Fractional Dehn twists, topological monodromies, and uniformization

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Abstract

Part I The homological monodromy of a degeneration whose singular fiber has at most normal crossings was described by C. H. Clemens. In his work, local monodromies were described in detail. It is actually a classical result that the local monodromy around a node is a Dehn twist. For higher-dimensional case, we describe local monodromies alternatively: On a local smooth fiber of dimension $n \geq 2$, we construct $n+1$ singular foliations and then describe the action of the local monodromy on each leaf. Here the ith singular foliation is used for describing its action on the ith face of the boundary of a local smooth fiber.

Part II Motivated by the canonical model of degenerations of Riemann surfaces, we consider a quotient space A_{d-1}/Γ of a 'multiplicative' A-singularity A_{d-1} in $\mathbb{C}^{n+1}$ under a certain cyclic group action Γ on A_{d-1} (unlike an 'additive' A-singularity, the singular locus of A_{d-1} is not isolated if $n \geq 3$). We *explicitly* construct a small finite abelian subgroup G of $GL(n, \mathbb{C})$ such that A_{d-1}/Γ is isomorphic to $\mathbb{C}^n/G$ (*uniformization theorem*). Moreover: (1) We give a numerical criterion for a certain subgroup of $GL(n, \mathbb{C})$ to be small. (2) For a certain family of subgroups of $GL(n, \mathbb{C})$, we show that if one subgroup of this family is small, then all subgroups of this family are small (*equi-smallness theorem*).

Future work We plan to generalize the method in Part I, to apply to describe the topological monodromy of the degeneration $A_{d-1}/\Gamma \to \mathbb{C}$ constructed in Part II. This topological monodromy is regarded as a higher-dimensional fractional Dehn twist.

2010 Mathematics Subject Classification: 32G05, 32Q30, 14D05

Keywords: Uniformization of singularity, Degeneration of complex manifolds, Topological monodromy, Dehn twist, Fractional Dehn twist, Small group, Pseudo-reflection group

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Background material

Historical background

Let M be an n-dimensional complex manifold. A proper surjective holomorphic map $\pi : M \to D := \{s \in \mathbb{C} \,:\, |s| < 1\}$ is said to be a *degeneration* of complex manifolds if $\pi^{-1}(0)$ is singular and all other fibers $\pi^{-1}(s)$ $(s \neq 0)$ are smooth. The *topological monodromy* $h : \pi^{-1}(s) \to \pi^{-1}(s)$ of a degeneration $\pi : M \to D$ is a lift of a counterclockwise rotation around $0 \in D$ (Figure 1), which describes how M is 'twisted' around the singular fiber $\pi^{-1}(0)$. The automorphism h_* of $H_1(\pi^{-1}(s), \mathbb{Z})$ induced from h is called the (*homological*) *monodromy* of this degeneration.

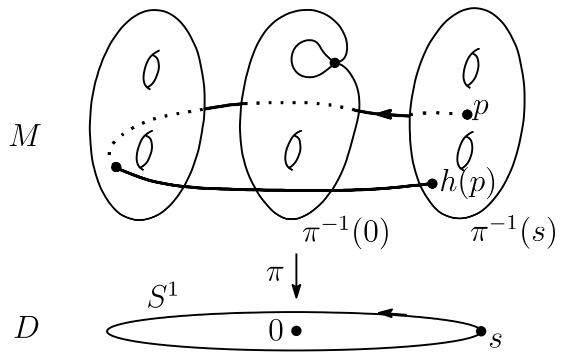


Figure 1:

For degenerations of elliptic curves, K. Kodaira [Kod] showed that there are 8 degenerations. For degenerations of Riemann surfaces of genus 2, Y. Namikawa and K. Ueno [NU] showed that there are about 120 degenerations.

A topological monodromy has more information than a homological monodromy. In fact, Y. Matsumoto and J. M. Montesinos [MM2] showed that the degenerations of Riemann surfaces of genus ≥ 2 are topologically classified by their topological monodromies. Here note that homeomorphisms appearing as topological monodromies are pseudo-periodic homeomorphisms of negative type [Ima], [ST], [ES]. S. Takamura [Tak2] reconstructed Matsumoto-Montesinos' results from algebro-geometric viewpoint.

A fractional Dehn twist f is a self-homeomorphism of an annulus such that for some integer $N \geq 2$, f^N is a Dehn twist. Depending on whether its twist is positive or negative, it is said to be positive or negative. Any negative fractional Dehn twist appears as the restriction of a pseudo-periodic homeomorphisms of negative type.

Set $A_{d-1} := \{(z, w, t) \,:\, zw = t^d\}$, then $A_{d-1} \to \mathbb{C}$, $(z, w, t) \mapsto t$ is a degeneration whose topological monodromy is a $(-d)$-Dehn twist. More generally, given a negative fractional Dehn twist, we may construct a degeneration whose topological monodromy is it. Its total space is the quotient of A_{d-1} under the action of some cyclic group Γ. Here A_{d-1}/Γ is actually isomorphic to some cyclic quotient singularity $\mathbb{C}^2/G$ [Tak2], where G may be explicitly constructed. The minimal resolution of A_{d-1}/Γ is thus given by the *Hirzeburch-Jung resolution*. In Part II, we generalize this to higher-dimensions, while in Part I, we describe the monodromy of a relevant degeneration in terms of foliations on a smooth fiber.

We plan to generalize the method in Part I, to apply to describe the topological monodromy of the degeneration $A_{d-1}/\Gamma \to \mathbb{C}$ constructed in Part II. This topological monodromy is regarded as a higher-dimensional fractional Dehn twist.

Background material

Throughout this paper, manifolds are assumed to be orientable and homeomorphisms are assumed to be orientation-preserving.

Degenerations and topological monodromies

A *degeneration* is a proper surjective holomorphic map $\pi : M \to D$ from a smooth complex manifold M to an open disk D such that $\pi^{-1}(0)$ is singular, and all other fibers $\pi^{-1}(s)$ $(s \neq 0)$ are smooth complex manifolds. Let $h : \pi^{-1}(s) \to \pi^{-1}(s)$ be a homeomorphism given as a lift of a counterclockwise rotation around $0 \in D$. Then the homeomorphism h (precisely, its isotopy class) is called the *topological monodromy* of a degeneration $\pi : M \to D$, and the automorphism h_* of $H_*(\pi^{-1}(s), \mathbb{Z})$ induced from h is called the (*homological*) *monodromy* of this degeneration.

The topological monodromies of degenerations of closed surfaces are periodic or negative pseudo-periodic homeomorphisms [Ima], [ST], [ES]. Here a homeomorphism $h : \Sigma \to \Sigma$ of a closed surface Σ of genus $g \geq 1$ is *pseudo-periodic* if there exist disjoint simple closed curves $l_1, l_2, \ldots, l_q$ on Σ such that

(1) h preserves $\{l_1, l_2, \ldots, l_q\}$ as a set, and

(2) the restriction of h to $\Sigma \setminus \{l_1, l_2, \ldots, l_q\}$ is isotopic to a periodic homeomorphism of $\Sigma \setminus \{l_1, l_2, \ldots, l_q\}$.

The set $\{l_1, l_2, \ldots, l_q\}$ is called the *admissible system* of h. (Note: A pseudo-periodic homeomorphism is a "reducible map" of the Nielsen-Thurston classification of homeomorphisms of a compact surface.)

The order of the periodic homeomorphism in (2) is called the *pseudo-order* of h. It is alternatively defined as the least positive integer N such that h^N is generated by Dehn twists along $l_1, l_2, \dots, l_q$. If these Dehn twists are all positive (resp. negative), h is said to be *positive* (resp. *negative*).

A degeneration of Riemann surfaces is *minimal* if its singular fiber does not contain (-1)-curves. Here a (-1)-curve is a projective line whose self-intersection number is -1. Matsumoto and Montesinos [MM2] proved that for a closed surface Σ of genus ≥ 2 and a periodic or negative pseudo-periodic homeomorphism $h : \Sigma \to \Sigma$, there exists a minimal degeneration of Σ with topological monodromy h. They also showed that two minimal degenerations of genus ≥ 2 are topologically equivalent if and only if their topological monodromies are conjugate. Here two degenerations $\pi : M \to D$ and $\pi' : M' \to D'$ are *topologically equivalent* if there exist homeomorphisms $F : M \to M'$ and $f : D \to D'$ such that $f(0) = 0$ and that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\pi"'] & M' \arrow[d, "\pi'"] \\ D \arrow[r, "f"] & D'. \end{tikzcd}$$

Note that in this case, the topological monodromies h and h' of π and π' are conjugate. Indeed, $h' \circ F = F \circ h$.

Construction of local models of degenerations

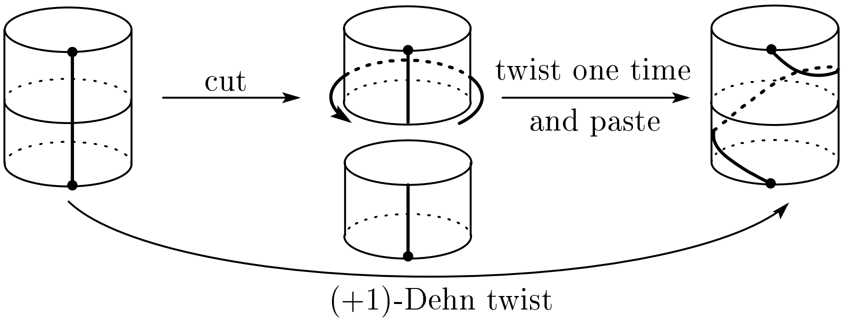
For a periodic or negative pseudo-periodic homeomorphism h of a closed surface Σ, a degeneration $\pi : M \to D$ of Σ with topological monodromy h is explicitly constructed by Takamura's *cyclic quotient construction* in [Tak2]. We explain it.

For the case that h is negative pseudo-periodic, let $\{l_1, l_2, \dots, l_q\}$ denote the admissible system of h. For a loop l_i, let r_i be the least positive integer such that $h^{r_i}(l_i) = l_i$. Then h^{r_i} is, in a neighborhood (an annulus) of l_i in Σ, either a fractional Dehn twist (non-amphidrome) or a Nielsen twist (amphidrome) along l_i. In particular, some power of h^{r_i} is a *negative* Dehn twist along l_i.

We first outline the construction of a degeneration of annuli whose topological monodromy is a (negative) fractional Dehn twist or a Nielsen twist. This is a local model of a degeneration of Riemann surfaces, and used for the construction of a global degeneration with the prescribed topological monodromy.

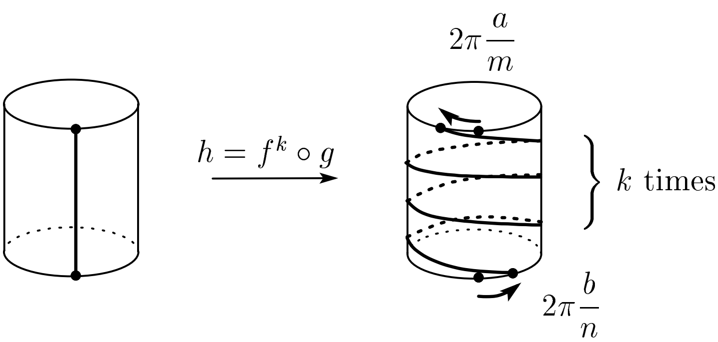
Fractional Dehn twists and Nielsen twists

A $(+1)$-*Dehn twist* is a self-homeomorphism of an annulus given as follows:



Set $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ (a unit circle), then a $(+1)$-Dehn twist f of an annulus $[0,1] \times S^1$ is explicitly given by $f(t,z) = (t, e^{2\pi \mathrm{i} t} z)$. The k times composition f^k of f is called a $(+k)$-Dehn twist, the inverse f^{-1} is a (-1)-Dehn twist, and $(f^{-1})^k$ is a $(-k)$-Dehn twist. Note that $(+1)$-Dehn twist (resp. (-1)-Dehn twist) is also called a *left* (resp. *right*) Dehn twist.

Next let a, m $(0 < a < m)$ and b, n $(0 < b < n)$ be relatively prime integers. A self-homeomorphism g of an annulus $[0,1] \times S^1$ given by $g(t,\, z) = (t,\, e^{2\pi \mathrm{i}\{(t-1)a/m + tb/n\}} z)$ is called a $\left(\frac{a}{m},\, \frac{b}{n}\right)$-*fractional Dehn twist*. More generally, the composition $h = f^k \circ g$ is said to be an $\left(\frac{a}{m},\, \frac{b}{n},\, k\right)$-*fractional Dehn twist*. Explicitly, $h(t,\, z) = (t,\, e^{2\pi \mathrm{i}\{(t-1)a/m + tb/n + tk\}} z)$. Note that f^k and g are commutative.



A self-homeomorphism f of an annulus $[0,1] \times S^1$ is a *Nielsen twist* if the action of f interchanges the boundary components $\{0\} \times S^1$ and $\{1\} \times S^1$, and f^2 is a fractional Dehn twist of $[0,1] \times S^1$. (Roughly speaking, it is the composition of an involution and a fractional Dehn twist.) Explicitly, an $\frac{a}{2m}$-*Nielsen twist* is given by

$$(t,\, z) \in [0,1] \times S^1 \longmapsto (t,\, e^{2\pi \mathrm{i}\{(2t-1)a/2m\}} \overline{z}) \in [0,1] \times S^1.$$

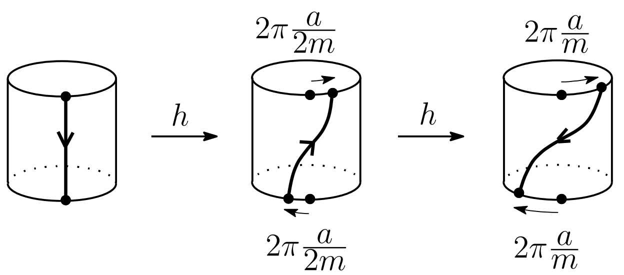


Figure 2: An $\frac{a}{2m}$-Nielsen twist h.

Its square is an $-\left(\frac{a}{m}, \frac{a}{m}\right)$-fractional Dehn twist.

More generally, an $\left(\frac{a}{2m}, \kappa\right)$-*Nielsen twist* is the composition of an $\frac{a}{2m}$-*Nielsen twist* and a $(-\kappa)$-Dehn twist (*not* $+\kappa$), explicitly given by

$$(t,\, z) \in [0,1] \times S^1 \longmapsto (t,\, e^{2\pi \mathrm{i}\{(2t-1)a/2m+t\kappa\}}\overline{z}) \in [0,1] \times S^1.$$

Its square is a $-\left(\frac{a}{m}, \frac{a}{m}, 2\kappa\right)$-fractional Dehn twist.

Local model of degenerations

A fractional Dehn twist appears as the topological monodromy of a degeneration: Set $c := \gcd(m,n)$, $m' := m/c$, $n' := n/c$, and let $\gamma : \mathbb{C}^3 \to \mathbb{C}^3$ be an automorphism defined by

$$\gamma :\, (z,w,t) \longmapsto (e^{2\pi \mathrm{i}a/m}z,\, e^{2\pi \mathrm{i}b/n}w,\, e^{2\pi \mathrm{i}/m'n'c}t). \tag{1}$$

Suppose that γ preserves $A_{d-1} := \{(z,w,t) \in \mathbb{C}^3 \,:\, zw = t^d\}$; this is the case precisely when $e^{2\pi \mathrm{i}a/m}e^{2\pi \mathrm{i}b/n} = e^{2\pi \mathrm{i}d/m'n'c}$, that is, $\frac{a}{m} + \frac{b}{n} \equiv \frac{d}{m'n'c} \bmod \mathbb{Z}$. Write $d = m'n'c\left(\frac{a}{m} + \frac{b}{n} + \kappa\right)$ for some integer κ such that $\frac{a}{m} + \frac{b}{n} + \kappa > 0$. Let Γ be the cyclic group generated by γ. Define a holomorphic map $\Phi : A_{d-1} \to \mathbb{C}$ by $\Phi(z,w,t) = t^{m'n'c}$. Then Φ is Γ-invariant, so descends to a holomorphic map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$, which is a degeneration of annuli whose topological monodromy is a $-\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist.

A Nielsen twist also appears as the topological monodromy of a degeneration: Let $\gamma' : \mathbb{C}^3 \to \mathbb{C}^3$ be an automorphism defined by

$$\gamma' :\, (z,w,t) \longmapsto (e^{2\pi \mathrm{i}a/2m}w, e^{2\pi \mathrm{i}a/2m}z, e^{2\pi \mathrm{i}/2m}t). \tag{2}$$

Suppose that γ' preserves A_{d-1}; this is the case precisely when $e^{2\pi \mathrm{i}a/m} = e^{2\pi \mathrm{i}d/2m}$, that is, $\frac{a}{m} \equiv \frac{d}{2m} \bmod \mathbb{Z}$. Write $d = 2a + 2m\kappa$ for some integer $\kappa \geq 0$. Let Γ' be the cyclic group generated by γ'. Define a holomorphic map $\Phi' : A_{d-1} \to \mathbb{C}$ by $\Phi'(z,w,t) = t^{2m}$. Then Φ' is Γ'-invariant, so descends to a holomorphic map $\overline{\Phi}' : A_{d-1}/\Gamma' \to \mathbb{C}$, which is a degeneration of annuli whose topological monodromy is an $\left(\frac{a}{2m}, \kappa\right)$-Nielsen twist.

The total spaces A_{d-1}/Γ and A_{d-1}/Γ' of the above degenerations are singular, so in order to obtain degenerations whose total spaces are smooth, we have to resolve them. Takamura [Tak2] showed that A_{d-1}/Γ is actually a cyclic quotient singularity $\mathbb{C}^2/G$, and moreover he showed that A_{d-1}/Γ' is a quotient of $A_{d-1}/\widetilde{\Gamma}$ by an involution, where $\widetilde{\Gamma}$ is a cyclic group generated by $(\gamma')^2$ (and thus $A_{d-1}/\widetilde{\Gamma} \cong \mathbb{C}^2/\widetilde{G}$ for some cyclic group $\widetilde{G}$). We can explicitly construct the above subgroups G, G', $\widetilde{G} \subset GL(2,\mathbb{C})$.

These groups are *small*, that is, do not contain pseudo-reflections. Here a matrix $A \in GL(n,\mathbb{C})$ is called a *pseudo-reflection* if it is diagonalizable and its eigenvalues are $\mu, 1, 1, \ldots, 1$, where $\mu \neq 1$ is a root of unity. (In particular if $\mu = -1$, then A is called a *reflection.*) If G is generated by pseudo-reflections, then $\mathbb{C}^n/G \cong \mathbb{C}^n$ (Chevalley-Shephard-Todd theorem).

Remark 1. If $A \in GL(n,\mathbb{C})$ is a pseudo-reflection, a linear map $h : \mathbb{C}^n \to \mathbb{C}^n$, $h(\vec{x}) = A\vec{x}$ is also called a *pseudo-reflection.* Let $\mathrm{Fix}(h)$ denote the hyperplane in $\mathbb{C}^n$ fixed by h. For example, for a root ζ of unity, a linear map $h : \mathbb{C}^n \to \mathbb{C}^n$ given by $h(z_1, z_2, \ldots, z_n) = (z_1, \ldots, \zeta z_i, \ldots, z_n)$ is a pseudo-reflection and $\mathrm{Fix}(h)$ is a hyperplane defined by $z_i = 0$ (see Figure 3).

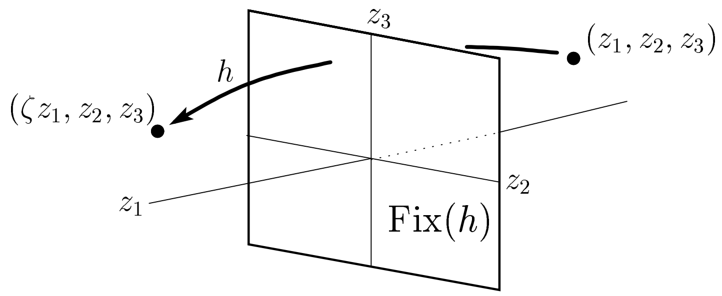


Figure 3:

Cyclic quotient construction

Given a periodic homeomorphism h of a closed surface C, we can construct a degeneration $\pi : M \to D := \{s \in \mathbb{C} : |s| < 1\}$ with topological monodromy h as follows: Let m denote the order of h and define an automorphism $g : C \times \mathbb{C} \to C \times \mathbb{C}$ by

$$g : \ (x, t) \ \longmapsto \ \big(h^{-1}(x),\, e^{2\pi \mathrm{i}/m} t\big).$$

Let G be a cyclic group generated by g and take a holomorphic function $\phi : C \times \mathbb{C} \to \mathbb{C}$ given by $\phi(x,t) = t^m$. Then ϕ is G-invariant, and it thus determines a holomorphic function $\overline{\phi} : (C \times \mathbb{C})/G \to \mathbb{C}$. Here note that $(C \times \mathbb{C})/G$ has cyclic quotient singularities. They are resolved by the *Hirzeburch-Jung resolution* explained below. Let $\mathfrak{r} : M \to (C \times \mathbb{C})/G$ be the minimal resolution. The composite map $\pi : \overline{\phi} \circ \mathfrak{r} : M \to \mathbb{C}$ is then a degeneration with topological monodromy h.

The singular fiber $X := \pi^{-1}(0)$ is star-shaped (*stellar*): X consists of a central irreducible component (the core) and chains of projective lines emanating from it (the branches). Let $v_1, v_2, \ldots, v_l$ denote the branch points of a quotient map $\psi : C \to B := C/\langle h \rangle$. Then B is the core of X, and the branches E_j emanate from $v_1, v_2, \ldots, v_l$: $X = mB + \sum_{j=1}^{l} E_j$.

Hirzeburch-Jung resolution

Let l and q $(0 < q < l)$ be relatively prime integers and let g be an automorphism of $\mathbb{C}^2$ given by

$$g : (u, v) \ \mapsto \ (e^{2\pi \mathrm{i}/l} u, \, e^{2\pi \mathrm{i} q/l} v).$$

Let G denote the cyclic group of order l generated by g. Here the origin is the fixed point of the action of G, and its image $\overline{0} \in \mathbb{C}^2/G$ under the quotient map $\mathbb{C}^2 \to \mathbb{C}^2/G$ is an isolated singularity (a *cyclic quotient singularity*) of $\mathbb{C}^2/G$. The minimal resolution of $\mathbb{C}^2/G$ is explicitly constructed as follows (*Hirzeburch-Jung resolution*): First take the negative continued fraction expansion of $\frac{l}{q}$:

$$\frac{l}{q} = r_1 - \cfrac{1}{r_2 - \cfrac{1}{r_3 - \ddots - \cfrac{1}{r_\delta}}} \, ,$$

where $r_1, r_2, \ldots, r_\delta$ are integers greater than 1. Take projective lines Θ_1, Θ_2, $\ldots$, Θ_δ. Let $\Theta_i = U_i \cup V_i$ be an open covering by two complex lines $(U_i = V_i = \mathbb{C})$ where $w_i \in U_i \setminus \{0\}$ and $z_i \in V_i \setminus \{0\}$ are glued via $z_i = \frac{1}{w_i}$. Next construct a line bundle N_i on Θ_i by gluing $(w_i, \eta_i) \in (U_i \setminus \{0\}) \times \mathbb{C}$ and $(z_i, \zeta_i) \in (V_i \setminus \{0\}) \times \mathbb{C}$ via

$$z_i = \frac{1}{w_i}, \qquad \zeta_i = w_i^{r_i} \eta_i.$$

Let R denote the smooth complex surface given by patching N_i and N_{i+1} $(i = 1, 2, \ldots, \delta - 1)$ by $(w_{i+1}, \eta_{i+1}) = (\zeta_i, z_i)$. Then R is the minimal resolution space of $\mathbb{C}^2/G$: There exists a holomorphic map (resolution map) $\mathfrak{r} : R \to \mathbb{C}^2/G$ such that the restriction $\mathfrak{r} : R \backslash \mathfrak{r}^{-1}(0) \to (\mathbb{C}^2/G)\backslash\{0\}$ is an isomorphism and the exceptional set $\mathfrak{r}^{-1}(0)$ of R is the chain of projective lines $\Theta_1 + \Theta_2 + \cdots + \Theta_\delta$. Here note that the self-intersection number $\Theta_i \cdot \Theta_i$ of Θ_i in R is the degree of N_i, that is, $-r_i$. In particular, Θ_i is not an exceptional curve in R, because $r_i \neq 1$.

