $\langle q'_i; i < n \rangle \in I \cap N$ such that for each $i < n$, $(q'_i)^d \setminus d_i \subseteq V_i$. Then for each $i < n$,

$$q'_i \cup p_i \leq_{\mathbf{T}(X)} p_i.$$

Since $q'_i \subseteq N^{(5)}$ and $(p_i^d \setminus d_i) \cap N = \emptyset$ for each $i < n$, we notice that

$$q'_i \cup p_i \leq_{\mathbf{T}(X)} q'_i.$$

Thus the condition $\langle q'_i \cup p_i; i < n \rangle$ is a common extension of conditions $\langle p_i; i < n \rangle$ and $\langle q'_i; i < n \rangle$ in ${}^n\mathbf{T}(X)$. $\square$

2.2. The chapter IX of [6]: Souslin Hypothesis Does Not Imply “Every Aronszajn Tree Is Special. In this section, we summarize Shelah’s approach to show the consistency that Suslin’s Hypothesis holds and there exists a non-special Aronszajn tree. All of definitions and proofs in this section are in [6, IX. Souslin Hypothesis Does Not Imply “Every Aronszajn Tree Is Special].

Definition 2.3 (Shelah, [6, IX 3.3 Definition]). *For an Aronszajn tree T and a subset S of ω_1 , T is called S -st-special if there exists a function f from the set $\{t \in T; \text{rk}_T(t) \in S\}$ into ω such that for each $n \in \omega$, the set $f^{-1}[\{n\}]$ forms an antichain in T .*

We note that if S is uncountable and an Aronszajn tree T is S -st-special, then T is still Aronszajn in the forcing extension where S is still uncountable. And then T has an uncountable antichain, hence then T is not a Suslin tree. For a costationary subset S of ω_1 , if T is a special Aronszajn tree, then there exists an antichain A through T such that the set $\text{rk}_T[A] \setminus S^{(6)}$ is stationary. Therefore if S is an uncountable costationary subset of ω_1 and T^* satisfies the property

(*) for every antichain A through T^* , the set $\text{rk}_{T^*}[A] \setminus S$ is nonstationary,

then T^* is a non-special Aronszajn tree.

In [6, IX 4.8 Conclusion], Shelah introduced the iterated proper forcing which forces that Suslin’s Hypothesis holds and there are a stationary and costationary subset S of ω_1 and an S -st-special Aronszajn tree T^* which satisfies the property (*). The S -st-speciality of T^* guarantees that T^* is still Aronszajn in any proper forcing extension. To guarantee the property (*) of T^* , we shoot a club on ω_1 for the complement of $\text{rk}_{T^*}[A]$ which is disjoint from S in some intermediate stage of the iteration [6, IX 4.7, 4.8]. However, the iteration is required to be a proper forcing. To do this, Shelah introduced the following preservation property.

⁽⁵⁾ q'_i is a countable subset of X .

⁽⁶⁾ $\text{rk}_T[A] := \{\text{rk}_T(t); t \in A\}$.

Definition 2.4 (Shelah [6, IX 4.5 Definition]). *Let T be an Aronszajn tree and S a subset of ω_1 .*

A forcing notion $\mathbb{P}$ is (T, S) -preserving if for a large enough regular cardinal θ , a countable elementary submodel N of $H(\theta)$ which has the set $\{\mathbb{P}, T, S\}$ and $p \in \mathbb{P} \cap N$, there exists $q \leq_{\mathbb{P}} p$ which is $(N, \mathbb{P})$ -generic such that if $\omega_1 \cap N \not\subseteq S$, then

for any $x \in T$ of height $\omega_1 \cap N$,

if $\forall A \in \mathcal{P}(T) \cap N (x \in A \rightarrow \exists y \in A (y <_T x))$,

then for every $\mathbb{P}$ -name $\dot{A}$, which is in N , for a subset of T ,

$$q \Vdash_{\mathbb{P}} "x \in \dot{A} \rightarrow \exists y \in \dot{A} (y <_T x)".$$

If T^* is a Suslin tree, then for every countable elementary submodel N of $H(\theta)$ (for some large enough regular cardinal θ) and $x \in T^*$ of height $\omega_1 \cap N$ and $A \in \mathcal{P}(T^*) \cap N$, if $x \in A$, then there exists $y \in A$ such that $y <_{T^*} x$ ⁽⁷⁾. It follows that T^* satisfies (*). So we start from a Suslin tree T^* and a stationary and costationary subset S of ω_1 and make each Aronszajn tree to be S -st-special and T^* to be S -st-special which satisfies the property (*) by the iterated proper forcing extension such that each iterand is (T^*, S) -preserving and the whole iteration is also (T^*, S) -preserving. For Aronszajn trees T and T^* and a stationary subset S of ω_1 , Shelah introduced the forcing notion $Q(T, S)$ which forces T to be S -st-special and is (T^*, S) -preserving [6, IX 4.2, 4.3, 4.6]. Moreover, Shelah introduced the new forcing iteration, so called a free limit iteration, which preserves the (T^*, S) -preserving property [6, IX §1, §2 and 4.7].