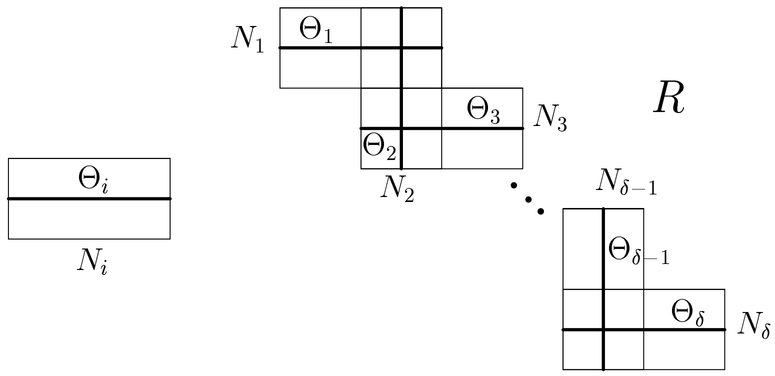


Figure 4: The resolution space R of $\mathbb{C}^2/G$ is constructed by "plumbing" the line bundles $N_1, N_2, \ldots, N_\delta$.

Here the resolution map $\mathfrak{r}$ is given as follows: First take a sequence of integers $m_0 > m_1 > \cdots > m_\delta \ (= 1) > m_{\delta+1} \ (= 0)$ inductively defined by

$$\begin{cases} m_0 = l, m_1 = q, \\ m_{i+1} = r_i m_i - m_{i-1} \quad (i = 1, 2, \ldots, \delta), \end{cases}$$

and define the *degeneration map* $\pi : R \to \mathbb{C}$ by

$$\begin{cases} \pi(w_i, \eta_i) = w_i^{m_{i-1}} \eta_i^{m_i} & (w_i, \eta_i) \in U_i \times \mathbb{C}, \\ \pi(z_i, \zeta_i) = z_i^{m_{i+1}} \zeta_i^{m_i} & (z_i, \zeta_i) \in V_i \times \mathbb{C}. \end{cases}$$

Next let q^* $(0 < q^* < l)$ be the integer satisfying $qq^* \equiv 1 \bmod l$ and take a sequence of integers $b_0 > b_1 > \cdots > b_\delta \ (= 1) > b_{\delta+1} \ (= 0)$ inductively defined by

$$\begin{cases} b_0 = l, b_1 = q^*, \\ b_{i+1} = r_i b_i - b_{i-1} \quad (i = 1, 2, \ldots, \delta). \end{cases}$$

Then define the *canonical projection* $\mathrm{pr} : R \to \mathbb{C}$ by

$$\begin{cases} \mathrm{pr}(w_i, \eta_i) = w_i^{b_{i-1}} \eta_i^{b_i} & (w_i, \eta_i) \in U_i \times \mathbb{C}, \\ \mathrm{pr}(z_i, \zeta_i) = z_i^{b_{i+1}} \zeta_i^{b_i} & (z_i, \zeta_i) \in V_i \times \mathbb{C}. \end{cases}$$

Define next an l-valued map $\widetilde{\mathfrak{r}} : R \to \mathbb{C}^2$ by

$$\widetilde{\mathfrak{r}}(x) = \left\{ (e^{2\pi ik/l}\mathrm{pr}(x)^{1/l}, e^{2\pi ik/l}\pi(x)^{1/l}) \ : \ k = 1, 2, \ldots, l \right\}.$$

Here note that for each point $x \in R$, the image $\widetilde{\mathfrak{r}}(x)$ is an orbit under the G-action, so the $\widetilde{\mathfrak{r}} : R \to \mathbb{C}$ determines a single-valued map $\mathfrak{r} : R \to \mathbb{C}^2/G$, which is the resolution map of $\mathbb{C}^2/G$.

Construction of degenerations with negative pseudo-periodic monodromies

Given a negative pseudo-periodic homeomorphism h, we may construct a degeneration with topological monodromy h. We outline its construction: Let h be a negative pseudo-periodic homeomorphism of a closed surface Σ, with admissible system $\{l_1, l_2, \ldots, l_q\}$. Let N be the pseudo-order of h, and suppose that h^N is generated by negative Dehn twist along $l_1, l_2, \ldots, l_q$. For each loop l_i ($i = 1, 2, \ldots, q$), let a_i be the least positive integer such that h^{a_i} is a Dehn twist (say a $(-d_i)$-Dehn twist) along l_i. Noting that a_i divides N, set $\nu_i = \dfrac{N}{a_i}$. Then in a neighborhood of l_i in Σ, h^N is a $(-\nu_i d_i)$-Dehn twist along l_i. Here d_i and ν_i are referred to as the *twist number* and *cotwist number* of h along l_i.

Now let C be a singular complex curve obtained from Σ by pinching each loop l_i to a point p_i; so C has nodes $p_1, p_2, \ldots, p_q$. Then there exist a stable degeneration $\varphi : S \to \Delta$ with topological monodromy h^N and a periodic automorphism $g : S \to S$ of order N such that

(i) $C = \varphi^{-1}(0)$ (so $p_i \in C \subset S$),

(ii) S has $A_{\nu_i d_i - 1}$-singularities at p_i ($i = 1, 2, \ldots, q$), and

(iii) the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "\pi"'] & S \arrow[d, "\pi"] \\ \Delta \arrow[r, "k"] & \Delta, \end{tikzcd}$$

where $k : t \mapsto e^{2\pi \mathrm{i}/N} t$ is a $\dfrac{2\pi}{N}$-rotation of Δ.

Here $\varphi : S \to \Delta$ is explicitly constructed by patching $C_j \times \mathbb{C}$ (where C_j is an irreducible component of C) and $A_{\nu_i d_i - 1}$-singularities. Let G be a cyclic group generated by g. Then $\psi := \varphi^N$ is a G-invariant holomorphic function on S, and it thus determines a holomorphic function $\overline{\psi} : S/G \to \Delta$. Let $\mathfrak{r} : M \to S/G$ be the resolution of the quotient space S/G, then the composite map $\pi := \overline{\psi} \circ \mathfrak{r} : M \to \Delta$ is the desired degeneration — its topological monodromy is h.

The configuration of the singular fiber $X := \pi^{-1}(0)$ is *constellar*: X consists of cores, branches, trunks and bihorns. The action of h is a rotation around a branch, a fractional Dehn twist around a trunk, and a Nielsen twist around a bihorn.

Example: Pseudo-periodic homeomorphisms and degenerations (non-amphidrome case)

Using a fractional Dehn twist, we can construct a pseudo-periodic homeomorphism of compact surfaces. Let Σ_1 be a torus and $f : \Sigma_1 \to \Sigma_1$ be a self-homeomorphism of order 6 that fixes a point p, interchanges two points q_1, q_2, and permutes three points r_1, r_2, r_3 as $r_1 \mapsto r_2$, $r_2 \mapsto r_3$, $r_3 \mapsto r_1$ (Figure 5). Next let Σ_2 be another torus, and $g : \Sigma_2 \to \Sigma_2$ be a self-homeomorphism of order 3 that fixes three points p', q', r' (Figure 5).

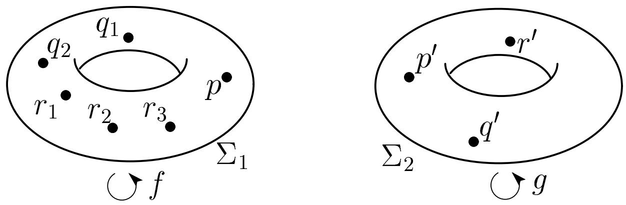


Figure 5:

Suppose that f is a $2\pi \times \frac{5}{6}$-rotation around p and g' is a $2\pi \times \frac{1}{3}$-rotation around p'. Let D_1, D_2 be disks in Σ_1, Σ_2 centered at p, p'. Set $\Sigma_1' := \Sigma_1 \setminus D_1$, $\Sigma_2' := \Sigma_2 \setminus D_2$ and let f', g' denote the restrictions of f, g to Σ_1', Σ_2'. As illustrated in Figure 6, combining $\Sigma_1 \setminus D_1$, $\Sigma_2 \setminus D_2$ and an annulus A, we obtain a compact surface Σ of genus 2. Take an integer κ and let $\varphi : A \to A$ be a $\left(\frac{5}{6}, \frac{2}{3}, \kappa\right)$-fractional Dehn twist. Then $f' : \Sigma_1' \to \Sigma_1'$, $g' : \Sigma_2' \to \Sigma_2'$ together with $\varphi : A \to A$ defines a self-homeomorphism h of Σ. Note that h is a pseudo-periodic homeomorphism. In fact, h preserves a loop l in Σ and the restriction $h|_{\Sigma \setminus l}$ is isotopic to a periodic homeomorphism of order 6. Moreover h is positive, indeed it is generated by a $\left(\frac{5}{6}, \frac{2}{3}\right)$-fractional Dehn twist (so h^6 is a $(+9)$-Dehn twist along l and $(h|_{\Sigma \setminus l})^6$ is isotopic to the identity map of $\Sigma \setminus l$).

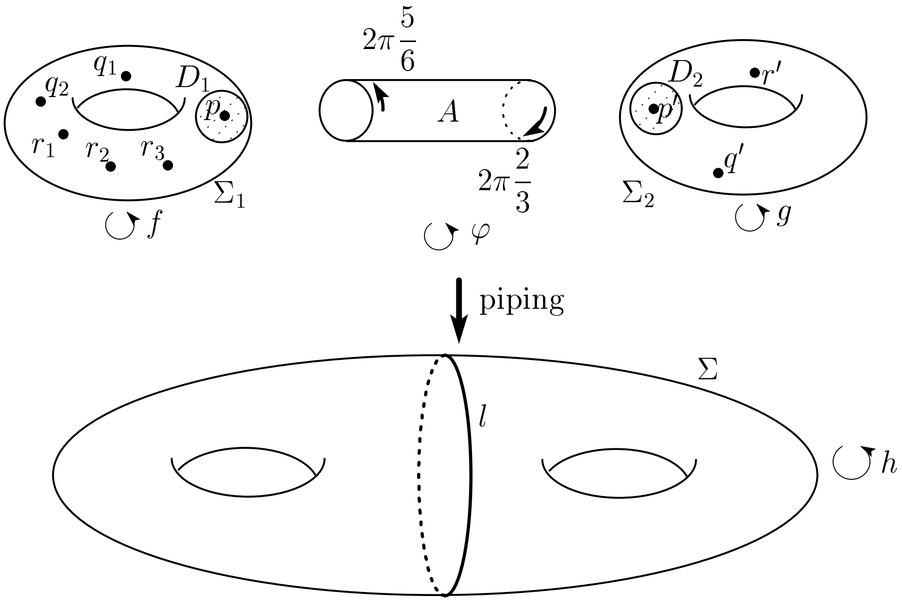


Figure 6:

Remark 2. An $\left(\frac{a}{m}, \frac{b}{n}\right)$-fractional Dehn twist φ is isotopic to the identity map. It is however not isotopic to the identity map through an isotopy fixing the boundaries, and the self-homeomorphism $h : \Sigma \to \Sigma$ (which is defined by 'patching' f', g' and φ) is not isotopic to the identity map of Σ. In particular, $[h]$ is a nontrivial element of the mapping class group of Σ.

Corresponding degenerations

We next consider the inverse $h^{-1} : \Sigma \to \Sigma$ of the positive pseudo-periodic homeomorphism h. Similarly as h, we can construct h^{-1} from periodic self-homeomorphisms f^{-1}, g^{-1} of tori and a $-\left(\frac{5}{6}, \frac{2}{3}, \kappa\right)$-fractional Dehn twist φ^{-1} of an annulus. Note that h^{-1} is a negative pseudo-periodic homeomorphism. Indeed it is generated by a $-\left(\frac{5}{6}, \frac{2}{3}, \kappa\right)$-fractional Dehn twist. If $\kappa \geq -1$, it appears as the topological monodromy of a degeneration of Σ. For $\kappa \geq 0$, a degeneration $\pi : R \to \mathbb{C}$ with topological monodromy h^{-1} is as illustrated in Figure 7.

The singular fiber X of $\pi : R \to \mathbb{C}$ is given by connecting singular fibers of two degenerations of tori whose topological monodromies are periodic homeomorphisms f^{-1}, g^{-1}:

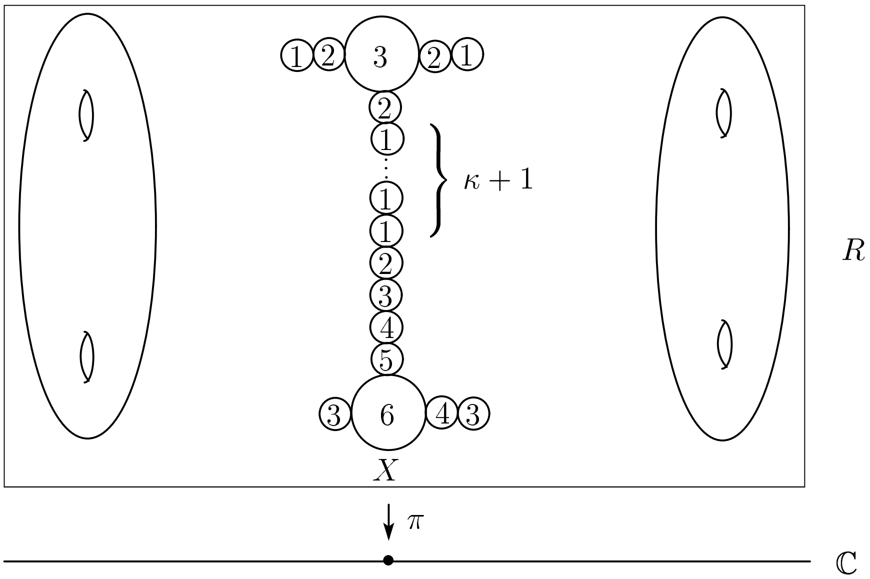
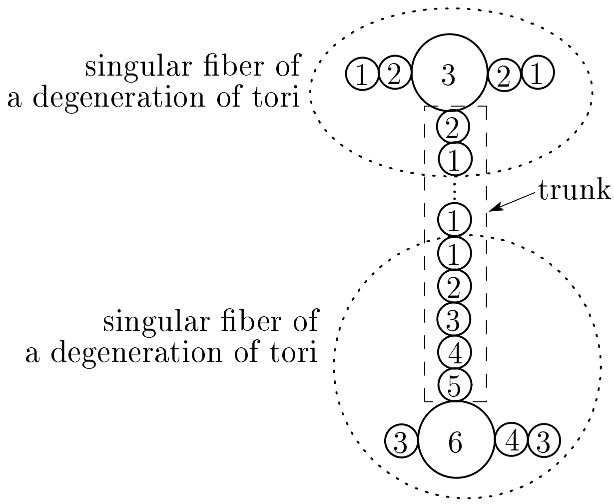


Figure 7: A circle means a projective line, and the number on it is its multiplicity.



The topological monodromy around the trunk connecting two singular fibers is a $-\left(\frac{5}{6}, \frac{2}{3}, \kappa\right)$-fractional Dehn twist.

On the other hand if $\kappa = -1$, the singular fiber of $\pi : R \to \mathbb{C}$ is given by omitting a part of the trunk of the singular fiber of the case $\kappa \geq 0$ (see Figure 8).

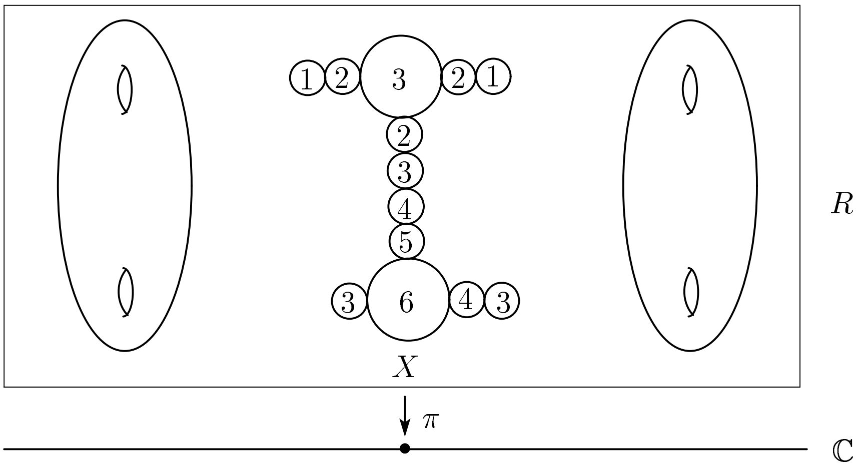


Figure 8:

Example: Pseudo-periodic homeomorphisms and degenerations (amphidrome case)

Consider the smooth surface Σ in Figure 6, and let h be its negative pseudo-periodic homeomorphism generated by a $\left(\frac{a}{2m}, \kappa\right)$-Nielsen twist along l. Then l is the fixed point set of h. Let C be the stable curve of genus 2 obtained by pinching l on Σ of Figure 6. Then an automorphism $f : C \to C$ induced from h interchanges irreducible components C_1, C_2 of C.

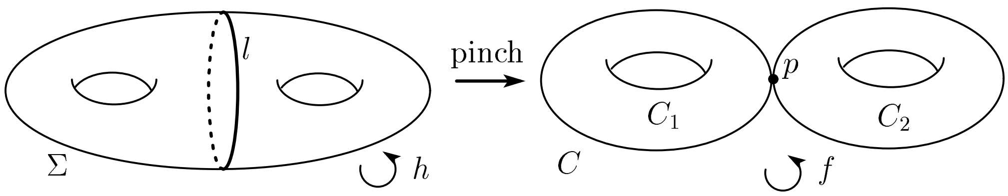


Figure 9:

Set $d := 2 + 2\kappa$, then there exists a stable degeneration $\varphi : S \to \Delta$ and a periodic automorphism $g : S \to S$ of order 2 such that $C = \varphi^{-1}(0)$, $g\,|_C = f$ and the complex surface S has A_{d-1}-singularity at p. Next let G be the cyclic group generated by g. (Note: $G \cong \mathbb{Z}_2$.) Then the quotient map $S \to S/G$ maps two irreducible components C_1, C_2 of C to a complex curve $\overline{C}_1$, and S/G has a dihedral singularity at $\overline{p}$, which is resolved by a bihorn containing $\kappa + 1$ copies of the projective line with multiplicity 2: Let $\mathfrak{r} : M \to S/G$ be the minimal resolution of S/G, then $\overline{\phi} \circ \mathfrak{r} : M \to D$ is a degeneration with topological monodromy h (Figure 10).

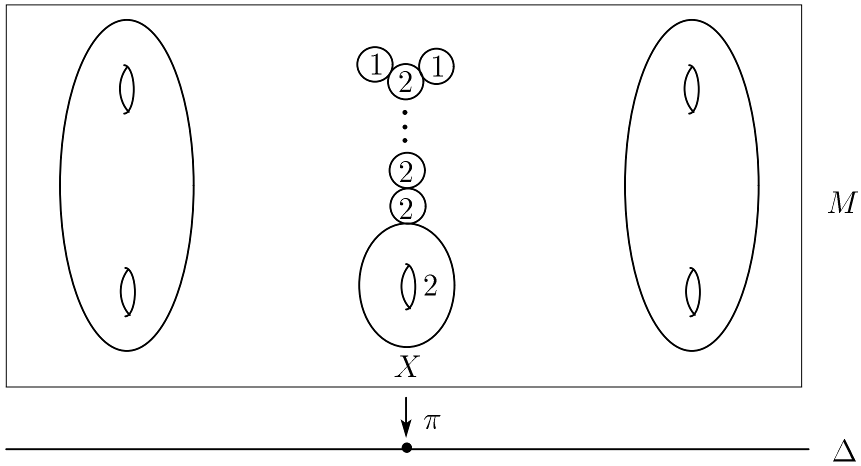


Figure 10:

Two kinds of A-singularities

There are two kinds of generalizations of A_{d-1}-singularity to higher-dimensions:

$$V_d := \{(z_1, z_2, \ldots, z_n, t) \in \mathbb{C}^{n+1} \ : \ z_1^2 + z_2^2 + \cdots + z_n^2 = t^d\},$$
$$W_d := \{(z_1, z_2, \ldots, z_n, t) \in \mathbb{C}^{n+1} \ : \ z_1 z_2 \cdots z_n = t^d\}.$$

They are respectively an *additive* and *multiplicative* A_{d-1}-singularities. The holomorphic maps $f : V_d \to \mathbb{C}$, $f(z_1, \ldots, z_n, t) = t$ and $g : W_d \to \mathbb{C}$, $g(z_1, \ldots, z_n, t) = t$ are two generalizations of the local model of degeneration whose topological monodromy is the $(-d)$-Dehn twist. Here note that the topological monodromies of f and g are different, in fact the singularity of V_d is the origin, while the singular locus of W_d is $\{z_i = z_j = t = 0 \ (i \neq j)\}$.

For the case $d = 1$, the topological monodromy of f is described by the *double covering method* (see [AGV]): Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a holomorphic function given by $f(z, w) = z^2 + w^2$. For $\lambda \in \mathbb{C}$, set $V_\lambda := f^{-1}(\lambda) = \{(z, w) : z^2 + w^2 = \lambda\}$. Then the map $(z, w) \in V_\lambda \mapsto z \in \mathbb{C}$ is a ramified double covering, whose branch points are $\pm\sqrt{\lambda}$. Joining two copies of $\mathbb{C}$ along the cut $(-\sqrt{\lambda}, \sqrt{\lambda})$, an annulus V_λ $(\lambda \neq 0)$ is obtained.

Take a positive number α and set $\lambda(t) := e^{2\pi \mathrm{i} t}\alpha$ $(0 \leq t \leq 1)$. As t varies from 0 to 1, the branch points $z = \pm\sqrt{\lambda(t)} = e^{\pi \mathrm{i} t}(\pm\sqrt{\alpha})$ move around the point $z = 0$ in the positive direction. The orbits are semicircles, so a

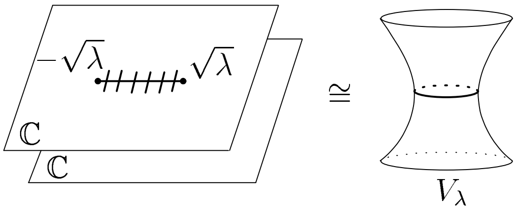
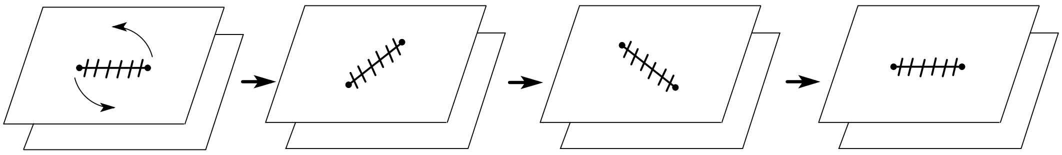


Figure 11:

branch point arrives at the other's place. Thus, a family of diffeomorphisms $\{\Gamma_t : V_{\lambda(0)} \to V_{\lambda(t)}\}_{0 \le t \le 1}$ is constructed, where Γ_0 is the identity map and Γ_1 is the *topological monodromy* of f.



Next let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function given by $f(z_1, z_2, \ldots, z_n) = z_1^2 + z_2^2 + \cdots + z_n^2$. For $\lambda \in \mathbb{C}$, set

$$V_\lambda := f^{-1}(\lambda) = \{(z_1, z_2, \ldots, z_n) \,:\, z_1^2 + z_2^2 + \cdots + z_n^2 = \lambda\}.$$

The map $(z_1, \ldots, z_n) \in V_\lambda \mapsto (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1}$ is a ramified double covering. Its branch locus is $z_1^2 + \cdots + z_{n-1}^2 = \lambda$. Then V_λ ($\lambda \neq 0$) is obtained by joining two copies of $\mathbb{C}^{n-1}$ along the loci defined by $z_1^2 + z_2^2 + \cdots + z_{n-1}^2 = \lambda$. The topological monodromy of f is constructed and described by using this model similarly as in the case $n = 2$.

Clemens' description

Clemens [Cle1] described the homological monodromy of a degeneration of compact complex manifolds whose singular fiber has at most normal crossings, that is, the underlying space of the singular fiber is locally defined by $z_1 z_2 \cdots z_n = 0$ (accordingly this degeneration is locally given by $(z_1, z_2, \ldots, z_n) \mapsto z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$, where $m_1, m_2, \ldots,\ m_n$ are positive integers). Initially he described its local monodromies: Consider a holomorphic map $\pi : \mathbb{C}^n \to \mathbb{C}$, $\pi(z_1, z_2, \ldots, z_n) = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$. Every smooth fiber of π consists of $m := \gcd(m_1, m_2, \ldots, m_n)$ connected components and each of them is homeomorphic to $H \times T^{n-1}$, where $H := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n\ :$

$\lambda_1+\cdots+\lambda_n = 1\}$ and T^{n-1} is an $(n-1)$-dimensional torus. For a smooth fiber $\pi^{-1}(s)$, Clemens constructed a map $\Phi : \pi^{-1}(s) \to \Delta_{n-1} := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \lambda_1+\cdots+\lambda_n = 1,\ 0 \le \lambda_i \le 1\}$ (the $(n-1)$-standard simplex). The map Φ is the composition of $\pi^{-1}(s) \to \mathbb{R}^n_{>0}$, $(z_1, \ldots, z_n) \mapsto (|z_1|, \ldots, |z_n|)$ and a homeomorphism $\mathbb{R}^n_{>0} \to H$ and a retraction $H \to \Delta_{n-1}$.

He then constructed a 1-parameter family of homeomorphisms $\{f_\theta : \pi^{-1}(s) \to \pi^{-1}(se^{\mathrm{i}\theta})\}_{0\le\theta\le 2\pi}$ as follows: For $(z_1, z_2, \ldots, z_n) \in \pi^{-1}(s)$, setting $(\lambda_1, \lambda_2, \ldots, \lambda_n) := \Phi(z_1, z_2, \ldots, z_n)$,

$$f_\theta(z_1, z_2, \ldots, z_n) := (e^{\mathrm{i}\theta\lambda_1/m_1} z_1,\ e^{\mathrm{i}\theta\lambda_2/m_2} z_2, \ldots, e^{\mathrm{i}\theta\lambda_n/m_n} z_n).$$

Then $f_{2\pi} : \pi^{-1}(s) \to \pi^{-1}(s)$ is the monodromy of $\pi : \mathbb{C}^n \to \mathbb{C}$. The restriction of $f_{2\pi}$ to the compact part (Figure 12) gives the desired local monodromy. Patching them together yields the global monodromy.

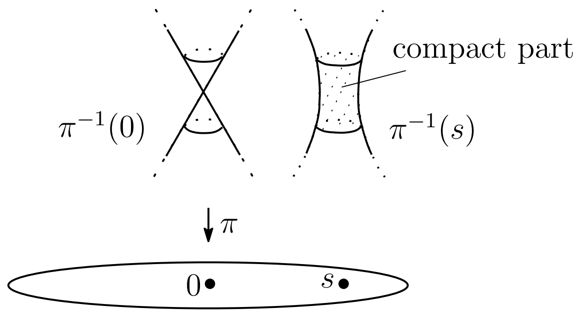


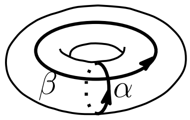
Figure 12:

Monodromy theorem [Cle1] *Let $\pi : M \to D = \{s \in \mathbb{C} : |s| < 1\}$ be a degeneration of compact complex manifolds whose singular fiber has at most normal crossings. For a smooth fiber $X_s := \pi^{-1}(s)$ $(s \neq 0)$, the homological monodromy $h_{*,k} : H_k(X_s, \mathbb{C}) \to H_k(X_s, \mathbb{C})$ $(k = 1, 2, \ldots)$ is quasi-unipotent.*

Here a square matrix P is *unipotent* if $(P - I)^b = O$ for some positive integer b, and *quasi-unipotent* if $(P^a - I)^b = O$ for some positive integers a and b.

Example 1. (1) If the topological monodromy h is a periodic homeomorphism of order l, then $h^l = \mathrm{id}$ and thus $h^l_{*,k} = I$ for any k, so the monodromy $h_{*,k}$ is quasi-unipotent.

(2) If X_s is a torus, then take a basis $[\alpha]$, $[\beta]$ of $H_1(X_s, \mathbb{C}) \cong \mathbb{C}^2$:



If the topological monodromy h is a $(-d)$-Dehn twist along α, then $h_{*,1} = \begin{pmatrix} 1 & -d \\ 0 & 1 \end{pmatrix}$. So $(h_{*,1} - I)^2 = O$, and thus $h_{*,1}$ is unipotent.

(3) If X_s is a torus and h is a negative pseudo-periodic homeomorphism, say that h^N is generated by negative Dehn twists, then

$$(h_{*,1}^N - I)^2 = O.$$

So $h_{*,1}$ is unipotent if $N = 1$, and quasi-unipotent if $N \geq 2$.

Similarly if X_s is a closed surface of genus g and h is negative pseudo-periodic, say that h^N is generated by negative Dehn twists, then $H_k(X_s, \mathbb{C}) \cong \mathbb{C}^{2g}$ and $h_{*,1} \in GL(2g, \mathbb{C})$ satisfies $(h_{*,1}^N - I)^{2g} = O$. So $h_{*,1}$ is unipotent if $N = 1$, and quasi-unipotent if $N \geq 2$.

Introduction

Introduction to Part I C. H. Clemens [Cle1] described the homological monodromy of a degeneration whose singular fiber has at most normal crossings, and showed that its action is quasi-unipotent. On the other hand, Y. Matsumoto and J. M. Montesinos [MM2] showed that the topological type of a degeneration of Riemann surfaces of genus ≥ 2 is determined by its topological monodromy. A simple example of a degeneration of Riemann surfaces is a Lefschetz fibration whose topological monodromy is a Dehn twist. In higher-dimensional case, there are two different generalizations of a Dehn twist: Consider *additive* and *multiplicative* A_{d-1}-singularities:

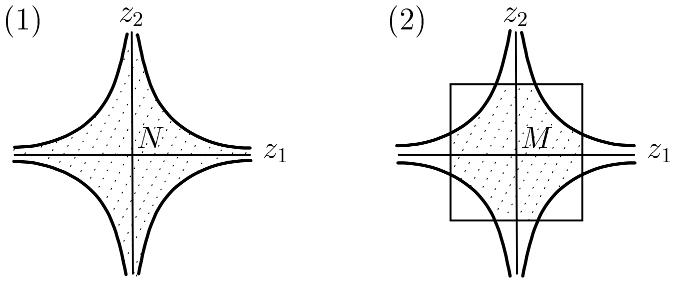
$$\begin{aligned} V_d &:= \{(z_1, z_2, \ldots, z_n, t) \in \mathbb{C}^{n+1} \ : \ z_1^2 + z_2^2 + \cdots + z_n^2 = t^d\}, \\ W_d &:= \{(z_1, z_2, \ldots, z_n, t) \in \mathbb{C}^{n+1} \ : \ z_1 z_2 \cdots z_n = t^d\}. \end{aligned}$$

Then $f : V_d \to \mathbb{C}$, $f(z_1, \ldots, z_n, t) = t$ and $g : W_d \to \mathbb{C}$, $g(z_1, \ldots, z_n, t) = t$ are degenerations, and their topological monodromies F_d and G_d are two different generalizations of a $(-d)$-Dehn twist. Note that $F_d = (F_1)^d$ and $G_d = (G_1)^d$, so it suffices to describe the case $d = 1$. In the sequel, we omit d to write V_1 and W_1 as V and W.

Remark If $n \geq 3$, V and W are *not* isomorphic (in contrast for $n = 2$, they are isomorphic). In fact, the singularities of the singular fiber of $g : W \to \mathbb{C}$ are not isolated, while that of $f : V \to \mathbb{C}$ is isolated.

The topological monodromy of the degeneration $f : V \to \mathbb{C}$ is described by the *double covering method* (see [AGV] p.6). We will describe the topological monodromy of the degeneration $g : W \to \mathbb{C}$. First note that W is isomorphic to $\mathbb{C}^n$ via $(z_1, z_2, \ldots, z_n, t) \mapsto (z_1, z_2, \ldots, z_n)$, accordingly $g : W \to \mathbb{C}$ is identified with a map $\pi : \mathbb{C}^n \to \mathbb{C}$ given by $\pi(z_1, z_2, \ldots, z_n) = z_1 z_2 \cdots z_n$.

Now set $N := \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \ : \ |z_1 z_2 \cdots z_n| \leq 1\}$ and $D := \{s \in \mathbb{C} \ : \ |s| \leq 1\}$. The restriction $\pi : N \to D$ of $\pi : \mathbb{C}^n \to \mathbb{C}$ to N is a 'shrinking' of $\pi : \mathbb{C}^n \to \mathbb{C}$. Here N is closed but not compact (see (1) of Schematic figure). To obtain a compact one, take $\rho > 1$ and set $M := \{(z_1, z_2, \ldots, z_n) \in N \ : \ |z_i| \leq \rho \ (i = 1, 2, \ldots, n)\}$. Note that M is compact, indeed M is the intersection of the closed set N and the polydisk of radius ρ (see (2) of Schematic figure). The restriction $\pi : M \to D$ of $\pi : \mathbb{C}^n \to \mathbb{C}$ to M is a local model of a degeneration of complex manifolds.



Schematic figure for $n = 2$

In order to describe the homological monodromy of a degeneration of complex manifolds, Clemens [Cle1] describes the monodromy of the degeneration $\pi : M \to D$ above. In this paper, we alternatively describe it from a different viewpoint. While for $n = 2$, the topological monodromy of π is known to be a (-1)-Dehn twist of an annulus, for $n \geq 3$, we describe the action of the topological monodromy in terms of n foliations constructed on a smooth fiber of π; precisely speaking, these are *singular* foliations — the dimension of some leaf is less than that of a generic leaf. The ith foliation is used for describing its action on the ith face of the boundary of a smooth fiber.

More explicitly, the description of $\pi^{-1}(s)$ $(s = re^{\mathrm{i}\xi} \neq 0)$ proceeds as follows:

Step 1 Express $\pi^{-1}(s) = J_r \times K_\xi$, where

$$J_r := \big\{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq \rho,\ x_1 \cdots x_n = r\big\},$$
$$K_\xi := \big\{(e^{\mathrm{i}\alpha_1}, \dots, e^{\mathrm{i}\alpha_n}) \in T^n : \alpha_1 + \cdots + \alpha_n \equiv \xi \bmod 2\pi\big\}$$

are respectively homeomorphic to the standard $(n-1)$-simplex Δ_{n-1} and an $(n-1)$-dimensional torus T^{n-1}.

Step 2 We explicitly construct homeomorphisms $\Psi : J_r \to \Delta_{n-1}$ and $\Phi : K_\xi \to T^{n-1}$. The construction of the former is quite involved and based on induction on the dimension. The latter is constructed as follows: Noting that $K_\xi \subset T^n$, let $\mathrm{pr} : T^n \to T^{n-1}$ be a projection given by $\mathrm{pr} : (e^{\mathrm{i}\alpha_1}, \dots, e^{\mathrm{i}\alpha_n}) \mapsto (e^{\mathrm{i}\alpha_1}, \dots, e^{\mathrm{i}\alpha_{n-1}})$. Then the restriction $\Phi = \mathrm{pr}\,|_{K_\xi} : K_\xi \to T^{n-1}$ is a homeomorphism.

Step 3 Using Ψ, we construct a 1-parameter family of homeomorphisms $\{f_\theta : M \to M\}_{0 \leq \theta \leq 2\pi}$ by $f_\theta : (z_1, \dots, z_n) \mapsto (e^{\mathrm{i}\theta\lambda_1} z_1, \dots, e^{\mathrm{i}\theta\lambda_n} z_n)$, where $(\lambda_1, \dots, \lambda_n) := \Psi(|z_1|, \dots, |z_n|)$. Then $h := f_{2\pi} : \pi^{-1}(s) \to \pi^{-1}(s)$ is the topological monodromy of π. Under the identification of $\pi^{-1}(s)$ with $\Delta_{n-1} \times$

T^{n-1} via $\Psi \times \Phi$, we show that h is given by

$$\begin{aligned} &\big((\lambda_1, \ldots, \lambda_n),\, (t_1, \ldots, t_{n-1})\big) \\ &\qquad \longmapsto \big((\lambda_1, \ldots, \lambda_n),\, (e^{2\pi \mathrm{i}\lambda_1} t_1, \ldots, e^{2\pi \mathrm{i}\lambda_{n-1}} t_{n-1})\big). \end{aligned} \tag{3}$$

(See Remark 3 in §4 for another 1-parameter family constructed by Clemens [Cle1].)

Step 4 As illustrated in Figure 13, we shrink the $(n-1)$-simplex Δ_{n-1} to obtain a family of $(n-1)$-simplexes $\Delta_{n-1|u}$ $(0 \le u \le 1)$ such that $\Delta_{n-1|1} = \Delta_{n-1}$ and $\Delta_{n-1|0}$ is the barycenter b of Δ_{n-1}.

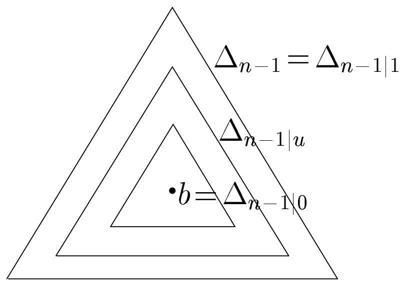


Figure 13:

Then $\{\partial \Delta_{n-1|u}\}_{0 \le u \le 1}$ is a singular foliation of Δ_{n-1}. We foliate $\Delta_{n-1} \times T^{n-1}$ by $\partial \Delta_{n-1|u} \times T^{n-1}$ $(0 \le u \le 1)$:

$$\Delta_{n-1} \times T^{n-1} = \coprod_{0 \le u \le 1} (\partial \Delta_{n-1|u} \times T^{n-1}) \quad \text{(disjoint union)}.$$

Now identify $\Delta_{n-1} \times T^{n-1}$ with a smooth fiber of $\pi : M \to D$ and regard the topological monodromy of π as a homeomorphism $h : \Delta_{n-1} \times T^{n-1} \to \Delta_{n-1} \times T^{n-1}$. By (3), h maps each leaf $\partial \Delta_{n-1|u} \times T^{n-1}$ to itself. To describe this action, we introduce n foliations of $\partial \Delta_{n-1|u} \times T^{n-1}$. First let $\mathcal{T}_i$ $(i = 1, 2, \ldots, n)$ be a foliation of T^{n-1} by parallel subtori $T^{n-2}_{i|v}$ $(v \in S^1)$ as illustrated in Figure 14 (for $n = 3$). Then $\partial \Delta_{n-1|u} \times T^{n-1}$ is foliated by $\partial \Delta_{n-1|u} \times T^{n-2}_{i|v}$ $(v \in S^1)$.

As mentioned above, h maps $\partial \Delta_{n-1|u} \times T^{n-1}$ to itself. We describe this action separately for $u = 0$ and $u \neq 0$.

(D1) Where $b = \partial \Delta_{n-1|0}$ is the barycenter of Δ_{n-1}, h acts on $\{b\} \times T^{n-1}$ as

$$(b, t_1, t_2, \ldots, t_{n-1}) \longmapsto (b, e^{2\pi \mathrm{i}/n} t_1, e^{2\pi \mathrm{i}/n} t_1, \ldots, e^{2\pi \mathrm{i}/n} t_{n-1}).$$

In particular, h maps $\{b\} \times T^{n-2}_{i|v}$ to $\{b\} \times T^{n-2}_{i|ve^{2\pi \mathrm{i}/n}}$.

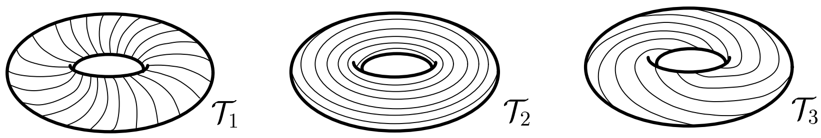


Figure 14:

Now let $\partial\Delta_{n-1}^{(j)}$ $(j = 1, 2, \ldots, n)$ denote the jth face of $\partial\Delta_{n-1}$, accordingly $\partial\Delta_{n-1|u}^{(j)}$ denotes the jth face of $\partial\Delta_{n-1|u}$ $(u \neq 0)$. See Figure 15.

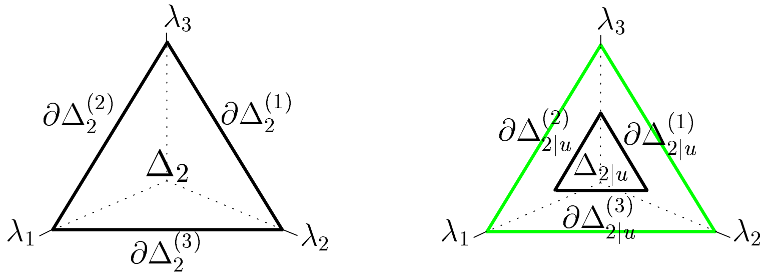


Figure 15:

(D2) The action of h on $\partial\Delta_{n-1|u} \times T^{n-1}$ $(u \neq 0)$ maps each face $\partial\Delta_{n-1|u}^{(j)} \times T^{n-1}$ $(j = 1, 2, \ldots, n)$ to itself. Setting $\mu := \dfrac{1-u}{n}$, this map is explicitly given by

$$\big((\lambda_1, \ldots, \lambda_{j-1}, \mu, \lambda_{j+1}, \ldots, \lambda_n), (t_1, t_2, \ldots, t_{n-1})\big) \longmapsto$$
$$\big((\lambda_1, \ldots, \lambda_{j-1}, \mu, \lambda_{j+1}, \ldots, \lambda_n), (e^{2\pi \mathrm{i}\lambda_1}t_1, \ldots, e^{2\pi\mathrm{i}\mu}t_j, \ldots, e^{2\pi\mathrm{i}\lambda_{n-1}}t_{n-1})\big).$$

Next fixing $u \neq 0$ and i, set $\mathcal{L}_v := \partial\Delta_{n-1|u} \times T^{n-2}_{i|v}$ and consider a foliation $\mathcal{F} = \{\mathcal{L}_v\}_{v \in S^1}$ of $\partial\Delta_{n-1|u} \times T^{n-1}$. Then $\mathcal{L}_v^{(j)} := \partial\Delta_{n-1|u}^{(j)} \times T^{n-2}_{i|v}$ is the *jth face* of $\mathcal{L}_v$. The following holds:

(D3) h does not preserve $\mathcal{F}$ — each face $\mathcal{L}_v^{(j)}$ $(j \neq i)$ of a leaf $\mathcal{L}_v$ of $\mathcal{F}$ is not mapped to a face of a leaf of $\mathcal{F}$. In contrast, for the case $j = i$, h maps $\mathcal{L}_v^{(i)}$ to $\mathcal{L}^{(i)}_{ve^{2\pi\mathrm{i}(1-u)/n}}$. *To describe the action of h on $\partial\Delta_{n-1|u}^{(i)} \times T^{n-1}$, we may thus describe its action on $\mathcal{L}_v^{(i)}$ for each $v \in S^1$ separately.*

For the case $n = 3$, each face of $\partial\Delta_{n-1|u} \times T^{n-2}_{i|v}$ $(u \neq 0)$ is an annulus, which as $u \to 0$, shrinks to a circle S^1 (Figure 16).

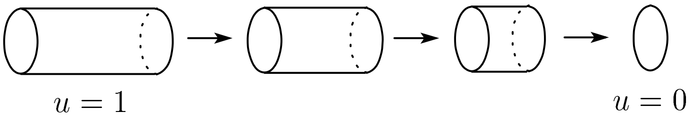


Figure 16:

Accordingly the action of the topological monodromy h varies from a (-1)-Dehn twist to the $\frac{2\pi}{3}$-rotation of S^1 (Proposition 2). A similar description is valid for arbitrary dimension.

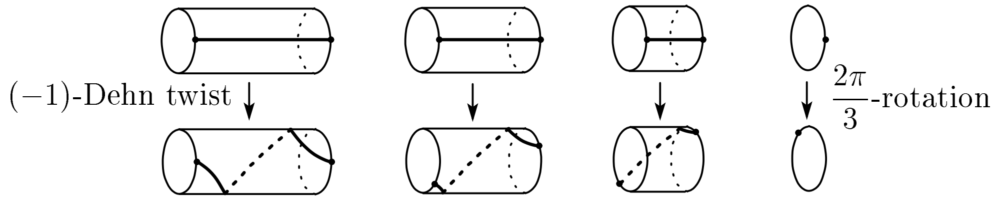


Figure 17: The variation of the action of h.

Introduction to Part II Let d be a positive integer and consider the following two complex varieties:

$$V = \{(x_1, x_2, \ldots, x_n, t) \in \mathbb{C}^{n+1} \ : \ x_1^2 + x_2^2 + \cdots + x_n^2 = t^d\},$$
$$W = \{(z_1, z_2, \ldots, z_n, t) \in \mathbb{C}^{n+1} \ : \ z_1 z_2 \cdots z_n = t^d\}.$$

We say that V is an *additive A-singularity* and that W is a *multiplicative A-singularity*. If $n = 2$, V and W are isomorphic via $(x_1, x_2) = (z_1 + \mathrm{i}z_2,\ z_1 - \mathrm{i}z_2)$. In contrast if $n \geq 3$, they are not isomorphic. In fact, the singular locus of V is isolated, while that of W is not isolated — the former is the origin, while the latter is the union of ${}_nC_2$ hyperplanes $H_{ij} = \{z_i = z_j = t = 0\}$, $1 \leq i < j \leq n$.

Now let $f : V \to \mathbb{C}$ be a holomorphic map given by $f(x_1, x_2, \ldots, x_n, t) = t$, and $g : W \to \mathbb{C}$ be a holomorphic map given by $g(z_1, z_2, \ldots, z_n, t) = t$. A smooth fiber $f^{-1}(s)$ $(s \neq 0)$ of f (resp. $g^{-1}(s)$ $(s \neq 0)$ of g) degenerates to the singular fiber $f^{-1}(0)$ (resp. $g^{-1}(0)$) as $s \to 0$, where the singular locus of $f^{-1}(0)$ (resp. $g^{-1}(0)$) coincides with that of V (resp. W).