The following is Shelah's iterated forcing in [6, Chapter IX, 4.8 Conclusion]⁽⁸⁾. We start in the ground model where $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, and there exists a Suslin tree T^* . Let S be a stationary and costationary subset of the set ω_1 . We define an $\aleph_1$ -free iteration $\langle P_\xi, Q_\eta; \xi \leq \omega_2 \ \& \ \eta < \omega_2 \rangle$ such that

- $Q_0 = Q(T^*, S)$,
- each Q_η satisfies one of the following:
 - (1) Q_η is proper and (T^*, S) -preserving of size $\aleph_1$,
 - (2) for some P_ξ -name of an antichain $\dot{A}$ of T^* , $\text{rk}_{T^*}[\dot{A}] \cap S = \emptyset$ and $Q_\eta = Q_{\text{club}}(\omega_1 \setminus \text{rk}_{T^*}[\dot{A}])$, which shoots a club through the set $\omega_1 \setminus \text{rk}_{T^*}[\dot{A}]$ by countable approximations.

In this extension (with some bookkeeping argument), S is still stationary and costationary, every Aronszajn tree is S -st-special (hence not Suslin), and T^* is an S -st-special Aronszajn tree which satisfies (*).

Combining Shelah's iteration above, some bookkeeping device, theorems in [16, 18] and the next section, we can conclude the following.

Theorem 2.5. *It is consistent that every forcing notions with $\aleph_{1, \aleph_1}$ has precaliber $\aleph_1$, every Todorćević ordering for any second countable Hausdorff space also has precaliber $\aleph_1$ and there exists a non-special Aronszajn tree.*

⁽⁷⁾Let $D := \{t \in T^*; t \in A \text{ or for every } s \in T^* \text{ with } t <_{T^*} s, s \notin A\}$. Since D is a dense subset of T^* and T^* is Suslin, there exists $y \in D \cap N$ which is compatible with x in T^* . Then it have to be true that $y <_{T^*} x$. Since $x \in A$, it have to be true that $y \in A$.

This statement is equivalent that there are no uncountable antichain through T^* .

⁽⁸⁾Shelah's proof uses an $\aleph_1$ -free iteration. This is different from a countable support iteration. But Schlindwein proved in [7] that the same proof works for a countable support iterations. So our theorem can be shown by a countable support iteration.

3. PROOF

Suppose that S is a stationary subset of ω_1 , X is a second countable Hausdorff space and I is an uncountable subset of $\mathbb{T}(X)$. By shrinking I if necessary, we may assume that

- the size of I is $\aleph_1$,
- the set $\{p^d; p \in I\}$ forms a Δ -system with root d ,
- for some $q \in \mathbb{T}(X)$,

$$q \Vdash_{\mathbb{T}(X)} "I \cap \dot{G} \text{ is uncountable}."$$

Let $\vec{M} = \langle M_\alpha; \alpha \in \omega_1 \rangle$ be a sequence of countable elementary submodels of $H(\aleph_2)$ such that $\{S, X, I\} \in M_0$, and for every $\alpha \in \omega_1$, $\langle M_\beta; \beta \in \alpha \rangle \in M_\alpha$. By shrinking I if necessary again, we may assume that

- for each $p \in I$ and $\alpha \in \omega_1$, if $p^d \cap (M_{\alpha+1} \setminus M_\alpha) \neq \emptyset$, then $p^d \subseteq M_{\alpha+1} \setminus M_\alpha$.

We have to notice then that it may happen that I does *not* belong to M_0 . From now on, we do *not* assume that $I \in M_\alpha$ for any $\alpha \in \omega_1$.