When $n = 2$, the topological monodromy of $f : V \to \mathbb{C}$ (and $g : W \to \mathbb{C}$) is a $(-d)$-*Dehn twist* (Figure 18).
When $n \geq 3$, the topological monodromy of $f : V \to \mathbb{C}$ is called a *generalized Dehn twist*, and is described by using the *double covering method* (see [AGV],

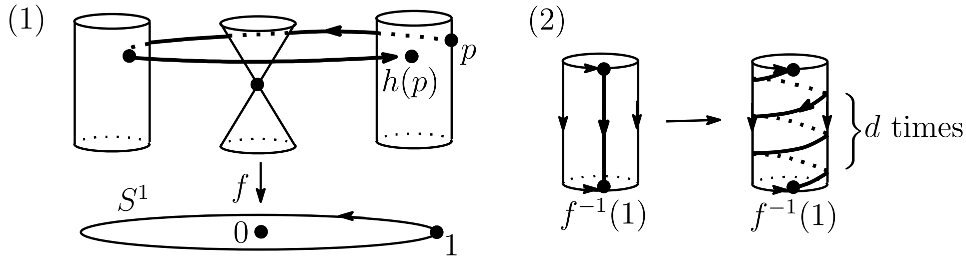


Figure 18: (1) The topological monodromy of $f : V \to \mathbb{C}$. (2) It is a $(-d)$-Dehn twist.

p.6). The topological monodromy of $g : W \to \mathbb{C}$ is another generalization of a Dehn twist, and is described in [Cle1]. *In what follows, we exclusively consider W, and write it as A_{d-1}.*

We next introduce a *fractional* Dehn twist. Where a and m $(0 < a < m)$ and b and n $(0 < b < n)$ are two pairs of relatively prime integers, an $\left(\frac{a}{m}, \frac{b}{n}\right)$*-fractional Dehn twist* is a self-homeomorphism of an annulus $[0,1] \times S^1$ illustrated in Figure 19. It is explicitly given by $(t, e^{\mathrm{i}\theta}) \mapsto (t, e^{2\pi\mathrm{i}\{-(1-t)a/m+tb/n\}}e^{\mathrm{i}\theta})$.

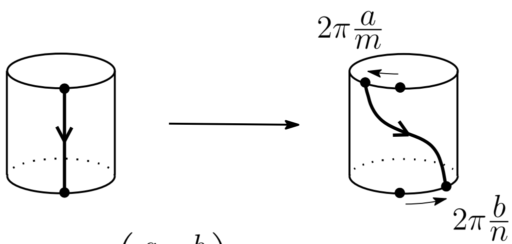


Figure 19: An $\left(\frac{a}{m}, \frac{b}{n}\right)$-fractional Dehn twist.

More generally, where κ is an integer, an $\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$*-fractional Dehn twist* is defined as the composite map of a $(+\kappa)$-Dehn twist and an $\left(\frac{a}{m}, \frac{b}{n}\right)$-fractional Dehn twist (it is illustrated in Figure 20). If $\frac{a}{m} + \frac{b}{n} + \kappa > 0$, a $-\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist appears as the topological monodromy of a degeneration: Set $c := \gcd(m, n)$, $m' := m/c$, $n' := n/c$ and $d := n'a + m'b + m'n'c\kappa$, or $d = m'n'c\left(\frac{a}{m} + \frac{b}{n} + \kappa\right)$. Let Γ be the cyclic group acting on A_{d-1} generated by an automorphism $\gamma : (z, w, t) \in A_{d-1} \mapsto (e^{2\pi\mathrm{i}a/m}z,\, e^{2\pi\mathrm{i}b/n}w,\, e^{2\pi\mathrm{i}/m'n'c}t) \in A_{d-1}$. Then the induced map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ by a Γ-invariant map $\Phi : (z, w, t) \in A_{d-1} \mapsto t^{m'n'c} \in \mathbb{C}$ is a degeneration whose topological monodromy is isotopic to the $-\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist.

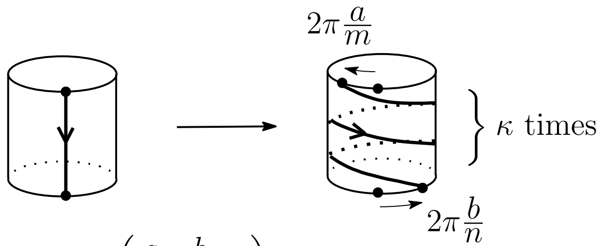


Figure 20: An $\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist.

We point out that $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ arises as a *local model* of a degeneration of Riemann surfaces; recall that a proper surjective holomorphic map $\pi : M \to \Delta$ from a smooth complex surface M to $\Delta := \{s \in \mathbb{C} : |s| < 1\}$ is a *degeneration of Riemann surfaces* (of genus g) if $\pi^{-1}(0)$ is singular and $\pi^{-1}(s)$ for $s \neq 0$ is a Riemann surface (of genus g). Figure 21 (1) illustrates an example of a singular fiber, which consists of cores, branches and a trunk. Contracting the branches and the trunk of this singular fiber yields the *canonical model* $\pi' : M' \to \Delta$ of $\pi : M \to \Delta$; the branches and the trunk become *cyclic quotient singularities* of M' (because the contraction of a chain of projective lines yields a cyclic quotient singularity). The singular fiber $(\pi')^{-1}(0)$ is thus as illustrated in Figure 21 (2). Let $p \in \pi^{-1}(0)$ be the point resulting from the contraction of the trunk. A neighborhood of $p \in M'$ is then isomorphic to A_{d-1}/Γ (for $a/m = 4/11$, $b/n = 3/5$, $\kappa = 0$). Moreover the restriction $\pi'|_{A_{d-1}/\Gamma}$ coincides with $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$, and the topological monodromy of $\pi'|_{A_{d-1}/\Gamma}$ is a $-\left(\frac{4}{11}, \frac{3}{5}, 0\right)$-fractional Dehn twist.

More generally, for any trunk (see Figure 22), the same result holds: A neighborhood of its contraction is isomorphic to A_{d-1}/Γ (for some a/m, b/n, κ), and the local topological monodromy is a $-\left(\frac{a}{m}, \frac{b}{n}, \kappa\right)$-fractional Dehn twist, and A_{d-1}/Γ is a cyclic quotient singularity.

In the above, the contraction of a trunk yields A_{d-1}/Γ, which is a cyclic quotient singularity. In fact, for *any* Γ (that is, for *any* a/m, b/n, κ), the quotient A_{d-1}/Γ is a cyclic quotient singularity, that is, $A_{d-1}/\Gamma \cong \mathbb{C}^2/G$ for some cyclic group $G = \langle g \rangle$, where the generator g is of the form $(u, v) \mapsto (e^{2\pi \mathrm{i}/l}u, e^{2\pi \mathrm{i}q/l}v)$ for some relatively prime positive integers l and q. This is the starting point of the present work — we generalize it to the higher-dimensional case in order to apply it to degenerations of complex manifolds.

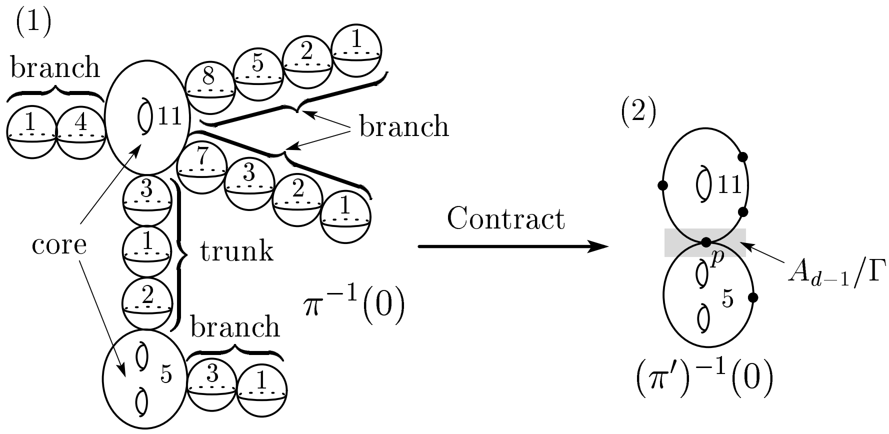


Figure 21: Intersections of irreducible components are *transversal.* The positive integer on an irreducible component denotes the *multiplicity* of that component. The five bold points on $(\pi')^{-1}(0)$ denote the *cyclic quotient singularities* of M'.

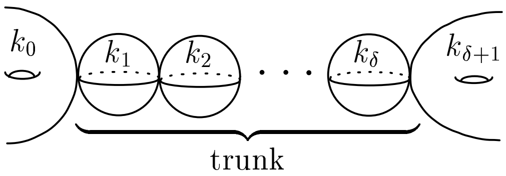


Figure 22: A trunk is a chain of projective lines connecting cores. (k_0, $k_1, \ldots, k_{\delta+1}$ are multiplicities.)

Let a_i and m_i $(i = 1, 2, \ldots, n)$ be relatively prime integers such that $0 < a_i < m_i$. Set $c := \gcd(m_1, m_2, \ldots, m_n)$ and $m_i' := m_i/c$. Take an integer κ such that $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa > 0$, and set

$$d := \left(\sum_{i=1}^{n} a_i m_1' \cdots \check{m}_i' \cdots m_n'\right) + m_1' m_2' \cdots m_n' c\kappa,$$

where $\check{m}_i'$ means the omission of m_i'. Or

$$d = m_1' m_2' \cdots m_n' c\Big(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa\Big). \tag{4}$$

Now let γ be an automorphism of $\mathbb{C}^{n+1}$ given by

$$\gamma:\ (x_1, \ldots, x_n, t) \longmapsto (e^{2\pi \mathrm{i} a_1/m_1} x_1,\ \ldots, e^{2\pi \mathrm{i} a_n/m_n} x_n,\ e^{2\pi \mathrm{i}/m_1' m_2' \cdots m_n' c} t).$$

Then (4) ensures that γ preserves $A_{d-1} := \{(x_1, x_2, \ldots, x_n, t) \in \mathbb{C}^{n+1} : x_1 x_2 \cdots x_n = t^d\}$. Let Γ be the cyclic group generated by γ. Let $\Phi : A_{d-1} \to \mathbb{C}$ be a Γ-invariant holomorphic map given by $\Phi(x_1, x_2, \cdots, x_n, t) = t^{m'_1 m'_2 \cdots m'_n c}$, and $\overline{\Phi}$ denote the holomorphic map on A_{d-1}/Γ induced by Φ. The topological monodromy of $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ is called a $-\left(\frac{a_1}{m_1}, \frac{a_2}{m_2}, \cdots, \frac{a_n}{m_n}, \kappa\right)$-*fractional Dehn twist*. This will be described in [SaTa]. (Remark: Clemens [Cle1] describes the monodromy of $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} = t$.)

In this paper, we will show that the cyclic quotient A_{d-1}/Γ is *uniformized* by a small abelian group. Here a finite subgroup of $GL(n, \mathbb{C})$ is *small* if it contains no pseudo-reflections, where a *pseudo-reflection* is a diagonalizable matrix such that one of its eigenvalues is a root of unity (distinct from 1) and all other eigenvalues are 1.

The following was originally proved by the second author:

(i) **Uniformization theorem for dimension 2** *There exists a small cyclic group $G \subset GL(2, \mathbb{C})$ such that $A_{d-1}/\Gamma \cong \mathbb{C}^2/G$* (Theorem 1). (This ensures that the minimal resolution of A_{d-1}/Γ is obtained by the *Hirzebruch-Jung resolution*.)

(ii) *Moreover under this isomorphism, $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ corresponds to the map $\overline{\phi} : \mathbb{C}^2/G \to \mathbb{C}$ induced by the G-invariant map $\phi : \mathbb{C}^2 \to \mathbb{C}$, $\phi(u, v) = u^n v^m$* (Lemma 6).

The construction of G appearing in the uniformization theorem is outlined as follows (see §5 for details): The universal covering $p : \mathbb{C}^2 \to A_{d-1}$ of A_{d-1} is a d-fold covering given by $p(X, Y) = (X^d, Y^d, XY)$. In the sequel, we often write the universal covering space $\mathbb{C}^2$ as $\widetilde{A}_{d-1}$. Next let $q : \widetilde{A}_{d-1} \to \mathbb{C}^2$ be an $m'n'$-fold covering given by $q(X, Y) = (X^{m'}, Y^{n'})$, and consider the following diagram:

$$\begin{tikzcd} & \widetilde{A}_{d-1} = \mathbb{C}^2 \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathbb{C}^2 & & A_{d-1}. \end{tikzcd} \tag{5}$$

Lift Γ to a group $\widetilde{\Gamma}$ (acting on $\widetilde{A}_{d-1}$) and then descend $\widetilde{\Gamma}$ to a group G (acting on $\mathbb{C}^2$). Then $A_{d-1}/\Gamma \cong \widetilde{A}_{d-1}/\widetilde{\Gamma} \cong \mathbb{C}^2/G$ and $G \subset GL(2, \mathbb{C})$ is a small cyclic group.

Remark In the paragraph subsequent to Lemma 4.2.2 of [Tak2], it is stated that $A = an'$ and $B = bm' + \kappa m'n'c$ are relatively prime to d. This is however generally not true, and moreover *unnecessary* in the construction of G.

The uniformization theorem for dimension 2 is generalized to an arbitrary dimension ≥ 3 as follows:

Main Theorem A (Theorem 4 and Corollary 14) *There exists a small finite abelian group $G \subset GL(n, \mathbb{C})$ such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$, where G is cyclic only when $n = 2$. Here G is explicitly given as follows: First set $l_i := \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}$ where $\check{m}'_i$ means the omission of m'_i (note that l_i is a (positive) integer; see Remark 7 below). Then G is generated by the following matrices Q, R_1, R_2, $\ldots$, R_{n-1}:*

$$Q = \begin{pmatrix} e^{2\pi\mathrm{i}l_1a_1/cd} & & & & O \\ & e^{2\pi\mathrm{i}l_2a_2/cd} & & & \\ & & \ddots & & \\ & & & e^{2\pi\mathrm{i}l_{n-1}a_{n-1}/cd} & \\ O & & & & e^{2\pi\mathrm{i}l_n(a_n+m_n\kappa)/cd} \end{pmatrix},$$

$$R_i = \begin{array}{cc} \begin{array}{c} \qquad\qquad\qquad \overset{i}{\smile} \end{array} & \\ \begin{pmatrix} 1 & & & \vdots & & & & O \\ & \ddots & & \vdots & & & & \\ & & 1 & \vdots & & & & \\ & & & e^{2\pi\mathrm{i}l_im'_i/d} & \cdots & \cdots & \cdots & \cdots \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ O & & & & & & & e^{-2\pi\mathrm{i}l_nm'_n/d} \end{pmatrix} & \begin{array}{c} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \\ \end{array} \end{array}.$$

Unlike $n = 2$, if $n \geq 3$, the construction of G requires *one more* descent, which uses the following diagram of coverings (cf. (5)):

$$\begin{tikzcd} & & \widetilde{A}_{d-1} = \mathbb{C}^n \arrow[dl, "q"'] \arrow[dr, "p"] & \\ & \mathbb{C}^n \arrow[dl, "r"'] & & A_{d-1}, \\ \mathbb{C}^n & & & \end{tikzcd} \tag{6}$$

where p, q and r are covering maps given by

- $p(X_1, X_2, \ldots, X_n) = (X_1^d, X_2^d, \ldots, X_n^d, X_1X_2 \cdots X_n)$ (note: $p : \widetilde{A}_{d-1} \to \mathbb{C}^n$ is the universal covering of A_{d-1}),
- $q(X_1, X_2, \ldots, X_n) = (X_1^{m'_1}, X_2^{m'_2}, \ldots, X_n^{m'_n})$,

- $r(u_1, u_2, \ldots, u_n) = (u_1^{l_1}, u_2^{l_2}, \ldots, u_n^{l_n})$, where l_i is the positive integer appearing in Main Theorem A.

We *lift and descend* Γ with respect to the diagram (6): Lift Γ to a group $\widetilde{\Gamma}$ (acting on $\widetilde{A}_{d-1}$), and then descend $\widetilde{\Gamma}$ to a group H (acting on $\mathbb{C}^n$), and next descend H to a group G (acting on $\mathbb{C}^n$). Then $A_{d-1}/\Gamma \cong \widetilde{A}_{d-1}/\widetilde{\Gamma} \cong \mathbb{C}^n/H \cong \mathbb{C}^n/G$ and $G \subset GL(n, \mathbb{C})$ is a small finite abelian group.

Actually H may be small, in which case the second descent is unnecessary and it suffices to take H as G. This case is characterized by the following numerical criterion of smallness:

Main Theorem B (**Theorem 3 (2)**) *The finite abelian group H is small if and only if* $\gcd(m'_i, m'_j) = 1$ *for any i, j such that $i \neq j$.*

Next let P be the pseudo-reflection subgroup of H, that is, P is generated by all pseudo-reflections of H. Regard κ as a 'parameter', and write $\widetilde{\Gamma}$, H, P as $\widetilde{\Gamma}_\kappa$, H_κ, P_κ. Then the following holds:

Main Theorem C (**Lemma 33 and Theorem 6**)

(1) *The pseudo-reflection subgroup P_κ of H_κ does not depend on κ: Let κ_0 denote the least integer among κ in the definition of d, then*

$$P_{\kappa_0} = P_{\kappa_0+1} = \cdots = P_\kappa = \cdots .$$

(2) (**Equi-smallness**) *If H_{κ_0} is small, then H_κ is small for any κ, and if H_{κ_0} is not small, then H_κ is not small for any κ.*

Now let $\Phi : A_{d-1} \to \mathbb{C}$ be a holomorphic map given by $\Phi(x_1, x_2, \ldots, x_n, t) = t^{m'_1 m'_2 \cdots m'_n c}$. Then Φ is Γ-invariant, thus induces a holomorphic map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$. Next set $k_i := \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)c$ where $\check{m}'_i$ means the omission of m'_i, and let $\phi : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic map given by $\phi(v_1, v_2, \ldots v_n) = v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n}$. Then ϕ is G-invariant (Theorem 5 (1)), thus induces a holomorphic map $\overline{\phi} : \mathbb{C}^n/G \to \mathbb{C}$. The following then holds:

Main Theorem D (**Theorem 5 (2)**) *Under the isomorphism $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ in Main Theorem A, $\overline{\Phi}$ corresponds to $\overline{\phi}$.*

Part I

1 Local models of degenerations and their fibers

Let $\pi : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic map given by $\pi(z_1, z_2, \ldots, z_n) = z_1 z_2 \cdots z_n$. Its singular fiber $Y := \pi^{-1}(0)$ is a complex analytic variety $z_1 z_2 \cdots z_n = 0$ in $\mathbb{C}^n$. Set $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. Each smooth fiber $B_s := \pi^{-1}(s)$ (where $s \neq 0$) is biholomorphic to $(\mathbb{C}^\times)^{n-1}$ via

$$(z_1, z_2, \ldots, z_n) \in B_s \ \longmapsto\ (z_1, z_2, \ldots, z_{n-1}) \in (\mathbb{C}^\times)^{n-1}. \tag{7}$$

Next set $N := \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_1 z_2 \cdots z_n| \leq 1\}$ and $D := \{s \in \mathbb{C} : |s| \leq 1\}$. Take $\rho > 1$ and set $\Delta := \big\{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_i| \leq \rho \ (i = 1, 2, \ldots, n)\big\}$. Set $M := N \cap \Delta$, then the restriction $\pi : M \to D$ of $\pi : \mathbb{C}^n \to \mathbb{C}$ to M is a degeneration. Its singular fiber is $X := Y \cap \Delta$ while $C_s := B_s \cap \Delta$ $(s \neq 0)$ is a smooth fiber. They are compact. Note that since C_s is a domain in B_s, the positive orientation of B_s naturally defines the positive orientation of the complex manifold C_s.

Write a nonzero complex number s as $re^{\mathrm{i}\xi}$ $(r > 0,\ 0 \leq \xi < 2\pi)$. Then set

$$\begin{aligned} J_r &= \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \begin{array}{l} 0 \leq x_i \leq \rho \ (i = 1, 2, \ldots, n) \\ x_1 x_2 \cdots x_n = r \end{array} \right\}, \\ K_\xi &= \big\{ (e^{\mathrm{i}\alpha_1}, e^{\mathrm{i}\alpha_2}, \ldots, e^{\mathrm{i}\alpha_n}) \in T^n : \alpha_1 + \alpha_2 + \cdots + \alpha_n \equiv \xi \bmod 2\pi \big\}, \end{aligned}$$

where $T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}$ is an n-dimensional torus. (Note that K_ξ is homeomorphic to T^{n-1} via $(e^{\mathrm{i}\alpha_1}, e^{\mathrm{i}\alpha_2}, \ldots, e^{\mathrm{i}\alpha_n}) \in K_\xi \mapsto (e^{\mathrm{i}\alpha_1}, e^{\mathrm{i}\alpha_2}, \ldots, e^{\mathrm{i}\alpha_{n-1}}) \in T^{n-1}$.) A smooth fiber $C_s = \pi^{-1}(s)$ of $\pi : M \to D$ is homeomorphic to $J_r \times K_\xi$ via

$$(z_1, \ldots, z_n) = (x_1 e^{\mathrm{i}\alpha_1}, \ldots x_n e^{\mathrm{i}\alpha_n}) \mapsto \big((x_1, \ldots, x_n),\ (e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\alpha_n})\big).$$

We say that J_r is the *real slice* of C_s, which is a part of a higher dimensional hyperboloid in $\mathbb{R}^n$ (Figure 23).

Note that J_r is homeomorphic to the standard $(n-1)$-simplex Δ_{n-1} (Figure 24). In §2, we explicitly construct a homeomorphism between them.

The *barycentric divisions* of J_r and Δ_{n-1} are given by $J_r = \bigcup_{i=1}^n A_i$ and $\Delta_{n-1} = \bigcup_{i=1}^n B_i$, where

$$\begin{aligned} A_i &= \{(x_1, \ldots, x_n) \in J_r : x_i \geq x_j \text{ for any } j \neq i\}, \\ B_i &= \{(\lambda_1, \ldots, \lambda_n) \in \Delta_{n-1} : \lambda_i \leq \lambda_j \text{ for any } j \neq i\}. \end{aligned}$$

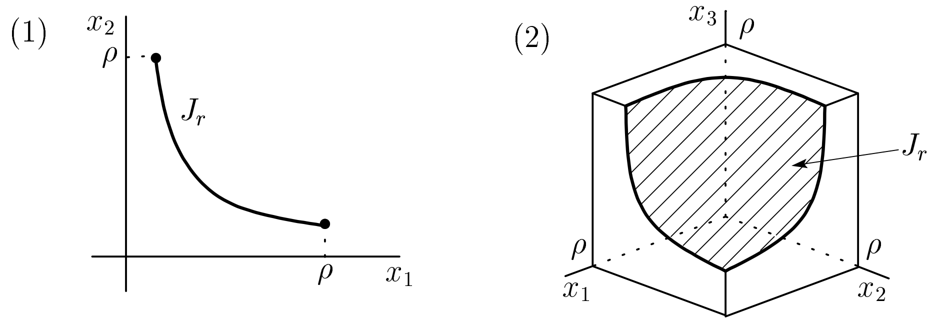


Figure 23: (1) $n = 2$. (2) $n = 3$.

Figure 24:

The boundary $\partial\Delta_{n-1}$ of Δ_{n-1} consists of n faces $\partial\Delta_{n-1}^{(l)}$ $(l = 1, 2, \ldots, n)$, where $\partial\Delta_{n-1}^{(l)}$ is defined by $\lambda_l = 0$ in Δ_{n-1}. Similarly the boundary ∂J_r of J_r consists of n faces $\partial J_r^{(l)}$ $(l = 1, 2, \ldots, n)$, where $\partial J_r^{(l)}$ is defined by $x_l = \rho$ in J_r.

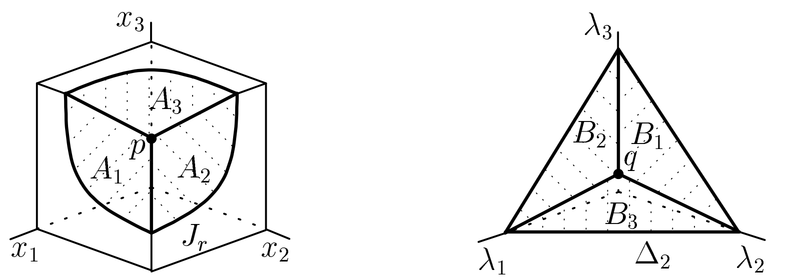
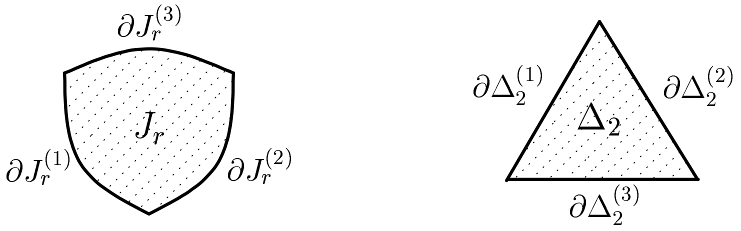


Figure 25: The barycentric divisions of J_r and Δ_{n-1} for $n = 3$. The points p and q are the *barycenters* of J_r and Δ_2.



Subsequently we explicitly construct a homeomorphism from J_r to Δ_{n-1} that maps the barycentric division of J_r to that of Δ_{n-1} and maps $\partial J_r^{(l)}$ to $\partial\Delta_{n-1}^{(l)}$, $l = 1, 2, \ldots, n$.

2 Construction of a simplicial homeomorphism Ψ

We construct a homeomorphism $\Psi : J_r \to \Delta_{n-1}$. The construction is based on induction on n, so it is convenient to write $n-1$ as m and Ψ, J_r, Δ_{n-1} as Ψ_m, J_m, Δ_m. We first construct Ψ_1 (below, x_1 and x_2 are denoted by x and y). Let $\sigma : J_1 \to [0, \infty]$ and $\tau : [0, \infty] \to [0, 1]$ be maps given by $\sigma(x, y) = \dfrac{\rho - y}{\rho - x}$ and $\tau(t) = \begin{cases} \frac{1}{2}t & (t \in [0, 1]) \\ 1 - \dfrac{1}{2t} & (t \in [1, \infty]) \end{cases}$, then the composite map $\lambda := \tau \circ \sigma : J_1 \to [0, 1]$ is a homeomorphism.

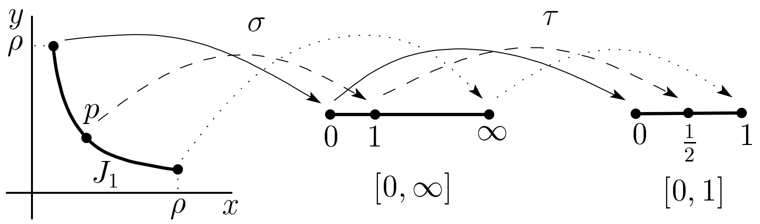


Figure 26: $\lambda := \tau \circ \sigma$ maps the midpoint of J_1 to that of $[0, 1]$.

Next define a homeomorphism $\varphi : [0, 1] \to \Delta_1$ by $\varphi(t) = (1 - t, t)$. The composite map $\Psi_1 := \varphi \circ \lambda : J_1 \to \Delta_1$ is the desired homeomorphism:

$$\Psi_1 : \ (x, y) \in J_1 \ \longmapsto \ \big(1 - \lambda(x, y),\ \lambda(x, y)\big) \in \Delta_1. \tag{8}$$

As illustrated in Figure 27, Ψ_1 maps the barycentric division of J_1 to that of Δ_1.

The construction of $\Psi_m : J_m \to \Delta_m$ $(m = 2, 3, \ldots)$ proceeds as follows:

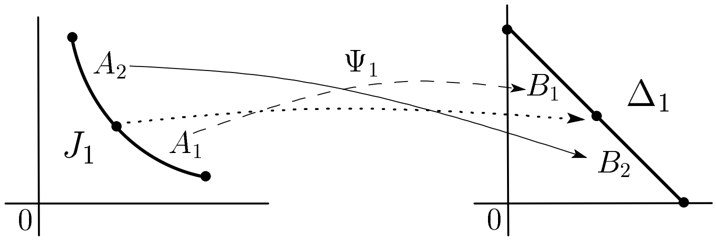


Figure 27: Ψ_1 maps A_i to B_i $(i = 1, 2)$.

Step 1 Let $J_m = \bigcup_{i=1}^{m+1} A_i$ and $\Delta_m = \bigcup_{i=1}^{m+1} B_i$ be the barycentric divisions. We then construct homeomorphisms $\psi_i : A_i \to B_i$ $(i = 1, 2, \dots, m+1)$ by using the homeomorphism $\Psi_{m-1} : J_{m-1} \to \Delta_{m-1}$.

Step 2 We show that $\psi_i = \psi_j$ on $A_i \cap A_j$. Thus $\psi_i : A_i \to B_i$ $(i = 1, 2, \dots, m+1)$ together define a homeomorphism $\Psi_m : J_m \to \Delta_m$ that maps the barycentric division of J_m to that of Δ_m.

Construction of $\Psi_2 : J_2 \to \Delta_2$: Let $J_2 = A_1 \cup A_2 \cup A_3$ and $\Delta_2 = B_1 \cup B_2 \cup B_3$ be the barycentric divisions. We first construct $\psi_1 : A_1 \to B_1$ (below, x_1, x_2, x_3 are denoted by x, y, z). Foliate A_1, B_1 as illustrated in Figure 28: $A_1 = \{L_c\}_{\sqrt[3]{r} \le c \le \rho}$, $B_1 = \{L'_d\}_{0 \le d \le 1/3}$.

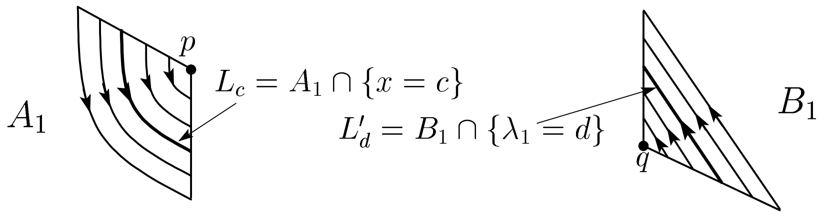


Figure 28:

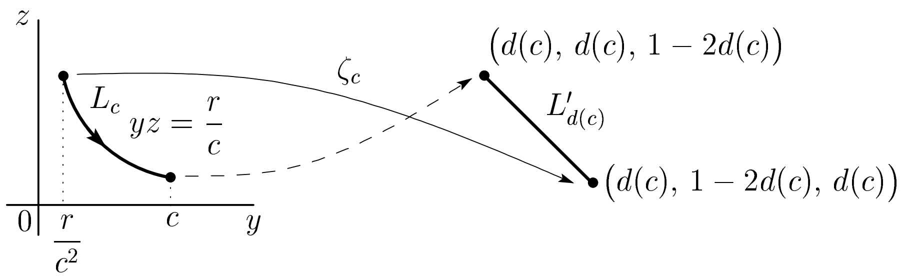
Note that $L_{\sqrt[3]{r}} = p$, $L'_{1/3} = q$, and L'_d $(d \neq 1/3)$ is homeomorphic to Δ_1 via $\varphi(d, y, z) \mapsto \Big(1 - 2d - \big(1 - 3d\big)z,\ d + \big(1 - 3d\big)z\Big)$. For $c \neq \sqrt[3]{r}$, a homeomorphism $\mu : J_1 \to L_c$ is given as follows: First take L_c as subsets of $\mathbb{R}^2$ by ignoring the x coordinate:

$$L_c = \Big\{(y, z) \in \mathbb{R}^2 \ :\ \frac{r}{c^2} \le y \le c,\ \frac{r}{c^2} \le z \le c,\ yz = \frac{r}{c}\Big\}.$$

Let $r : \mathbb{R}^2 \to \mathbb{R}^2$ be a parallel transport given by $r(y, z) = (y+\rho-c,\ z+\rho-c)$. For $(y, z) \in J_1$, denote by $\mu'(y, z)$ the intersection of $r(L_c)$ and the line connecting (y, z) and (ρ, ρ). Then we have a homeomorphism $\mu' : J_1 \to r(L_c)$, and hence the composition $\mu := r^{-1} \circ \mu' : J_1 \to L_c$ is a homeomorphism.

Now let $d : [\sqrt[3]{r},\ \rho] \to \left[0,\ \dfrac{1}{3}\right]$ be a function given by $d(c) := \dfrac{\rho - c}{3(\rho - \sqrt[3]{r})}$, and for each $c \in (\sqrt[3]{r},\ \rho]$, set $\zeta_c := \varphi^{-1} \circ \Psi_1 \circ \mu^{-1} : L_c \to L'_{d(c)}$, where $\Psi_1 : J_1 \to \Delta_1$ is the simplicial homeomorphism in (8):

$$\zeta_c(c, y, z) = \varphi^{-1} \circ \Psi_1 \circ \mu^{-1}(y, z).$$



The homeomorphism $\zeta_{\sqrt[3]{r}} : L_{\sqrt[3]{r}} \to L'_{1/3}$ is naturally defined by $\zeta_{\sqrt[3]{r}}(p) = q$. Then the family of homeomorphisms $\{\zeta_c : L_c \to L_{d(c)}\}_{\sqrt[3]{r} \le c \le \rho}$ determines the homeomorphism $\psi_1 : A_1 \to B_1$. In particular,

$$\psi_1(\rho, y, z) = \zeta_\rho(\rho, y, z) = \big(0, \Psi_1 \circ \mu^{-1}(y, z)\big). \tag{9}$$

Similarly ψ_2 and ψ_3 are constructed. By construction, $\psi_i = \psi_j$ on $A_i \cap A_j$, so they together define a homeomorphism $\Psi_2 : J_2 \to \Delta_2$.

For $m \geq 3$, we can similarly construct $\Psi_m : J_m \to \Delta_m$ from $\Psi_{m-1} : J_{m-1} \to \Delta_{m-1}$. We refer to Ψ_m as a *simplicial homeomorphism*. The positive orientation of Δ_m induces the positive orientation of J_m via $\Psi_m : J_m \to \Delta_m$.

3 One-parameter family of homeomorphisms and topological monodromy

Fix $\rho > 1$ and set

$$M = \big\{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n\ :\ |z_i| \leq \rho\ (i = 1, 2, \ldots, n),\ |z_1 z_2 \cdots z_n| \leq 1\big\}.$$

Consider the degeneration $\pi : M \to D$ defined by $\pi(z_1, z_2, \ldots z_n) = z_1 z_2 \cdots z_n$. Then $C_s := \pi^{-1}(s)$ for nonzero s is a smooth fiber and $X := \pi^{-1}(0)$ is its singular fiber. The projection $p : M \to J_r$ of M to its real slice J_r is given by

$p : (z_1, z_2, \dots, z_n) \mapsto (|z_1|, |z_2|, \dots, |z_n|)$. Let $\Psi : J_r \to \Delta_{n-1}$ be the simplicial homeomorphism constructed in §2, where $\Delta_{n-1} := \big\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \,:\, 0 \le \lambda_i \le 1,\ \lambda_1 + \cdots + \lambda_n = 1\big\}$ is the standard $(n-1)$-simplex. Consider the composite map $F := \Psi \circ p : M \to \Delta_{n-1}$. Set $(\lambda_1, \lambda_2, \dots, \lambda_n) := F(z_1, z_2, \dots, z_n)$, and for any $\theta \in \mathbb{R}$, define a map $f_\theta : M \to M$ by

$$f_\theta(z_1, z_2, \dots, z_n) = \big(e^{\mathrm{i}\theta\lambda_1} z_1,\ e^{\mathrm{i}\theta\lambda_2} z_2,\ \dots, e^{\mathrm{i}\theta\lambda_n} z_n\big). \tag{10}$$

Lemma 1. $f_\theta : C_s \to C_{se^{\mathrm{i}\theta}}$. *In particular,* $f_{2\pi} : C_s \to C_s$.

Proof. It suffices to show that if $z_1 z_2 \cdots z_n = s$, then $(e^{\mathrm{i}\theta\lambda_1} z_1)(e^{\mathrm{i}\theta\lambda_2} z_2) \cdots (e^{\mathrm{i}\theta\lambda_n} z_n) = se^{\mathrm{i}\theta}$. Since $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta_{n-1}$, $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, thus $(e^{\mathrm{i}\theta\lambda_1} z_1)(e^{\mathrm{i}\theta\lambda_2} z_2) \cdots (e^{\mathrm{i}\theta\lambda_n} z_n) = e^{\mathrm{i}\theta} z_1 z_2 \cdots z_n = e^{\mathrm{i}\theta} s$. □

Note that $\{f_\theta : M \to M\}_{\theta \in \mathbb{R}}$ is a 1-parameter family of homeomorphisms: (i) f_0 is the identity map and (ii) $f_{\theta_1 + \theta_2} = f_{\theta_1} \circ f_{\theta_2}$. (i) is obvious. (ii) is confirmed as follows: Note first that

$$\begin{aligned}
(z_1, z_2, \dots, z_n) &\xrightarrow{f_{\theta_2}} \big(e^{\mathrm{i}\theta_2\lambda_1} z_1,\ e^{\mathrm{i}\theta_2\lambda_2} z_2,\ \dots, e^{\mathrm{i}\theta_2\lambda_n} z_n\big)\\
&\xrightarrow{f_{\theta_1}} \big(e^{\mathrm{i}\theta_1\lambda'_1} e^{\mathrm{i}\theta_2\lambda_1} z_1,\ e^{\mathrm{i}\theta_1\lambda'_2} e^{\mathrm{i}\theta_2\lambda_2} z_2,\ \dots, e^{\mathrm{i}\theta_1\lambda'_n} e^{\mathrm{i}\theta_2\lambda_n} z_n\big),
\end{aligned}$$

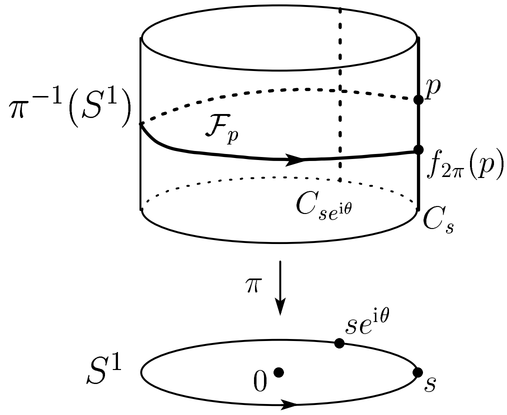
where

$$\begin{cases} (\lambda_1, \lambda_2, \dots, \lambda_n) = \Psi(|z_1|, |z_2|, \dots, |z_n|),\\ (\lambda'_1, \lambda'_2, \dots, \lambda'_n) = \Psi(|e^{\mathrm{i}\theta_2\lambda_1} z_1|, |e^{\mathrm{i}\theta_2\lambda_2} z_2|, \dots, |e^{\mathrm{i}\theta_2\lambda_n} z_n|). \end{cases}$$

Since $|e^{2\pi\mathrm{i}\lambda_i} z_i| = |z_i|$ $(i = 1, 2, \dots, n)$, we have $(\lambda'_1, \lambda'_2, \dots, \lambda'_n) = (\lambda_1, \lambda_2, \dots, \lambda_n)$. Hence

$$\begin{aligned}
f_{\theta_1} \circ f_{\theta_2}(z_1, z_2, \dots, z_n) &= \big(e^{\mathrm{i}\theta_1\lambda_1} e^{\mathrm{i}\theta_2\lambda_1} z_1,\ e^{\mathrm{i}\theta_1\lambda_2} e^{\mathrm{i}\theta_2\lambda_2} z_2,\ \dots, e^{\mathrm{i}\theta_1\lambda_n} e^{\mathrm{i}\theta_2\lambda_n} z_n\big)\\
&= \big(e^{\mathrm{i}(\theta_1+\theta_2)\lambda_1} z_1,\ e^{\mathrm{i}(\theta_1+\theta_2)\lambda_2} z_2,\ \dots, e^{\mathrm{i}(\theta_1+\theta_2)\lambda_n} z_n\big)\\
&= f_{\theta_1+\theta_2}(z_1, z_2, \dots, z_n).
\end{aligned}$$

In what follows, we consider $\{f_\theta : M \to M\}_{0 \le \theta \le 2\pi}$. Take a circle $S^1 = \{z \in \mathbb{C} \,:\, |z| < r\}$ contained in the unit disk D. For $s \in S^1$, the flow $\mathcal{F}_p := \big\{f_\theta(p) \,:\, 0 \le \theta \le 2\pi\big\}$ starting at $p \in C_s$ transversely intersects each smooth fiber $C_{se^{\mathrm{i}\theta}}$ $(0 < \theta < 2\pi)$, which defines a "parallel transport":



The homeomorphism $h := f_{2\pi} : C_s \to C_s$ is the *topological monodromy* of $\pi : M \to D$. For $\big((x_1, \ldots, x_n), (w_1, \ldots, w_n)\big) \in J_r \times K_\xi$ $(= C_s)$, set $(\lambda_1, \ldots, \lambda_n) := \Psi(x_1, \ldots, x_n)$, then

$$\begin{aligned} &h\big((x_1, \ldots, x_n), (w_1, \ldots, w_n)\big) \\ &\quad = \big((x_1, \ldots, x_n), (e^{2\pi \mathrm{i}\lambda_1} w_1, \ldots, e^{2\pi \mathrm{i}\lambda_n} w_n)\big). \end{aligned} \tag{11}$$

Recall that for each $l = 1, 2, \ldots, n$, Ψ maps the lth face $\partial J_r^{(l)}$ of ∂J_r (defined by $x_l = \rho$) to the lth face $\partial\Delta_{n-1}^{(l)}$ of $\partial\Delta_{n-1}$ (defined by $\lambda_l = 0$). Hence the action of h on the lth face $\partial J_r^{(l)} \times K_\xi$ of the boundary $\partial J_r \times K_\xi$ of $J_r \times K_\xi$ is given by setting $x_l = \rho$ and $\lambda_l = 0$ in (11):

$$\begin{aligned} &h\big((x_1, \ldots, x_{l-1}, \rho, x_{l+1}, \ldots, x_n), (w_1, \ldots, w_n)\big) \\ &\quad = \Big((x_1, \ldots, x_{l-1}, \rho, x_{l+1}, \ldots, x_n), \\ &\qquad\quad (e^{2\pi\mathrm{i}\lambda_1} w_1, \ldots, e^{2\pi\mathrm{i}\lambda_{l-1}} w_{l-1}, w_l, e^{2\pi\mathrm{i}\lambda_{l+1}} w_{l+1}, \ldots, e^{2\pi\mathrm{i}\lambda_n} w_n)\Big). \end{aligned} \tag{12}$$

We next construct a homeomorphism between C_s and $\Delta_{n-1} \times T^{n-1}$, and regard the topological monodromy $h : C_s \to C_s$ as a homeomorphism $\Delta_{n-1} \times T^{n-1} \to \Delta_{n-1} \times T^{n-1}$, which is more easy to describe.

First let $\mathrm{pr} : T^n \to T^{n-1}$ be the projection given by $\mathrm{pr} : (e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\alpha_n}) \mapsto (e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\alpha_{n-1}})$. The restriction $\mathrm{pr} : K_\xi \to T^{n-1}$ is a homeomorphism, in fact its inverse is given by

$$(e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\alpha_{n-1}}) \in T^{n-1} \mapsto \big(e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\alpha_{n-1}}, e^{\mathrm{i}(\xi - \alpha_1 - \cdots - \alpha_{n-1})}\big) \in K_\xi. \tag{13}$$

The positive orientation of T^{n-1} induces the positive orientation of K_ξ via $\mathrm{pr} : K_\xi \to T^{n-1}$. Then $\phi := \Psi \times \mathrm{pr} : C_s = J_r \times K_\xi \to \Delta_{n-1} \times T^{n-1}$ is an orientation-preserving homeomorphism. We say that ϕ is a *datum* *homeomorphism*, which gives a trivialization of C_s (this is analogous to the trivialization of vector bundle). Similarly, for $C_{se^{\mathrm{i}\theta}}$ $(0 < \theta \leq 2\pi)$, define the datum homeomorphism $\phi_\theta : C_{se^{\mathrm{i}\theta}} = J_r \times K_{\xi+\theta} \to \Delta_{n-1} \times T^{n-1}$ by $\phi_\theta = \Psi \times \mathrm{pr}$. Set $F_\theta := \phi_\theta \circ f_\theta \circ \phi^{-1}$; then $F_{2\pi} = \phi \circ f_{2\pi} \circ \phi^{-1}$ as $\phi_{2\pi} = \phi$. The following diagram commutes, and thus, to describe f_θ, it suffices to describe F_θ.

$$\begin{tikzcd}
C_s \arrow[r, "f_\theta"] \arrow[d, "\phi"'] & C_{se^{\mathrm{i}\theta}} \arrow[d, "\phi_\theta"] \\
\Delta_{n-1} \times T^{n-1} \arrow[r, "F_\theta"] & \Delta_{n-1} \times T^{n-1}.
\end{tikzcd}$$

Lemma 2. *$F_\theta : \Delta_{n-1} \times T^{n-1} \to \Delta_{n-1} \times T^{n-1}$ is given by*

$$F_\theta\big((\lambda_1, \ldots, \lambda_n),\, (t_1, \ldots, t_{n-1})\big) = \big((\lambda_1, \ldots, \lambda_n),\, (e^{\mathrm{i}\lambda_1\theta}t_1, \ldots, e^{\mathrm{i}\lambda_{n-1}\theta}t_{n-1})\big).$$

Proof. Write t_i as $e^{\mathrm{i}\alpha_i}$ $(i = 1, 2, \ldots, n-1)$. For $(\lambda_1, \ldots, \lambda_n) \in \Delta_{n-1}$, set $(x_1, \ldots, x_n) := \Psi^{-1}(\lambda_1, \ldots, \lambda_n)$. Then

$$\begin{aligned}
&F_\theta\big((\lambda_1, \ldots, \lambda_n),\, (e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\alpha_{n-1}})\big) \\
&= \phi_\theta \circ f_\theta \circ \phi^{-1}\big((\lambda_1, \ldots, \lambda_n),\, (e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\alpha_{n-1}})\big) \\
&= \phi_\theta \circ f_\theta\big((x_1, \ldots, x_n),\, (e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\alpha_{n-1}},\, e^{\mathrm{i}(\xi-\alpha_1-\cdots-\alpha_{n-1})})\big) \\
&= \phi_\theta\big((x_1, \ldots, x_n),\, (e^{\mathrm{i}\lambda_1\theta}e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\lambda_{n-1}\theta}e^{\mathrm{i}\alpha_{n-1}},\, e^{\mathrm{i}\lambda_n\theta}e^{\mathrm{i}(\xi-\alpha_1-\cdots-\alpha_{n-1})})\big) \\
&= \big((\lambda_1, \ldots, \lambda_n),\, (e^{\mathrm{i}\lambda_1\theta}e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\lambda_{n-1}\theta}e^{\mathrm{i}\alpha_{n-1}})\big).
\end{aligned}$$

$\square$

The lth face $\partial\Delta_{n-1}^{(l)}$ $(l = 1, 2, \ldots, n)$ of the boundary $\partial\Delta_{n-1}$ of Δ_{n-1} is defined by $\lambda_l = 0$. Hence:

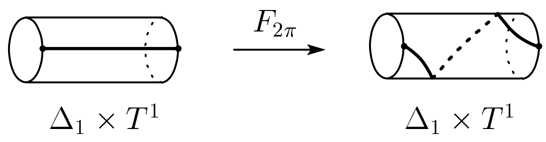
Corollary 1. *The action of F_θ on the lth face $\partial\Delta_{n-1}^{(l)} \times T^{n-1}$ of the boundary $\partial\Delta_{n-1} \times T^{n-1}$ of $\Delta_{n-1} \times T^{n-1}$ is given by*

$$\begin{aligned}
&F_\theta\big((\lambda_1, \ldots, \lambda_{l-1}, 0, \lambda_{l+1}, \ldots, \lambda_n),\, (t_1, \ldots, t_{n-1})\big) \\
&\quad = \big((\lambda_1, \ldots, \lambda_{l-1}, 0, \lambda_{l+1}, \ldots, \lambda_n),\, (e^{\mathrm{i}\lambda_1\theta}t_1, \ldots, t_l, \ldots, e^{\mathrm{i}\lambda_{n-1}\theta}t_{n-1})\big).
\end{aligned}$$

For $n = 2$, $\Delta_{n-1} \times T^{n-1}$ is an annulus $\Delta_1 \times T^1$ and by Lemma 2,

$$F_{2\pi} : \big((\lambda_1, \lambda_2),\, t_1\big) \in \Delta_1 \times T^1 \;\mapsto\; \big((\lambda_1, \lambda_2),\, e^{2\pi\mathrm{i}\lambda_1}t_1\big) \in \Delta_1 \times T^1. \tag{14}$$

Here $\lambda_1 + \lambda_2 = 1$, λ_2 varies from 0 to 1 and $\varphi : \lambda_2 \in [0, 1] \mapsto (1-\lambda_2, \lambda_2) \in \Delta_1$ is an orientation-preserving homeomorphism, so $F_{2\pi} : \big((1-\lambda_2, \lambda_2),\, t_1\big) \mapsto \big((1-\lambda_2, \lambda_2),\, e^{-2\pi\mathrm{i}\lambda_2}t_1\big)$, which is a (-1)-Dehn twist of $\Delta_1 \times T^1$.