We define the forcing notion $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ which consists of pairs $\langle h, f \rangle$ such that

- h is a finite partial function from ω_1 into ω_1 ,
- for any $\alpha, \beta \in \text{dom}(h)$, $\alpha \leq h(\alpha)$, and if $\alpha < \beta$, then $h(\alpha) < \beta$,
- for any $\alpha \in \text{dom}(h) \cap S$, $h(\alpha) = \alpha$,
- f is a finite partial function from I into ω ,
- for any $\alpha \in \text{dom}(h)$ and $p \in \text{dom}(f)$,

$$p^d \cap (M_{h(\alpha)} \setminus M_\alpha) = \emptyset,$$

- for any $p \in \text{dom}(f)$, the set $\bigcup f^{-1}[\{f(p)\}]$ is a common extension of members of the set $f^{-1}[\{f(p)\}]$ in $\mathbb{T}(X)$,

ordered by extension, that is, for any $\langle h, f \rangle$ and $\langle h', f' \rangle$ in $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$,

$$\langle h, f \rangle \leq_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)} \langle h', f' \rangle : \iff h \supseteq h' \ \& \ f \supseteq f'.$$

By a density argument, if $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ is proper, then $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ adds an uncountable subset of I which satisfies the finite compatibility property. Therefore, under the approach due to Shelah in §2, it suffices to show that $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ is proper and (T^*, S) -preserving.

Lemma 3.1. $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ is proper.

Proof. Let θ be a large enough regular cardinal, a countable elementary submodel N of $H(\theta)$ which has the set $\{X, I, \vec{M}, S\}$, $\langle h, f \rangle \in \mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$, and δ a countable ordinal not smaller than the ordinal $\omega_1 \cap N$ (if $\omega_1 \cap N \in S$, then we define $\delta := \omega_1 \cap N$). We show that $\langle h \cup \{\langle \omega_1 \cap N, \delta \rangle\}, f \rangle$ is $(N, \mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S))$ -generic.

Let $\langle h', f' \rangle \leq_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)} \langle h \cup \{\langle \omega_1 \cap N, \delta \rangle\}, f \rangle$ and D a dense open subset of $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$. We will find a condition in $D \cap N$ which is compatible with $\langle h', f' \rangle$ in $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$.

By extending the condition $\langle h', f' \rangle$ if necessary, we may assume that $\langle h', f' \rangle \in D$. We note that $\langle h' \upharpoonright N, f' \upharpoonright N \rangle$ is in $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S) \cap N$ because $\omega_1 \cap N \in \text{dom}(h')$.

Let

$$D' := \left\{ \langle k, g \rangle \in D; \langle k, g \rangle \leq_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)} \langle h' \restriction N, f' \restriction N \rangle \ \& \ \text{ran}(g) = \text{ran}(f') \right\}.$$

We note that D' is in $N^{(9)}$, $\langle h', f' \rangle \in D'$ and D' is dense in $\mathcal{Q}(\mathbb{Q}, I, \vec{M})$ below $\langle h' \restriction N, f' \restriction N \rangle$. Since the product forcing ${}^{\text{ran}(f')} \mathbb{T}(X)$ of $\mathbb{T}(X)$ is ccc in the model N , by the elementarity of N , there exists a countable subset J of ${}^{\text{ran}(f')} \mathbb{T}(X)$ in N such that

- J is a subset of the set

$$\left\{ \left\langle \bigcup g^{-1}[\{n\}]; n \in \text{ran}(f') \right\rangle; \langle k, g \rangle \in D' \right\},$$

- for every $\langle k, g \rangle \in D'$, there exists $\langle \mu_n; n \in \text{ran}(f') \rangle \in J$ such that for each $n \in \text{ran}(f')$, μ_n and $\bigcup g^{-1}[\{n\}]$ are compatible in $\mathbb{T}(X)$.

Since $\langle h', f' \rangle \in D'$, there exists $\langle \mu_n; n \in \text{ran}(f') \rangle \in J$ such that for each $n \in \text{ran}(f')$, μ_n and $\bigcup (f')^{-1}[\{n\}]$ are compatible in $\mathbb{T}(X)$. Since $\langle \mu_n; n \in \text{ran}(f') \rangle \in J$ holds in N , there exists $\langle k, g \rangle \in D' \cap N$ such that

$$\left\langle \bigcup g^{-1}[\{n\}]; n \in \text{ran}(f') \right\rangle = \langle \mu_n; n \in \text{ran}(f') \rangle.$$

Then $\langle h' \cup k, f' \cup g \rangle$ is a common extension of $\langle h', f' \rangle$ and $\langle k, g \rangle$ in $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$. $\square$

Lemma 3.2. *For any Aronszajn tree T , $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ is (T, S) -preserving.*

Proof. Let T, θ, N be as in the statement of the definition of the (T, S) -preservation, (moreover we suppose $\vec{M} \in N$) and $\langle h, f \rangle \in \mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S) \cap N$. Suppose that $\omega_1 \cap N \notin S$, because if $\omega_1 \cap N \in S$, then the condition $\langle h \cup \{\langle \omega_1 \cap N, \omega_1 \cap N \rangle\}, f \rangle$ is as desired.