Remark 3. Instead of $\pi : M \to D$, Clemens [Cle1] began with the 'whole map' $\pi : \mathbb{C}^n \to \mathbb{C}$; so J_r is replaced with an (infinite) hyperboloid $J_r^{\text{non-cpt}}$ defined by $x_1 x_2 \cdots x_n = r$ in $\mathbb{R}^n_{>0}$. Then instead of $\Psi : J_r \to \Delta_{n-1}$, he constructed a map $J_r^{\text{non-cpt}} \to \Delta_{n-1}$ that is a composition of a homeomorphism and a retraction as illustrated in Figure 29, and then he restricted this map to a compact domain in $J_r^{\text{non-cpt}}$. Using this map, he constructed a 1-parameter family of homeomorphisms $F'_\theta : \Delta_{n-1} \times T^{n-1} \to \Delta_{n-1} \times T^{n-1}$ $(0 \le \theta \le 2\pi)$ different from ours.

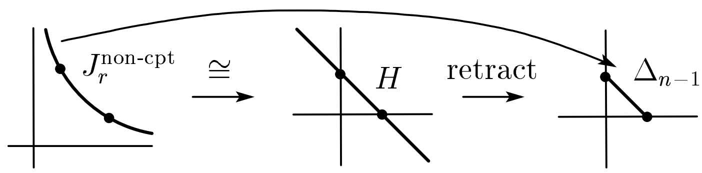


Figure 29: H is the hyperplane in $\mathbb{R}^n$ containing Δ_{n-1}.

4 Description of the action of the topological monodromy on foliations

We describe the action of the topological monodromy $F_{2\pi}$ on $\Delta_{n-1} \times T^{n-1}$ $(n \ge 3)$ in terms of foliations. As illustrated in Figure 30, we shrink the standard $(n-1)$-simplex Δ_{n-1} to obtain a family of $(n-1)$-simplexes $\Delta_{n-1|u}$ $(0 \le u \le 1)$ such that $\Delta_{n-1|1} = \Delta_{n-1}$ and $\Delta_{n-1|0}$ is the barycenter b of Δ_{n-1}.

Then $\{\partial \Delta_{n-1|u}\}_{0 \le u \le 1}$ is a (singular) foliation of Δ_{n-1}, and $\{\partial \Delta_{n-1|u} \times T^{n-1}\}_{0 \le u \le 1}$ gives a foliation of a smooth fiber $\Delta_{n-1} \times T^{n-1}$.

The topological monodromy $h := F_{2\pi} : \Delta_{n-1} \times T^{n-1} \to \Delta_{n-1} \times T^{n-1}$ maps each leaf $\partial \Delta_{n-1|u} \times T^{n-1}$ to itself. To describe it explicitly, we introduce n foliations on T^{n-1}. First for each $i = 1, 2, \ldots, n$, let $\mathcal{M}_{i|v}$ $(v \in S^1)$ be the $(n-2)$-dimensional subtorus in K_ξ defined by $e^{\mathrm{i}\alpha_i} = v$, then $\mathcal{K}_i = \{\mathcal{M}_{i|v}\}_{v \in S^1}$ is a foliation on K_ξ by parallel subtori (Figure 31 for $n = 3$).
The homeomorphism $\mathrm{pr} : K_\xi \,(\subset T^n) \to T^{n-1}$ given by $(e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\alpha_n}) \mapsto (e^{\mathrm{i}\alpha_1}, \ldots, e^{\mathrm{i}\alpha_{n-1}})$ transforms the foliation $\mathcal{K}_i = \{\mathcal{M}_{i|v}\}_{v \in S^1}$ to a foliation $\mathcal{T}_i =$

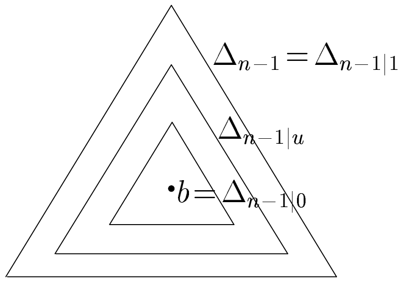


Figure 30:

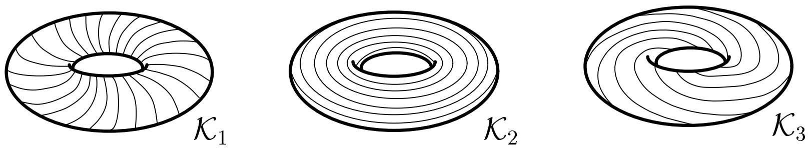


Figure 31:

$\{T^{n-2}_{i|v}\}_{v \in S^1}$ of T^{n-1}. Then $\{\partial\Delta_{n-1|u} \times T^{n-2}_{i|v}\}_{v \in S^1}$ is a foliation of $\partial\Delta_{n-1|u} \times T^{n-1}$. Here varying u yields a singular foliation of $\Delta_{n-1} \times T^{n-1}$.

The monodromy action on $\{b\} \times T^{n-1}$: The barycenter b $(= \partial\Delta_{n-1|0})$ of Δ_{n-1} is $\left(\frac{1}{n}, \dots \frac{1}{n}\right)$, and the topological monodromy $h : \Delta_{n-1} \times T^{n-1} \to \Delta_{n-1} \times T^{n-1}$ acts on $\{b\} \times T^{n-1}$ as

$$(b, t_1, t_2, \dots, t_{n-1}) \longmapsto (b, e^{2\pi \mathrm{i}/n} t_1, e^{2\pi \mathrm{i}/n} t_2, \dots, e^{2\pi \mathrm{i}/n} t_{n-1}).$$

In particular h maps each leaf $\{b\} \times T^{n-2}_{i|v}$ to $\{b\} \times T^{n-2}_{i|ve^{2\pi \mathrm{i}/n}}$.

The monodromy action on $\partial\Delta_{n-1|u} \times T^{n-1}$ $(u \neq 0)$: For each $j = 1, 2, \dots, n$, let $\partial\Delta^{(j)}_{n-1|u}$ denote the jth face of $\partial\Delta_{n-1|u} \cong \partial\Delta_{n-1}$; then $\partial\Delta^{(j)}_{n-1|u} \times T^{n-1}$ is the jth face of $\partial\Delta_{n-1|u} \times T^{n-1}$. Fixing $u \neq 0$ and i, set $\mathcal{L}_v := \partial\Delta_{n-1|u} \times T^{n-2}_{i|v}$ and consider a foliation $\mathcal{F} = \{\mathcal{L}_v\}_{v \in S^1}$ of $\partial\Delta_{n-1|u} \times T^{n-1}$. Then $\mathcal{L}^{(j)}_v := \partial\Delta^{(j)}_{n-1|u} \times T^{n-2}_{i|v}$ is the jth face of $\mathcal{L}_v$. *While h maps each face of $\partial\Delta_{n-1|u} \times T^{n-1}$ to itself, h does not map a leaf $\mathcal{L}_v$ to itself. Moreover, as we will see below, h maps $\mathcal{L}^{(j)}_v$ to itself only if $j = i$ and $u = 1$.*

We specify the parameter u such that the side length of $\partial\Delta_{n-1|u}$ $(u \neq 0)$ is $\sqrt{2}u$. Then for $u \neq 0$, the jth face $\partial\Delta^{(j)}_{n-1|u}$ of $\partial\Delta_{n-1|u}$ is defined by $\lambda_j = \dfrac{1-u}{n}$ in $\partial\Delta_{n-1|u}$. We first consider the case $j = i$; we describe the action of h on $\mathcal{L}_v^{(i)} = \partial\Delta_{n-1|u}^{(i)} \times T_{i|v}^{n-2}$. Set $\mu := \dfrac{1-u}{n}$. Then by Lemma 2, the restriction of h to $\mathcal{L}_v^{(i)}$ is explicitly given by

$$\begin{aligned}&\big((\lambda_1, \ldots, \lambda_{i-1}, \mu, \lambda_{i+1}, \ldots, \lambda_n), (t_1, \ldots, t_{i-1}, v, t_{i+1}, \ldots . t_{n-1})\big) \longmapsto \\ &\big((\lambda_1, \ldots, \lambda_{i-1}, \mu, \lambda_{i+1}, \ldots, \lambda_n), (e^{2\pi \mathrm{i}\lambda_1} t_1, \ldots, e^{2\pi\mathrm{i}\mu} v, \ldots, e^{2\pi\mathrm{i}\lambda_{n-1}} t_{n-1})\big).\end{aligned}$$

We thus obtain the following:

Lemma 3. *h maps $\mathcal{L}_v^{(i)}$ to $\mathcal{L}_{ve^{2\pi\mathrm{i}(1-u)/n}}^{(i)}$. In particular when $u = 1$, it maps $\mathcal{L}_v^{(i)}$ to itself, and is explicitly given by*

$$\begin{aligned}h : &\big((\lambda_1, \ldots \lambda_{i-1}, 0, \lambda_{i+1}, \ldots, \lambda_n),\ (t_1, \ldots t_{i-1}, v, t_{i+1}, \ldots, t_{n-1})\big) \longmapsto \\ &\big((\lambda_1, \ldots \lambda_{i-1}, 0, \lambda_{i+1}, \ldots, \lambda_n),\ (e^{2\pi\mathrm{i}\lambda_1} t_1, \ldots, v, \ldots, e^{2\pi\mathrm{i}\lambda_{n-1}} t_{n-1})\big).\end{aligned}$$

Remark 4. While h maps $\mathcal{L}_v^{(i)}$ to $\mathcal{L}_{ve^{2\pi\mathrm{i}(1-u)/n}}^{(i)}$, for $j \neq i$, h does *not* map $\mathcal{L}_v^{(j)}$ $(j \neq i)$ to a face of a leaf of $\mathcal{F}$. Indeed,

$$\begin{aligned}h(\mathcal{L}_v^{(j)}) &= \{\big((\lambda_1, \ldots, \lambda_n), (t_1, \ldots, t_{n-1})\big) \in \partial\Delta_{n-1|u}^{(j)} \times T^{n-1} \,:\, t_i = ve^{2\pi\mathrm{i}\lambda_i}\} \\ &\cong \partial\Delta_{n-1|u}^{(j)} \times T^{n-2}.\end{aligned}$$

To emphasize n, write h as $h_n : \Delta_{n-1} \times T^{n-1} \to \Delta_{n-1} \times T^{n-1}$. Here $\partial\Delta_{n-1|1}^{(i)} \times T_{i|v}^{n-2}$ is homeomorphic to $\Delta_{n-2} \times T^{n-2}$ via

$$\begin{aligned}\varphi : &\big((\lambda_1, \ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \ldots, \lambda_n),\ (t_1, \ldots, t_{i-1}, v, t_{i+1}, \ldots t_{n-1})\big) \\ &\longmapsto \big((\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n),\ (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots t_{n-1})\big).\end{aligned}$$

Identify $\partial\Delta_{n-1|1}^{(i)} \times T_{i|v}^{n-2}$ with $\Delta_{n-2} \times T^{n-2}$ via φ. From Lemma 3,

$$\begin{aligned}&\varphi \circ h_n \circ \varphi^{-1}\big((\lambda_1, \ldots, \lambda_{n-1}),\ (t_1, \ldots, t_{n-2})\big) \\ &= \varphi \circ h_n\big((\lambda_1, \ldots \lambda_{i-1}, 0, \lambda_i, \ldots, \lambda_{n-1}),\ (t_1, \ldots t_{i-1}, v, t_i, \ldots, t_{n-2})\big) \\ &= \varphi\big((\lambda_1, \ldots \lambda_{i-1}, 0, \lambda_i, \ldots, \lambda_{n-1}),\ (e^{2\pi\mathrm{i}\lambda_1} t_1, \ldots, v, \ldots, e^{2\pi\mathrm{i}\lambda_{n-2}} t_{n-2})\big) \\ &= \big((\lambda_1, \ldots, \lambda_{n-1}),\ (e^{2\pi\mathrm{i}\lambda_1} t_1, \ldots, e^{2\pi\mathrm{i}\lambda_{n-2}} t_{n-2})\big) \\ &= h_{n-1}\big((\lambda_1, \ldots, \lambda_{n-1}),\ (t_1, \ldots, t_{n-2})\big).\end{aligned}$$

Thus $\varphi \circ h_n \circ \varphi^{-1} = h_{n-1}$, so

Proposition 1. *The restriction of h_n to $\partial\Delta_{n-1|1}^{(i)} \times T_{i|v}^{n-2}$ is h_{n-1}.*

Variation of topological monodromy As $u \to 0$, the ith face $\mathcal{L}_v^{(i)} = \partial\Delta_{n-1|u}^{(i)} \times T_{i|v}^{n-2}$ of the leaf $\mathcal{L}_v = \partial\Delta_{n-1|u} \times T_{i|v}^{n-2}$ shrinks to the $(n-2)$-dimensional torus T^{n-2}. Accordingly $h_u := h\,|_{\mathcal{L}_v^{(i)}}$ varies. In the case $n = 3$, $\mathcal{L}_v^{(i)}$ for $u \neq 0$ is an annulus and for $u = 0$ a circle, and as we see below, h_u varies from a (-1)-Dehn twist to a rotation of S^1, where recall that for each integer k, a k-Dehn twist is a self-homeomorphism of an annulus $[0,1] \times S^1$ given by $(x,t) \in [0,1] \times S^1 \to (x, e^{2\pi \mathrm{i} kx} t) \in [0,1] \times S^1$.

Proposition 2 $(n = 3)$**.** *As u varies from* 1 *to* 0*, h_u varies from a* (-1)*-Dehn twist to the $\frac{2\pi}{3}$-rotation of S^1 as illustrated in Figure 32.*

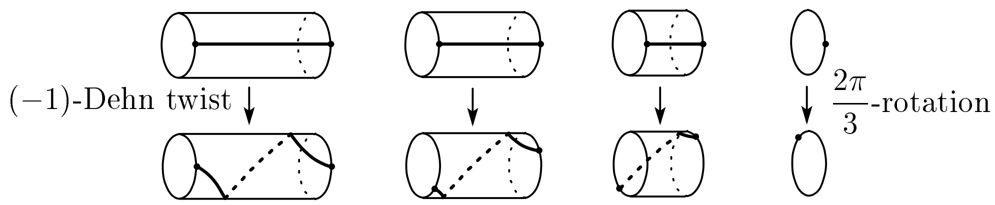


Figure 32:

Proof. We show the assertion for $i = 1$ (it is similarly shown for other i). Identify $\mathcal{L}_v^{(1)}$ and $\mathcal{L}_{ve^{2\pi \mathrm{i}(1-u)/3}}^{(1)}$ with the annulus $\left[\frac{1-u}{3}, \frac{1+2u}{3}\right] \times S^1$ via the homeomorphism $\big((\lambda_1, \lambda_2, \lambda_3), (t_1, t_2)\big) \mapsto (\lambda_3,\, t_2)$. Regard then $h_u : \mathcal{L}_v^{(1)} \to \mathcal{L}_{ve^{2\pi \mathrm{i}(1-u)/3}}^{(1)}$ as a homeomorphism

$$h_u : (x,t) \in \left[\tfrac{1-u}{3}, \tfrac{1+2u}{3}\right] \times S^1 \mapsto (x, e^{-2\pi \mathrm{i}\{x+(1-u)/3\}} t) \in \left[\tfrac{1-u}{3}, \tfrac{1+2u}{3}\right] \times S^1.$$

As u varies from 1 to 0, this varies from a (-1)-Dehn twist of $[0,1] \times S^1$ to the $\frac{2\pi}{3}$-rotation of S^1 as illustrated in Figure 32. $\square$

Part II

5 Uniformization theorem for dimension 2

5.1 Proof

Let a and m $(0 < a < m)$ and b and n $(0 < b < n)$ be two pairs of relatively prime integers, and set $c := \gcd(m,n)$, $m' := \frac{m}{c}$, $n' := \frac{n}{c}$. (Note that m' and n' are integers.) Take an integer κ such that $\frac{a}{m} + \frac{b}{n} + \kappa > 0$, and set $d := an' + bm' + m'n'c\kappa$. Let γ be the automorphism of $\mathbb{C}^3$ given by $\gamma : (z,w,t) \mapsto (e^{2\pi \mathrm{i}a/m}z,\ e^{2\pi \mathrm{i}b/n}w,\ e^{2\pi \mathrm{i}/m'n'c}t)$. Then γ preserves $A_{d-1} := \{zw = t^d\}$ in $\mathbb{C}^3$. Let Γ be the cyclic group generated by γ. Then:

Theorem 1 (**Uniformization theorem** [Tak2])**.** *There exists a small cyclic group $G \subset GL(2,\mathbb{C})$ such that $A_{d-1}/\Gamma \cong \mathbb{C}^2/G$. Here G is explicitly given as follows: Let a^* $(0 < a^* < m)$ be the integer such that $aa^* \equiv 1 \bmod m$, and let q $(0 < \mathsf{q} < cd)$ be the integer such that $\mathsf{q} \equiv \dfrac{a^*d - n'}{m'} \bmod cd$* (the right hand side is indeed an integer; see Remark 5 below)*. Then G is generated by the automorphism g of $\mathbb{C}^2$ given by $g : (u,v) \mapsto (e^{2\pi \mathrm{i}/cd}u,\ e^{2\pi \mathrm{i}\mathsf{q}/cd}v)$.*

Remark 5. Substituting $d := an' + bm' + m'n'c\kappa$ into $\dfrac{a^*d - n'}{m'}$ yields $\dfrac{aa^* - 1}{m'}n' + a^*b + a^*n'c\kappa$. Here since $aa^* \equiv 1 \bmod m$, we may write $aa^* - 1 = Km\,(= Km'c)$, where K is an integer. Then $\dfrac{a^*d - n'}{m'} = Kn'c + a^*b + a^*n'c\kappa$.

Proof. Note first that the universal covering $p : \widetilde{A}_{d-1}\ (= \mathbb{C}^2) \to A_{d-1}$ of A_{d-1} is a d-fold covering given by $p(X,Y) = (X^d, Y^d, XY)$. Next let $q : \widetilde{A}_{d-1} \to \mathbb{C}^2$ be an $m'n'$-fold covering given by $q(X,Y) = (X^{m'}, Y^{n'})$, and consider the following diagram:

$$\begin{tikzcd} & \widetilde{A}_{d-1} = \mathbb{C}^2 \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathbb{C}^2 & & A_{d-1}. \end{tikzcd} \tag{15}$$

Let $\widetilde{\Gamma}$ be the lift of Γ with respect to p, and G be the descent of $\widetilde{\Gamma}$ with respect to q. Then $A_{d-1}/\Gamma \cong \widetilde{A}_{d-1}/\widetilde{\Gamma} \cong \mathbb{C}^2/G$.

We next show that G is generated by g. For $j = 1,2,\dots,m'n'c$ and $k = 1,2,\dots,d$, let $\widetilde{\gamma}_{j,k} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ be the automorphism given by $\widetilde{\gamma}_{j,k} : (X,Y) \mapsto (e^{2\pi \mathrm{i}(ja+km)/md}X,\ e^{2\pi \mathrm{i}\{j(b+n\kappa)-kn\}/nd}Y)$, and $g_{j,k} : \mathbb{C}^2 \to \mathbb{C}^2$ be the automorphism given by $g_{j,k} : (u,v) \mapsto (e^{2\pi \mathrm{i}(ja+km)/cd}u,\ e^{2\pi \mathrm{i}\{j(b+n\kappa)-kn\}/cd}v)$. Then for each $j = 1,2,\dots,m'n'c$, the set of all lifts of $\gamma^j \in \Gamma$ with respect to p is $\{\widetilde{\gamma}_{j,k} : k = 1,2,\dots,d\}$, and for any j, k, the descent of $\widetilde{\gamma}_{j,k}$ with respect to q is $g_{j,k}$. Hence $\widetilde{\Gamma}$ and G are explicitly given by

$$\widetilde{\Gamma} = \big\{\widetilde{\gamma}_{j,k} : j = 1,2,\dots,m'n'c,\ k = 1,2,\dots,d\big\},$$
$$G = \big\{g_{j,k} : j = 1,2,\dots,m'n'c,\ k = 1,2,\dots,d\big\}.$$

Therefore G is generated by the following two automorphisms α, β:

$$\begin{aligned}\alpha : \ (u,v) &\longmapsto (e^{2\pi \mathrm{i} \mathsf{a}/cd}u, e^{2\pi\mathrm{i}(b+n\kappa)/cd}v),\\ \beta : \ (u,v) &\longmapsto (e^{2\pi\mathrm{i}m'/d}u, e^{-2\pi\mathrm{i}n'/d}v).\end{aligned}$$

Let l $(0 < l < cd)$ be the integer such that $l \equiv \dfrac{1-aa^*}{m} \bmod cd$. Then by Corollary 15,

$$\alpha^{a^*}\beta^l = g, \qquad g^a = \alpha, \qquad g^m = \beta.$$

Hence $g \in G$ and G is generated by g.

We next show that G is small. Recall that G is generated by $g : (u,v) \mapsto (e^{2\pi\mathrm{i}/cd}u,\ e^{2\pi\mathrm{i}\mathsf{q}/cd}v)$. Here q and cd are relatively prime (Lemma 4 (2) below), so G is small. $\square$

Supplement Let a^* $(0 < a^* < m)$ be the integer such that $aa^* \equiv 1 \bmod m$, and let q $(0 < \mathsf{q} < cd)$ be the integer such that $\mathsf{q} \equiv \dfrac{a^*d-n'}{m'} \bmod cd$, where the right hand side is indeed an integer (Remark 5). Similarly let b^* $(0 < b^* < n)$ be the integer such that $bb^* \equiv 1 \bmod n$, and let r $(0 < \mathsf{r} < cd)$ be the integer such that $\mathsf{r} \equiv \dfrac{b^*d-m'}{n'} \bmod cd$, where the right hand side is an integer as for q.

Lemma 4. (1) $\mathsf{qr} \equiv 1 \bmod cd$, *that is,* $\mathsf{r} = \mathsf{q}^*$.

(2) q *and cd are relatively prime.*

Proof. (1): It suffices to show that $\dfrac{a^*d-n'}{m'}\,\dfrac{b^*d-m'}{n'} \equiv 1 \bmod cd$. Here

$$\frac{a^*d-n'}{m'}\,\frac{b^*d-m'}{n'} = d\left(\frac{aa^*-1}{m'}n'b^* + \frac{bb^*-1}{n'}m'a^* + a^*b^*m'n'c\kappa\right) + 1.$$

Write $aa^*-1 = Km \,(= Km'c)$ and $bb^*-1 = Ln \,(= Ln'c)$. Then

$$\begin{aligned}\frac{a^*d-n'}{m'}\,\frac{b^*d-m'}{n'} &= cd\big(Kn'b^* + Lm'a^* + a^*b^*m'n'\kappa\big) + 1\\ &\equiv 1 \bmod cd.\end{aligned}$$

(2): Since $\mathsf{qr} \equiv 1 \bmod cd$, $\mathsf{qr} = 1 + Mcd$ for some integer M. Then $\mathsf{qr} - Mcd = 1$. Note that $\gcd(\mathsf{q}, cd)$ divides the left hand side, hence divides 1. Thus $\gcd(\mathsf{q}, cd) = 1$, that is, q and cd are relatively prime. $\square$

5.2 Comments on uniformization theorem

We explicitly give the isomorphism $A_{d-1}/\Gamma \cong \mathbb{C}^2/G$ in the uniformization theorem. To that end, let $p : \widetilde{A}_{d-1} \to A_{d-1}$ and $q : \widetilde{A}_{d-1} \to \mathbb{C}^2$ be the covering maps appearing in the diagram (15). Note that:

- Since $\widetilde{\Gamma}$ is the lift of Γ with respect to p, the map p induces an isomorphism $\overline{p} : \widetilde{A}_{d-1}/\widetilde{\Gamma} \to A_{d-1}/\Gamma$.
- Since G is the descent of $\widetilde{\Gamma}$ with respect to q, the map q induces an isomorphism $\overline{q} : \widetilde{A}_{d-1}/\widetilde{\Gamma} \to \mathbb{C}^2/G$.

The isomorphism $A_{d-1}/\Gamma \cong \mathbb{C}^2/G$ in the uniformization theorem (Theorem 1) is then given by

$$\Psi := \overline{q} \circ \overline{p}^{\,-1} : \; A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^2/G. \tag{16}$$

Explicitly:

Lemma 5. $\Psi\big([x, y, t]\big) = \big[x^{m'/d},\, y^{n'/d}\big]$,
where we denote the image of a point $(X, Y) \in \widetilde{A}_{d-1}$ *under the quotient map* $\widetilde{A}_{d-1} \to \widetilde{A}_{d-1}/\widetilde{\Gamma}$ *by* $[X, Y] \in \widetilde{A}_{d-1}/\widetilde{\Gamma}$ *and so on.*

Proof. Since $p(X, Y) = (X^d, Y^d, XY)$, we have $\overline{p}\big([X, Y]\big) = \big[X^d, Y^d, XY\big]$, so $\overline{p}^{\,-1}\big([x, y, t]\big) = \big[x^{1/d},\, y^{1/d}\big]$. Next since $q(X, Y) = (X^{m'}, Y^{n'})$, we have $\overline{q}\big([x^{1/d},\, y^{1/d}]\big) = \big[x^{m'/d},\, y^{n'/d}\big]$. Hence $\Psi := \overline{q} \circ \overline{p}^{\,-1}$ is explicitly given by $\Psi\big([x, y, t]\big) = \big[x^{m'/d},\, y^{n'/d}\big]$. $\square$

Correspondence between functions Let $\Phi : A_{d-1} \to \mathbb{C}$ be a holomorphic map given by $\Phi(z, w, t) = t^{m'n'c}$. Then Φ is Γ-invariant, so induces a holomorphic map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$. As explained in § Introduction, the topological monodromy of $\overline{\Phi}$ is a $-\Big(\dfrac{a}{m}, \dfrac{b}{n}, \kappa\Big)$-fractional Dehn twist.

Under the isomorphism $\Psi : A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^2/G$ in (16), the holomorphic map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ corresponds to a holomorphic map on $\mathbb{C}^2/G$. *This map is explicitly given.* First let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be a holomorphic map defined by $\phi(u, v) = u^n v^m$. Then ϕ is G-invariant. To see this, recall that by Theorem 1, the cyclic group G is generated by $g : (u, v) \mapsto (e^{2\pi \mathrm{i}/cd}u,\, e^{2\pi \mathrm{i}\mathsf{q}/cd}v)$, where q $(0 < \mathsf{q} < cd)$ is the integer such that $\mathsf{q} \equiv \dfrac{a^*d - n'}{m'} \bmod cd$. Then

$$\begin{aligned}
\phi \circ g(u, v) &= \phi(e^{2\pi \mathrm{i}/cd}u,\, e^{2\pi \mathrm{i}\mathsf{q}/cd}v) = e^{2\pi \mathrm{i}c(n' + m'\mathsf{q})/cd}u^n v^m \\
&= e^{2\pi \mathrm{i}ca^*d/cd}u^n v^m \qquad \text{by } n' + m'\mathsf{q} \equiv a^*d \bmod cd \\
&= e^{2\pi \mathrm{i}a^*}u^n v^m = u^n v^m \\
&= \phi(u, v).
\end{aligned}$$

Thus ϕ is G-invariant, so induces a holomorphic map $\overline{\phi} : \mathbb{C}^2/G \to \mathbb{C}$.

Lemma 6 ([Tak2]). *Under the isomorphism* $\Psi : A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^2/G$ *in (16),* $\overline{\Phi}$ *corresponds to* $\overline{\phi}$*, that is,* $\overline{\Phi} = \overline{\phi} \circ \Psi$*.*

Proof. Note first that

$$\begin{aligned} \overline{\phi} \circ \Psi\big([x,y,t]\big) &= \overline{\phi}\left([x^{m'/d},\, y^{n'/d}]\right) \qquad \text{by Lemma 5} \\ &= x^{m'n/d} y^{n'm/d} = (xy)^{m'n'c/d}. \end{aligned}$$

Here $xy = t^d$ (because $(x,y,t) \in A_{d-1}$), so $\overline{\phi} \circ \Psi\big([x,y,t]\big) = t^{m'n'c}$. Thus $\overline{\phi} \circ \Psi\big([x,y,t]\big) = \overline{\Phi}\big([x,y,t]\big)$. $\square$

Where $\mathfrak{r} : R \to A_{d-1}/\Gamma$ is the minimal resolution of A_{d-1}/Γ, the composite map $\pi := \overline{\Phi} \circ \mathfrak{r} : R \to \mathbb{C}$ is a degeneration. As we see immediately, *thanks to* the uniformization theorem, the degeneration $\pi : R \to \mathbb{C}$ is isomorphic to a degeneration which is easy to describe.

Where $\mathfrak{r}' : R' \to \mathbb{C}^2/G$ is the minimal resolution of $\mathbb{C}^2/G$, the composite map $\pi' := \overline{\phi} \circ \mathfrak{r}' : R' \to \mathbb{C}$ is a degeneration. Since A_{d-1}/Γ and $\mathbb{C}^2/G$ are isomorphic (Theorem 1), two minimal resolutions $\mathfrak{r} : R \to A_{d-1}/\Gamma$ and $\mathfrak{r}' : R' \to \mathbb{C}^2/G$ are isomorphic, that is, there exists an isomorphism $\widetilde{\Psi} : R \to R'$ that makes the following diagram commute:

$$\begin{tikzcd} R \arrow[r, "\widetilde{\Psi}", "\cong"'] \arrow[d, "\mathfrak{r}"'] & R' \arrow[d, "\mathfrak{r}'"] \\ A_{d-1}/\Gamma \arrow[r, "\Psi", "\cong"'] & \mathbb{C}^2/G. \end{tikzcd} \tag{17}$$

Theorem 2. *The following diagram commutes*:

$$\begin{tikzcd} R \arrow[rr, "\widetilde{\Psi}", "\cong"'] \arrow[dr, "\pi"'] & & R' \arrow[dl, "\pi'"] \\ & \mathbb{C}. & \end{tikzcd} \tag{18}$$

Hence two degenerations $\pi := \overline{\Phi} \circ \mathfrak{r} : R \to \mathbb{C}$ *and* $\pi' := \overline{\phi} \circ \mathfrak{r}' : R' \to \mathbb{C}$ *are isomorphic.*

Proof. By Lemma 6, the following diagram commutes:

$$\begin{tikzcd} A_{d-1}/\Gamma \arrow[rr, "\Psi", "\cong"'] \arrow[dr, "\overline{\Phi}"'] & & \mathbb{C}^2/G \arrow[dl, "\overline{\phi}"] \\ & \mathbb{C}. & \end{tikzcd} \tag{19}$$

Combining the commutative diagrams (17) and (19) yields the commutative diagram (18). $\square$

The degeneration $\pi' := \overline{\phi} \circ \mathfrak{r}' : R' \to \mathbb{C}$ may be described as follows: Since G is cyclic, $\mathbb{C}^2/G$ has a (unique) cyclic quotient singularity, which is resolved by a chain of projective lines (*Hirzebruch-Jung resolution*). Accordingly the singular fiber $(\pi')^{-1}(0)$ of $\pi' : R' \to \mathbb{C}$ is as illustrated in Figure 33 (see also Remark 6).

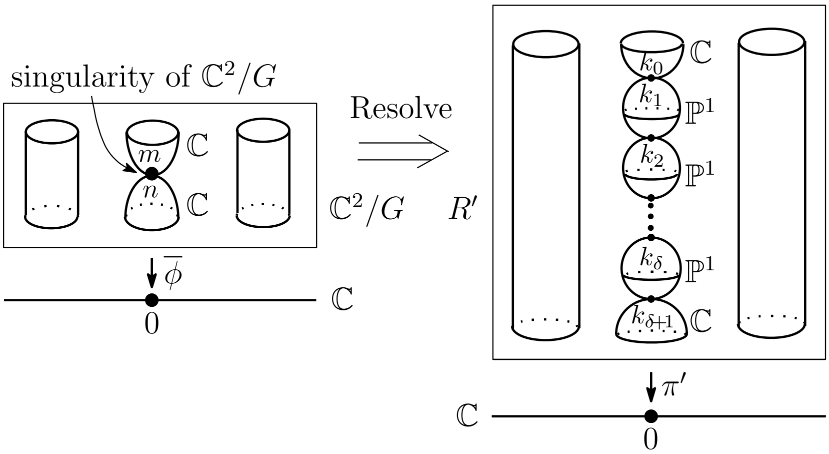


Figure 33: The positive integers $k_0, k_1, \ldots, k_{\delta+1}$ are multiplicities. They are explicitly determined from Γ, more specifically, from m, n, a, b, κ (Remark 6).

Remark 6. The multiplicities of the singular fiber $(\pi')^{-1}(0)$ in Figure 33 is explicitly determined from m, n, a, b, κ. Let a^* and b^* ($0 < a^* < m$, $0 < b^* < n$) be the integers such that $aa^* \equiv 1 \bmod m$ and $bb^* \equiv 1 \bmod n$. Define then two sequences of integers $m_0 > m_1 > \cdots > m_\lambda = 1$ and $n_0 > n_1 > \cdots > n_\nu = 1$ inductively by the division algorithm with *negative* residues:

$$\begin{cases} m_0 := m, \quad m_1 := a^*, \\ m_{i-1} = s_i m_i - m_{i+1} \quad (0 < m_{i+1} < m_i), \qquad i = 1, 2, \ldots, \lambda - 1, \end{cases}$$

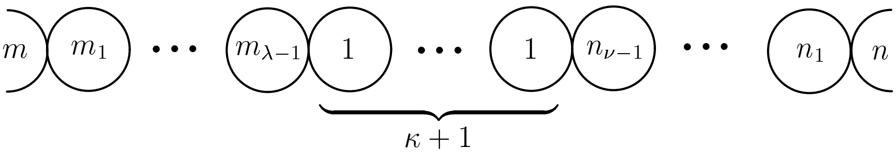
$$\begin{cases} n_0 := n, \quad n_1 := b^*, \\ n_{i-1} = t_i n_i - n_{i+1} \quad (0 < n_{i+1} < n_i), \qquad i = 1, 2, \ldots, \nu - 1. \end{cases}$$

Then:

(i) If $\kappa \geq 0$, then $(\pi')^{-1}(0)$ is as illustrated in (1) of Figure 34.

(ii) If $\kappa = -1$, then there exists a unique pair of integers λ_0 and ν_0 ($0 < \lambda_0 < \lambda$, $0 < \nu_0 < \nu$) such that $m_{\lambda_0+1} + n_{\nu_0+1} = m_{\lambda_0} = n_{\lambda_0}$, and $(\pi')^{-1}(0)$ is as illustrated in (2) of Figure 34.

(1) $\kappa \geq 0$



(2) $\kappa = -1$

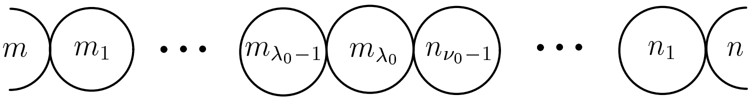


Figure 34: The singular fibers for (1) $\kappa \geq 0$ and (2) $\kappa = -1$. A circle stands for $\mathbb{P}^1$ and a hemicircle for $\mathbb{C}$. (Each intersection is a node.)

6 Lifting and descent

6.1 Diagram of covering maps

We generalize the uniformization theorem for dimension 2 (Theorem 1) to an arbitrary dimension. To that end, we begin with some preparation.

Lemma 7. *Let a_i and m_i $(i = 1, 2, \ldots, n)$ be relatively prime integers such that $0 < a_i < m_i$. If κ is an integer satisfying $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa > 0$, then $\kappa \geq -n + 1$.*

Proof. Since $0 < a_i < m_i$, we have $0 < \frac{a_i}{m_i} < 1$ $(i = 1, 2, \ldots, n)$. In particular, $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} < n$, so

$$\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa < n + \kappa.$$

Here the left hand side is positive by assumption, so $0 < n + \kappa$, that is, $-n + 1 \leq \kappa$. $\square$

We keep the notation of Lemma 7. Let κ be an integer satisfying $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa > 0$. Set $c := \gcd(m_1, m_2, \ldots, m_n)$, $m'_i := m_i/c$ and

$$d := \left(\sum_{i=1}^{n} a_i m'_1 \cdots \check{m}'_i \cdots m'_n \right) + m'_1 m'_2 \cdots m'_n c\kappa, \tag{20}$$

where $\check{m}'_i$ means the omission of m'_i.

Lemma 8. $d > 0$.

Proof. Rewrite (20) as

$$d = m_1' m_2' \cdots m_n' c \left(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa \right). \tag{21}$$

Here $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa > 0$ by assumption, so $d > 0$. $\square$

We next show that

$$e^{2\pi \mathrm{i}(a_1/m_1 + a_2/m_2 + \cdots + a_n/m_n)} = e^{2\pi \mathrm{i} d/m_1' m_2' \cdots m_n' c}. \tag{22}$$

To show this, rewrite (21) as

$$\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} = \frac{d}{m_1' m_2' \cdots m_n' c} - \kappa.$$

Thus

$$e^{2\pi \mathrm{i}(a_1/m_1 + a_2/m_2 + \cdots + a_n/m_n)} = e^{2\pi \mathrm{i} d/m_1' m_2' \cdots m_n' c} e^{-2\pi \mathrm{i}\kappa}.$$

Here κ is an integer, so $e^{-2\pi \mathrm{i}\kappa} = 1$, confirming (22).

Note next that:

Lemma 9. *Let γ be an automorphism of $\mathbb{C}^{n+1}$ given by*

$$\gamma :\ (x_1, \ldots, x_n, t) \ \mapsto\ (e^{2\pi \mathrm{i} a_1/m_1} x_1,\ \ldots,\ e^{2\pi \mathrm{i} a_n/m_n} x_n,\ e^{2\pi \mathrm{i}/m_1' m_2' \cdots m_n' c} t).$$

Then γ preserves $A_{d-1} := \left\{ (x_1, x_2, \ldots, x_n, t) \in \mathbb{C}^{n+1} \ :\ x_1 x_2 \cdots x_n = t^d \right\}$, that is, γ maps A_{d-1} to itself.

Proof. It is sufficient to show that if $(x_1, \ldots, x_n, t) \in A_{d-1}$, then

$$(e^{2\pi \mathrm{i} a_1/m_1} x_1, \ldots, e^{2\pi \mathrm{i} a_n/m_n} x_n, e^{2\pi \mathrm{i}/m_1' m_2' \cdots m_n' c} t) \in A_{d-1},$$

or explicitly if $x_1 x_2 \cdots x_n = t^d$, then

$$(e^{2\pi \mathrm{i} a_1/m_1} x_1)(e^{2\pi \mathrm{i} a_2/m_2} x_2) \cdots (e^{2\pi \mathrm{i} a_n/m_n} x_n) = (e^{2\pi \mathrm{i}/m_1' m_2' \cdots m_n' c} t)^d,$$

that is,

$$e^{2\pi \mathrm{i}(a_1/m_1 + a_2/m_2 + \cdots + a_n/m_n)} x_1 x_2 \cdots x_n = e^{2\pi \mathrm{i} d/m_1' m_2' \cdots m_n' c} t^d. \tag{23}$$

The assumption $x_1 x_2 \cdots x_n = t^d$ and (22) ensure that (23) indeed holds. $\square$

In what follows, regard γ as an automorphism of A_{d-1} (Lemma 9). Let Γ be the cyclic group generated by γ.

The universal covering $p : \widetilde{A}_{d-1}\ (= \mathbb{C}^n) \to A_{d-1}$ of A_{d-1} is a d^{n-1}-fold covering given by $p : (X_1, X_2, \ldots, X_n) \mapsto (X_1^d,\ X_2^d, \ldots, X_n^d, X_1X_2\cdots X_n)$.

Consider the following diagram of coverings:

$$\begin{tikzcd}[column sep=small] & & \widetilde{A}_{d-1} = \mathbb{C}^n \arrow[dl, "q"'] \arrow[dr, "p"] & \\ & \mathbb{C}^n \arrow[dl, "r"'] & & A_{d-1}, \\ \mathbb{C}^n & & & \end{tikzcd} \tag{24}$$

where

- $q : (X_1, X_2, \ldots, X_n) \mapsto (X_1^{m_1'},\ X_2^{m_2'}, \ldots, X_n^{m_n'})$ is an $m_1'm_2'\cdots m_n'$-fold covering,
- $r : (u_1, u_2, \ldots, u_n) \mapsto (u_1^{l_1},\ u_2^{l_2}, \ldots, u_n^{l_n})$ is an $l_1l_2\cdots l_n$-fold covering. Here
$$l_i := \frac{m_1'\cdots \check{m}_i' \cdots m_n'}{\operatorname{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')} \qquad (i = 1, 2, \ldots, n),$$
where $\check{m}_i'$ means the omission of m_i'. Note that l_i is a positive integer (see Remark 7 below).

Remark 7. $l_i := \dfrac{m_1'\cdots \check{m}_i' \cdots m_n'}{\operatorname{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')}$ is a (positive) integer, because from the definition of lcm, the denominator $\operatorname{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')$ divides the numerator $m_1'\cdots \check{m}_i' \cdots m_n'$.

Now let $\widetilde{\Gamma}$ be the lift of Γ with respect to the covering p, H be the lift of $\widetilde{\Gamma}$ with respect to the covering q, and G be the lift of H with respect to the covering r. We will show that G is a small finite abelian group such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ (the uniformization theorem). We begin with some preparation.

6.2 $\widetilde{\Gamma}$, H and G are finite groups

We first show that $\widetilde{\Gamma}$ is a group.

(i) $1 \in \widetilde{\Gamma}$: This is the trivial lift of $1 \in \Gamma$ (that is the identity map of $\widetilde{A}_{d-1}$).

(ii) $\xi \in \widetilde{\Gamma} \Rightarrow \xi^{-1} \in \Gamma$: If ξ is a lift of $\gamma^j \in \Gamma$, then ξ^{-1} is a lift of $\gamma^{-j} \in \Gamma$.

(iii) $\xi_1, \xi_2 \in \widetilde{\Gamma} \Rightarrow \xi_1\xi_2 \in \widetilde{\Gamma}$: If ξ_1, ξ_2 are lifts of γ^j, $\gamma^k \in \Gamma$, then $\xi_1\xi_2$ is a lift of $\gamma^{j+k} \in \Gamma$.

We next show that H is a group as follows (similarly we can show that G is a group):

(i)' $1 \in H$: This is the descent of $1 \in \widetilde{\Gamma}$.

(ii)' $h \in H \Rightarrow h^{-1} \in H$: If h is the descent of $\xi \in \widetilde{\Gamma}$, then h^{-1} is the descent of $\xi^{-1} \in \widetilde{\Gamma}$.

(iii)' $h_1,\, h_2 \in H \Rightarrow h_1 h_2 \in H$: If $h_1,\, h_2$ are the descents of $\xi_1,\, \xi_2 \in \widetilde{\Gamma}$, then $h_1 h_2$ is the descent of $\xi_1 \xi_2 \in \widetilde{\Gamma}$.

The orders of $\widetilde{\Gamma}$, H and G are determined as follows (below, $|\widetilde{\Gamma}|$, $|H|$ and $|G|$ denote the orders):

Order of $\widetilde{\Gamma}$: Since $\widetilde{\Gamma}$ is the lift of Γ with respect to the d^{n-1}-fold covering p, we have $|\widetilde{\Gamma}| = d^{n-1}|\Gamma|$. Here $|\Gamma| = m_1' m_2' \cdots m_n' c$, so $|\widetilde{\Gamma}| = m_1' m_2' \cdots m_n' c d^{n-1}$.

Order of H: Since H is the descent of $\widetilde{\Gamma}$ (or $\widetilde{\Gamma}$ is the lift of H) with respect to the $m_1' m_2' \cdots m_n'$-fold covering q, we have $|\widetilde{\Gamma}| = m_1' m_2' \cdots m_n' |H|$. Here $|\widetilde{\Gamma}| = m_1' m_2' \cdots m_n' c d^{n-1}$ so $|H| = c d^{n-1}$.

Order of G: Since G is the descent of H (or H is the lift of G) with respect to the $l_1 l_2 \cdots l_n$-fold covering r, we have $|H| = l_1 l_2 \cdots l_n |G|$. Here $|H| = c d^{n-1}$, so $|G| = \dfrac{c d^{n-1}}{l_1 l_2 \cdots l_n}$. (This is indeed an integer. See Remark 8 below.)

The results obtained in this section are summarized as follows:

Proposition 3. *Let $\widetilde{\Gamma}$ be the lift of Γ with respect to the covering p. Let H be the descent of $\widetilde{\Gamma}$ with respect to the covering q, and let G be the descent of H with respect to the covering r. Then*:

(1) *The lift $\widetilde{\Gamma}$ of Γ is a finite group of order $m_1' m_2' \cdots m_n' c d^{n-1}$.* (In fact, $\widetilde{\Gamma}$ is abelian. See Lemma 13.)

(2) *The descent H of $\widetilde{\Gamma}$ is a finite group of order $c d^{n-1}$.* (In fact, H is abelian. See Lemma 15.)