Let

$$\delta := \sup \{F(\omega_1 \cap N) + 1; F \in ({}^{\omega_1} \omega_1) \cap N\}.$$

Since N is countable, δ is a countable ordinal. We will show that the condition $\langle h \cup \{\langle \omega_1 \cap N, \delta \rangle\}, f \rangle$ of $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$ is our desired one.

As seen in the proof of the previous lemma, the condition $\langle h \cup \{\langle \omega_1 \cap N, \delta \rangle\}, f \rangle$ is $(N, \mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S))$ -generic. Suppose that $x \in T$ of height $\omega_1 \cap N$ such that for any subset $A \in N$ of T , if $x \in A$, then there is $y \in A$ such that $y <_T x$. Let $\dot{A} \in N$ be a $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ -name for a subset of T . We will show that

$$\langle h \cup \{\langle \omega_1 \cap N, \delta \rangle\}, f \rangle \Vdash_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)} "x \notin \dot{A} \text{ or } \exists y \in \dot{A} (y <_T x)".$$

Let $\langle h', f' \rangle \leq_{\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)} \langle h \cup \{\langle \omega_1 \cap N, \delta \rangle\}, f \rangle$, and assume that

$$\langle h', f' \rangle \nVdash_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)} "x \notin \dot{A}"/>.$$

By strengthening $\langle h', f' \rangle$ if necessary, we may assume that

$$\langle h', f' \rangle \Vdash_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)} "x \in \dot{A}"/>.$$

⁽⁹⁾ $\text{ran}(f')$ is a finite subset of ω .

We note that $\langle h' \upharpoonright N, f' \upharpoonright N \rangle$ is in N (because $\omega_1 \cap N \in \text{dom}(h')$), and by the definition of $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$, for every $p \in \text{dom}(f')$, if $\text{ran}(p) \notin N$, then

$$(p^d \setminus d) \cap M_\delta = \emptyset.$$

Let $\gamma \in \omega_1 \cap N$ be such that for every $p \in \text{dom}(f')$, if the set $p^d \setminus d$ intersects N , then $p^d \subseteq M_\gamma$ ⁽¹⁰⁾. Since X is second countable Hausdorff and N is an elementary submodel, there exists a finite set $\mathcal{U}$ of pairwise disjoint open subsets of X in N such that for each $n \in \text{ran}(f')$, the finite set $(\bigcup (f')^{-1}[\{n\}])^d$ is separated by $\mathcal{U}$. We define a function F with the domain

$$\{t \in T; \text{ht}_T(t) > \max(\text{dom}(h' \upharpoonright N))\}$$

such that for each $t \in T$ of height larger than $\max(\text{dom}(h' \upharpoonright N))$,

$$F(t) := \sup \left\{ \beta \in \omega_1; \text{there exists } \langle k, g \rangle \in \mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S) \text{ such that} \right. \\
\begin{aligned}
&\bullet \min(\text{dom}(k)) = \text{rk}_T(t), \\
&\bullet k(\text{rk}_T(t)) = \beta, \\
&\bullet \langle (h' \upharpoonright N) \cup k, (f' \upharpoonright N) \cup g \rangle \text{ is a condition of } \mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S), \\
&\bullet \text{for each } p \in \text{dom}(g), (p^d \setminus d) \cap M_\gamma = \emptyset, \\
&\bullet \text{ran}(g) = \text{ran}(f' \setminus N), \\
&\bullet \text{for each } n \in \text{ran}(f' \setminus N), \text{ the set } (\bigcup g^{-1}[\{n\}])^d \setminus d \text{ is separated} \\
&\quad \text{by } \mathcal{U}, \text{ and} \\
&\bullet \langle (h' \upharpoonright N) \cup k, (f' \upharpoonright N) \cup g \rangle \Vdash_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)} "t \in \dot{A}" \left. \right\}.
\end{aligned}$$

Then F belongs to N . Let

$$B := \{t \in T; \text{rk}_T(t) > \max(\text{dom}(h' \upharpoonright N)) \ \& \ F(t) = \omega_1\},$$

which is also in N . We define a function F' with the domain

$$[\max(\text{dom}(h' \upharpoonright N)) + 1, \omega_1)$$

such that for a countable ordinal β larger than $\max(\text{dom}(h' \upharpoonright N))$,

$$F'(\beta) := \sup \{F(t) + 1; t \in T \setminus B \ \& \ \text{rk}_T(t) \in (\max(\text{dom}(h' \upharpoonright N)), \beta]\}.$$

This F' is a function from ω_1 into ω_1 and also in N . Hence $F'(\omega_1 \cap N) < \delta$ by the definition of δ . Since $\langle h', f' \rangle \Vdash_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)} "x \in \dot{A}"$ and $h'(\text{rk}_T(x)) = h'(\omega_1 \cap N) = \delta$, $F(x) \geq \delta$ holds. Therefore x have to belong to B . Thus by our assumption, there exists $y \in B$ such that $y <_T x$.

Take $\varepsilon \in \omega_1$ such that $f' \subseteq M_\varepsilon$. Let

$$E := \left\{ \langle k, g \rangle \in \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S); \right. \\
\begin{aligned}
&\bullet \min(\text{dom}(k)) = \text{rk}_T(y), \\
&\bullet \langle (h' \upharpoonright N) \cup k, (f' \upharpoonright N) \cup g \rangle \text{ is a condition of } \mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S), \\
&\bullet \text{for each } p \in \text{dom}(g), (p^d \setminus d) \cap M_\gamma = \emptyset, \\
&\bullet \text{ran}(g) = \text{ran}(f'), \\
&\bullet \text{for each } n \in \text{ran}(f'), \text{ the set } (\bigcup g^{-1}[\{n\}])^d \setminus d \text{ is separated by} \\
&\quad \mathcal{U}, \text{ and} \\
&\bullet \langle (h' \upharpoonright N) \cup k, (f' \upharpoonright N) \cup g \rangle \Vdash_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)} "y \in \dot{A}" \left. \right\}.
\end{aligned}$$

⁽¹⁰⁾Then for every $p \in \text{dom}(f')$, $(p^d \setminus d) \cap M_\gamma = \emptyset$ iff $(p^d \setminus d) \cap M_\delta = \emptyset$.

We note that E is in N , and the set

$$\{k(\text{rk}_T(y)); \langle k, g \rangle \in E\}$$

is uncountable because $F(y) = \omega_1$. So there exists $\langle k, g \rangle \in E$ such that for each $p \in \text{dom}(g)$, $(p^d \setminus d) \cap M_\varepsilon = \emptyset$. Then for each $n \in \text{ran}(f' \setminus N)$,

$$\left(\left(\bigcup g^{-1}[\{n\}] \right)^d \setminus d \right) \cap \left(\bigcup (f' \setminus N)^{-1}[\{n\}] \right) = \emptyset.$$

Since X is second countable Hausdorff and N is an elementary submodel, there exists disjoint open subsets U and V of X in $N^{(11)}$ such that for each $n \in \text{ran}(f' \setminus N)$,

$$\left(\bigcup (f' \setminus N)^{-1}[\{n\}] \right)^d \setminus d \subseteq U,$$

$$\left(\bigcup g^{-1}[\{n\}] \right)^d \setminus d \subseteq V$$

and

$$V \cap \left(\left(\bigcup (f' \setminus N)^{-1}[\{n\}] \right) \setminus U \right) = \emptyset.$$

By the elementarity of N , we can find $\langle k', g' \rangle \in E$ such that

$$\left(\bigcup (g')^{-1}[\{n\}] \right)^d \setminus d \subseteq V.$$

Then for each $n \in \text{ran}(f' \setminus N)$, the set

$$\bigcup (f')^{-1}[\{n\}] \cup \bigcup (g')^{-1}[\{n\}] \leq_{\mathbb{T}(X)} \bigcup (f')^{-1}[\{n\}].$$

Since $g' \subseteq N$ and $\left(\bigcup (f' \setminus N)^{-1}[\{n\}] \right)^d \cap N = \emptyset$, we note that for each $n \in \text{ran}(f' \setminus N)$, the set

$$\bigcup (f')^{-1}[\{n\}] \cup \bigcup (g')^{-1}[\{n\}] \leq_{\mathbb{T}(X)} \bigcup (g')^{-1}[\{n\}].$$

Therefore $\langle k' \cup h', g' \cup f' \rangle$ is an extension of $\langle h', f' \rangle$ in $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ and

$$\langle k' \cup h', g' \cup f' \rangle \Vdash_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)} "y \in \dot{A}."$$

□

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⁽¹¹⁾This can be done because the set $\left(\bigcup (f' \setminus N)^{-1}[\{n\}] \right) \setminus U$ is finite if U satisfies that $\left(\bigcup (f' \setminus N)^{-1}[\{n\}] \right)^d \setminus d \subseteq U$.

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