(3) *The descent G of H is a finite group of order $\dfrac{c d^{n-1}}{l_1 l_2 \cdots l_n}$.* (In fact, G is abelian. See Lemma 28.)

Remark 8. The fact that $|G| = \dfrac{c d^{n-1}}{l_1 l_2 \cdots l_n}$ is an integer is reconfirmed as follows (we show this only for $n=3$): Using

$$\begin{cases} d = m_1' m_2' m_3' c \left(\dfrac{a_1}{m_1} + \dfrac{a_2}{m_2} + \dfrac{a_3}{m_3} + \kappa \right) \quad \text{(see (21))}, \\ l_1 := \dfrac{m_2' m_3'}{\operatorname{lcm}(m_2', m_3')},\ l_2 := \dfrac{m_1' m_3'}{\operatorname{lcm}(m_1', m_3')},\ l_3 := \dfrac{m_1' m_2'}{\operatorname{lcm}(m_1', m_2')}, \end{cases}$$

rewrite $|G| = \dfrac{cd^2}{l_1 l_2 l_3}$ as

$$
\begin{aligned}
|G| &= c\Big\{\prod_{i\neq j} \mathrm{lcm}(m_i',\, m_j')\Big\} c^2 \left(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} + \kappa\right)^2 \\
&= c \prod_{i\neq j} \mathrm{lcm}(m_i', m_j') \left\{ (c\kappa)^2 + \sum_{i=1}^{3} \left(\frac{2a_i c\kappa}{m_i'} + \frac{a_i^2}{(m_i')^2} \right) + \sum_{i\neq j} \frac{2a_i a_j}{m_i' m_j'} \right\}.
\end{aligned}
$$

Here $\prod_{i\neq j} \mathrm{lcm}(m_i', m_j') = \mathrm{lcm}(m_1',\, m_2')\mathrm{lcm}(m_1',\, m_3')\mathrm{lcm}(m_2',\, m_3')$ is divisible by $m_i',\ (m_i')^2,\ m_i' m_j'$, so the last expression is indeed an integer.

7 Determination of H

Recall the diagram (24) between the covering maps:

$$
\begin{tikzcd}[column sep=small]
 & & \widetilde{A}_{d-1} = \mathbb{C}^n \arrow[dl, "q"'] \arrow[dr, "p"] & \\
 & \mathbb{C}^n \arrow[dl, "r"'] & & A_{d-1}, \\
\mathbb{C}^n & & &
\end{tikzcd}
\tag{25}
$$

where the covering maps p, q and r are given by

- $p(X_1, X_2, \ldots, X_n) = (X_1^d,\, X_2^d, \ldots, X_n^d, X_1 X_2 \cdots X_n)$,
- $q(X_1, X_2, \ldots, X_n) = (X_1^{m_1'},\, X_2^{m_2'}, \ldots, X_n^{m_n'})$,
- $r(u_1, u_2, \ldots, u_n) = (u_1^{l_1},\, u_2^{l_2}, \ldots, u_n^{l_n})$, where $l_i := \dfrac{m_1' \cdots \check{m}_i' \cdots m_n'}{\mathrm{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')}$ ($\check{m}_i'$ means the omission of m_i') is a positive integer (see Remark 7).

We adopt the following notation: For $j = 1, 2, \ldots, m_1' m_2' \cdots m_n' c$,

- $\mathrm{Lift}^{(j)}$: The set of all lifts of $\gamma^j \in \Gamma$ with respect to the covering map $p : \widetilde{A}_{d-1} \to A_{d-1}$.
- $q_*\big(\mathrm{Lift}^{(j)}\big)$: The descent of $\mathrm{Lift}^{(j)}$ with respect to the covering map $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$.
- $r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big)$: The descent of $q_*\big(\mathrm{Lift}^{(j)}\big)$ with respect to the covering map $r : \mathbb{C}^n \to \mathbb{C}^n$.

Then

- $\widetilde{\Gamma} = \displaystyle\bigcup_{j=1}^{m_1' m_2' \cdots m_n' c} \mathrm{Lift}^{(j)}$ is the lift of Γ with respect to the covering map $p : \widetilde{A}_{d-1} \to A_{d-1}$.

- $H = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} q_*\bigl(\mathrm{Lift}^{(j)}\bigr)$ is the descent of $\widetilde{\Gamma}$ with respect to the covering map $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$.
- $G = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} r_* \circ q_*\bigl(\mathrm{Lift}^{(j)}\bigr)$ is the descent of H with respect to the covering map $r : \mathbb{C}^n \to \mathbb{C}^n$.

Actually, $\widetilde{\Gamma} = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} \mathrm{Lift}^{(j)}$ is a disjoint union. Namely, if $j \neq k$, then $\mathrm{Lift}^{(j)} \cap \mathrm{Lift}^{(k)} = \emptyset$. On the other hand, $H = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} q_*\bigl(\mathrm{Lift}^{(j)}\bigr)$ and $G = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} r_* \circ q_*\bigl(\mathrm{Lift}^{(j)}\bigr)$ are *not* disjoint unions. In fact, a descent of an element of $\mathrm{Lift}^{(j)}$ may coincide with that of an element of $\mathrm{Lift}^{(k)}$ $(j \neq k)$. In this case, $q_*\bigl(\mathrm{Lift}^{(j)}\bigr) \cap q_*\bigl(\mathrm{Lift}^{(k)}\bigr) \neq \emptyset$, and moreover, $r_* \circ q_*\bigl(\mathrm{Lift}^{(j)}\bigr) \cap r_* \circ q_*\bigl(\mathrm{Lift}^{(k)}\bigr) \neq \emptyset$.

In what follows, we write $\widetilde{\Gamma}$ as a disjoint union: $\widetilde{\Gamma} = \coprod_{j=1}^{m'_1 m'_2 \cdots m'_n c} \mathrm{Lift}^{(j)}$.

7.1 The lifts of each element of Γ

We next determine the set $\mathrm{Lift}^{(j)}$ of all lifts of $\gamma^j \in \Gamma$ with respect to the covering p. For $j = 1, 2, \ldots, m'_1 m'_2 \cdots m'_n c$, we first define a set $\Lambda^{(j)}$ of n-tuples of integers as follows:

$$\Lambda^{(j)} := \Bigl\{(p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n \, : \, 0 \leq p_i < d, \, \sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z}\Bigr\}. \tag{26}$$

Lemma 10. *The number of elements of $\Lambda^{(j)}$ is d^{n-1}.*

Proof. Setting $\Xi := \{(p_1, p_2, \ldots, p_{n-1}) \in \mathbb{Z}^{n-1} \, : \, 0 \leq p_i < d\}$, consider a map $\varphi : \Lambda^{(j)} \to \Xi$ given by $(p_1, p_2, \ldots, p_{n-1}, p_n) \longmapsto (p_1, p_2, \ldots, p_{n-1})$. Here Ξ consists of d^{n-1} elements, thus it suffices to show that φ is bijective.
Surjectivity: We show that for any $(p_1, p_2, \ldots, p_{n-1}) \in \Xi$, the inverse image $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ is not empty. Set $N := j\kappa - \sum_{i=1}^{n-1} p_i$ and let p_n $(0 \leq p_n < d)$ be the integer such that $p_n \equiv N \bmod d$. Then $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$. Moreover $\varphi(p_1, p_2, \ldots, p_n) = (p_1, p_2, \ldots, p_{n-1})$, thus $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ is not empty.

Injectivity: We show that for any $(p_1, p_2, \ldots, p_{n-1}) \in \Xi$, the inverse image $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ is a single point. Note that $(p_1, p_2, \ldots, p_n)$ is contained in $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ precisely when p_n satisfies $\sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z}$, that is, $p_n \equiv j\kappa - \sum_{i=1}^{n-1} p_i \bmod d$. Such an integer p_n $(0 \leq p_n < d)$ is unique, so $\varphi^{-1}(p_1, p_2, \ldots, p_{n-1})$ is a single point. $\square$

Let $\Lambda^{(j)}$ be the set given by (26). For each $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$, define an automorphism $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ by

$$\begin{aligned}&(X_1, X_2, \ldots, X_n) \longmapsto \\ &\qquad \left(e^{2\pi \mathrm{i}(ja_1+m_1p_1)/m_1d} X_1,\ e^{2\pi \mathrm{i}(ja_2+m_2p_2)/m_2d} X_2, \ldots, e^{2\pi \mathrm{i}(ja_n+m_np_n)/m_nd} X_n\right).\end{aligned}$$

Lemma 11. *For any $(p_1, p_2, \ldots, p_n)$, $(p'_1, p'_2, \ldots, p'_n) \in \Lambda^{(j)}$, the following hold:*

(1) *For $i = 1, 2, \ldots, n$,*

$$e^{2\pi \mathrm{i}(ja_i+m_ip_i)/m_id} = e^{2\pi \mathrm{i}(ja_i+m_ip'_i)/m_id} e^{2\pi \mathrm{i}(p_i-p'_i)/d}.$$

(2) *If $(p_1, p_2, \ldots, p_n) \neq (p'_1, p'_2, \ldots, p'_n)$, say $p_i \neq p'_i$ for some i, then $e^{2\pi \mathrm{i}(ja_i+m_ip_i)/m_id} \neq e^{2\pi \mathrm{i}(ja_i+m_ip'_i)/m_id}$.*

(3) *If $(p_1, p_2, \ldots, p_n) \neq (p'_1, p'_2, \ldots, p'_n)$, then $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \neq \widetilde{\gamma}^{(j)}_{p'_1,p'_2,\ldots,p'_n}$.*

Proof. (1): From $\frac{ja_i + m_ip_i}{m_id} - \frac{ja_i + m_ip'_i}{m_id} = \frac{p_i - p'_i}{d}$, we have $\frac{ja_i + m_ip_i}{m_id} = \frac{ja_i + m_ip'_i}{m_id} + \frac{p_i - p'_i}{d}$, which yields the equation in assertion.

(2): Since $0 \leq p_i < d$ and $0 \leq p'_i < d$, $p_i \neq p'_i$ implies $p_i \not\equiv p'_i \bmod d$, accordingly $\frac{p_i - p'_i}{d} \not\equiv 0 \bmod \mathbb{Z}$. Hence $e^{2\pi \mathrm{i}(p_i-p'_i)/d} \neq 1$ in (1), implying that $e^{2\pi \mathrm{i}(ja_i+m_ip_i)/m_id} \neq e^{2\pi \mathrm{i}(ja_i+m_ip'_i)/m_id}$.

(3): This follows from (2). $\square$

We next show the following:

Corollary 2. *The number of elements of $\left\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\right\}$ is d^{n-1}.*

Proof. By (3) of Lemma 11, the number of elements in the set $\left\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\right\}$ coincides with that of $\Lambda^{(j)}$, and by Lemma 10, it is d^{n-1}. $\square$

Recall that $d = m_1' m_2' \cdots m_n' c \left(\sum_{i=1}^{n} \frac{a_i}{m_i} + \kappa \right)$ (see (21)), so

$$\sum_{i=1}^{n} \frac{a_i}{m_i} = \frac{d}{m_1' m_2' \cdots m_n' c} - \kappa. \tag{27}$$

Lemma 12. *For any* $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$,

$$\sum_{i=1}^{n} \frac{ja_i + m_i p_i}{m_i d} \equiv \frac{j}{m_1' m_2' \cdots m_n' c} \bmod \mathbb{Z}.$$

Proof. Using (27), the left hand side is rewritten as

$$\sum_{i=1}^{n} \frac{ja_i + m_i p_i}{m_i d} \frac{j}{m_1' m_2' \cdots m_n' c} - \frac{j\kappa}{d} + \sum_{i=1}^{n} \frac{p_i}{d}.$$

Here $\sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \bmod \mathbb{Z}$ (because $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$), so $\sum_{i=1}^{n} \frac{ja_i + m_i p_i}{m_i d} \equiv \frac{j}{m_1' m_2' \cdots m_n' c} \bmod \mathbb{Z}$. □

Corollary 3. *For each j, let* $\mathrm{Lift}^{(j)}$ *be the set of all lifts of $\gamma^j \in \Gamma$ with respect to the covering $p : \widetilde{A}_{d-1} \to A_{d-1}$. Then the following hold*:

1. *The number of elements of* $\mathrm{Lift}^{(j)}$ *is d^{n-1}.*
2. *For any $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$, $\widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n} \in \mathrm{Lift}^{(j)}$.*
3. $\mathrm{Lift}^{(j)} = \left\{ \widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)} \right\}$.

Proof. (1): Since the covering p is d^{n-1}-fold, for each j, $\gamma^j \in \Gamma$ has d^{n-1} lifts, so $\mathrm{Lift}^{(j)}$ consists of d^{n-1} elements.

(2): It suffices to show that the following diagram commutes:

$$\begin{tikzcd}
\widetilde{A}_{d-1} \arrow[r, "\widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n}"] \arrow[d, "p"'] & \widetilde{A}_{d-1} \arrow[d, "p"] \\
A_{d-1} \arrow[r, "\gamma^j"] & A_{d-1}.
\end{tikzcd}$$

For $(X_1, \ldots, X_n) \in \widetilde{A}_{d-1}$,

$$\begin{aligned}
&p \circ \widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n}(X_1, \ldots, X_n) \\
&\quad = p\big(e^{2\pi \mathrm{i}(ja_1 + m_1 p_1)/m_1 d} X_1, \ \ldots, e^{2\pi \mathrm{i}(ja_n + m_n p_n)/m_n d} X_n\big) \\
&\quad = \big(e^{2\pi \mathrm{i} ja_1/m_1} X_1, \ \ldots, e^{2\pi \mathrm{i} ja_n/m_n} X_n, \ e^{2\pi \mathrm{i} \sum_{i=1}^{n} \{(ja_i + m_i p_i)/m_i d\}} X_1 X_2 \cdots X_n\big).
\end{aligned}$$

Here $e^{2\pi\mathrm{i}\sum_{i=1}^{n}\{(ja_i+m_ip_i)/m_id\}} = e^{2\pi\mathrm{i}j/m'_1m'_2\cdots m'_nc}$ by Lemma 12, thus

$$\begin{aligned} &p\circ\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}(X_1,\dots,X_n) \\ &\qquad = \left(e^{2\pi\mathrm{i}ja_1/m_1}X_1,\ \dots, e^{2\pi\mathrm{i}ja_n/m_n}X_n,\ e^{2\pi\mathrm{i}j/m'_1m'_2\cdots m'_nc}X_1X_2\cdots X_n\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\gamma^j\circ p(X_1,\dots,X_n) \\ &\qquad = \left(e^{2\pi\mathrm{i}ja_1/m_1}X_1,\ \dots, e^{2\pi\mathrm{i}ja_n/m_n}X_n,\ e^{2\pi\mathrm{i}j/m'_1m'_2\cdots m'_nc}X_1X_2\cdots X_n\right). \end{aligned}$$

Hence $p\circ\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} = \gamma^j\circ p$, confirming the assertion.

(3): From (2), $\left\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} : (p_1,p_2,\dots,p_n)\in\Lambda^{(j)}\right\} \subset \mathrm{Lift}^{(j)}$. Here "$\subset$" is "$=$", because the numbers of elements of both sets are equal, indeed they consist of d^{n-1} elements ((1) and Corollary 2). $\square$

The following will be used in later discussion:

Corollary 4. *$\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}$ descends to γ^j. Moreover if $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}$ is of the form $(X_1,\dots,X_i,\dots,X_n)\mapsto(X_1,\dots,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}X_i,\dots,X_n)$, then it descends to γ^j of the form*

$$(x_1,\dots,x_n,t)\longmapsto(x_1,\dots,e^{2\pi\mathrm{i}ja_i/m_i}x_i,\dots,x_n,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}t).$$

Proof. The first statement follows from Corollary 3 (3). To show the second statement, it suffices to show that $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}$ and γ^j of the above forms satisfy $p\circ\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} = \gamma^j\circ p$, which is confirmed as follows:

$$\begin{aligned} &p\circ\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}(X_1,\dots,X_i,\dots,X_n) \\ &\qquad = p(X_1,\dots,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}X_i,\dots,X_n) \\ &\qquad = (X_1^d,\dots,e^{2\pi\mathrm{i}ja_i/m_i}X_i^d,\dots,X_n^d,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}X_1X_2\cdots X_n) \\ &\qquad = \gamma^j\circ p(X_1,\dots,X_i,\dots,X_n). \end{aligned}$$

$\square$

By Corollary 3 (3), $\mathrm{Lift}^{(j)} = \left\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} : (p_1,p_2,\dots,p_n)\in\Lambda^{(j)}\right\}$. Since $\widetilde{\Gamma} = \coprod_{j=1}^{m'_1m'_2\cdots m'_nc}\mathrm{Lift}^{(j)}$ (disjoint union), we have

$$\widetilde{\Gamma} = \coprod_{j=1}^{m'_1m'_2\cdots m'_nc}\left\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} : (p_1,p_2,\dots,p_n)\in\Lambda^{(j)}\right\}.$$

Or

$$\widetilde{\Gamma} = \left\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} : (p_1,p_2,\dots,p_n)\in\Lambda^{(j)},\ j=1,2,\dots,m'_1m'_2\cdots m'_nc\right\}. \quad (28)$$

We thus obtain:

Lemma 13. *The lift* $\widetilde{\Gamma}$ *of* Γ *consists of the automorphisms* $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ *given by*

$$(X_1, X_2, \ldots, X_n) \longmapsto \big(e^{2\pi \mathrm{i}(ja_1+m_1p_1)/m_1d}X_1,\ e^{2\pi \mathrm{i}(ja_2+m_2p_2)/m_2d}X_2, \ldots, e^{2\pi \mathrm{i}(ja_n+m_np_n)/m_nd}X_n\big),$$

where $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$ *and* $j = 1, 2, \ldots m'_1m'_2 \cdots m'_nc$. (In particular, any two elements of $\widetilde{\Gamma}$ commute, so $\widetilde{\Gamma}$ is abelian.)

7.2 Determination of H

Recall that $\widetilde{\Gamma} = \coprod_{j=1}^{m'_1m'_2\cdots m'_nc} q_*\big(\mathrm{Lift}^{(j)}\big)$, where $\mathrm{Lift}^{(j)}$ denotes the set of all lifts of $\gamma^j \in \Gamma$ with respect to the covering $p : \widetilde{A}_{d-1} \to A_{d-1}$. Accordingly $H = \bigcup_{j=1}^{m'_1m'_2\cdots m'_nc} q_*\big(\mathrm{Lift}^{(j)}\big)$, where $q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of $\mathrm{Lift}^{(j)}$ with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$. We determine $q_*\big(\mathrm{Lift}^{(j)}\big)$. To that end, for $j = 1, 2, \ldots, m'_1m'_2 \cdots m'_nc$ and $(p_1,\ p_2,\ \ldots, p_n) \in \Lambda^{(j)}$, define an automorphism $h^{(j)}_{p_1,p_2,\ldots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ by

$$(u_1, u_2, \ldots, u_n) \longmapsto \big(e^{2\pi \mathrm{i}(ja_1+m_1p_1)/cd}u_1,\ e^{2\pi \mathrm{i}(ja_2+m_2p_2)/cd}u_2, \ldots, e^{2\pi \mathrm{i}(ja_n+m_np_n)/cd}u_n\big).$$

Lemma 14. (1) $h^{(j)}_{p_1,p_2,\ldots,p_n}$ *is the descent of* $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}$ *with respect to the covering* $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$.

(2) $q_*\big(\mathrm{Lift}^{(j)}\big) = \big\{h^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\big\}$.

(3) $H = \Big\{h^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)},\ j = 1, 2, \ldots, m'_1m'_2 \cdots m'_nc\Big\}$.

Proof. (1): Since $\big(e^{2\pi \mathrm{i}(ja_i+m_ip_i)/m_id}\big)^{m'_i} = e^{2\pi \mathrm{i}(ja_i+m_ip_i)/cd}$ for $i = 1, 2, \ldots, n$, the following diagram commutes:

$$\begin{tikzcd} \widetilde{A}_{d-1} \arrow[r, "\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}"] \arrow[d, "q"'] & \widetilde{A}_{d-1} \arrow[d, "q"] \\ \mathbb{C}^n \arrow[r, "h^{(j)}_{p_1,p_2,\ldots,p_n}"] & \mathbb{C}^n. \end{tikzcd}$$

Hence $h^{(j)}_{p_1,p_2,\ldots,p_n}$ is the descent of $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}$ with respect to the covering q.

(2): By Corollary 3 (3),

$$\mathrm{Lift}^{(j)} = \{\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} : (p_1, p_2, \dots, p_n) \in \Lambda^{(j)}\},$$

accordingly by (1),

$$q_*\big(\mathrm{Lift}^{(j)}\big) = \{h^{(j)}_{p_1,p_2,\dots,p_n} : (p_1, p_2, \dots, p_n) \in \Lambda^{(j)}\}.$$

(3): This follows from $H = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} q_*\big(\mathrm{Lift}^{(j)}\big)$ and (2). □

For later discussion, we restate (3) of Lemma 14 as follows:

Lemma 15. *The descent H of $\widetilde{\Gamma}$ consists of the automorphisms $h^{(j)}_{p_1,p_2,\dots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ given by*

$$(u_1, u_2, \dots, u_n) \longmapsto \big(e^{2\pi\mathrm{i}(ja_1+m_1p_1)/cd}u_1,\ e^{2\pi\mathrm{i}(ja_2+m_2p_2)/cd}u_2, \dots, e^{2\pi\mathrm{i}(ja_n+m_np_n)/cd}u_n\big),$$

where $(p_1, p_2, \dots, p_n) \in \Lambda^{(j)}$ and $j = 1, 2, \dots m'_1 m'_2 \cdots m'_n c$. (In particular, any two elements of H commute, so H is abelian.)

8 The pseudo-reflection subgroup of H

8.1 Cyclic subgroups Γ_i of Γ and $\widetilde{\Gamma}_i$ of $\widetilde{\Gamma}$

Let $\gamma : A_{d-1} \to A_{d-1}$ be the automorphism given by

$$\gamma : (x_1, \dots, x_n, t) \longmapsto (e^{2\pi\mathrm{i}a_1/m_1}x_1, \dots, e^{2\pi\mathrm{i}a_n/m_n}x_n,\ e^{2\pi\mathrm{i}/m'_1m'_2\cdots m'_nc}t). \quad (29)$$

(The order of γ is $m'_1 m'_2 \cdots m'_n c$.) Consider the cyclic group Γ generated by γ:

$$\Gamma = \{\gamma^j : j = 1, 2, \dots, m'_1 m'_2 \cdots m'_n c\}.$$

Let Γ_i $(i = 1, 2, \dots, n)$ be the subgroup of Γ consisting of automorphisms of the form

$$(x_1, \dots, x_i, \dots, x_n, t) \longmapsto (x_1, \dots, e^{2\pi\mathrm{i}ja_i/m_i}x_i, \dots, x_n,\ e^{2\pi\mathrm{i}j/m'_1m'_2\cdots m'_nc}t),$$

that is,

$$(\sharp) \qquad e^{2\pi\mathrm{i}ja_k/m_k} = 1 \quad (k = 1, 2, \dots, \check{i}, \dots, n).$$

Lemma 16. *For $j \in \mathbb{Z}$,*

$$\gamma^j \in \Gamma_i \iff j \textit{ is a multiple of } \mathrm{lcm}(m'_1, \dots, \check{m}'_i, \dots, m'_n)c.$$

Proof. $\Longrightarrow$: If $\gamma^j \in \Gamma_i$, then from ($\sharp$), ja_k is divisible by m_k ($k = 1, 2, \ldots, \check{i}, \ldots, n$). Here a_k and m_k are relatively prime, so j is divisible by m_k ($k = 1, 2, \ldots, \check{i}, \ldots, n$). In particular, j is a multiple of $\operatorname{lcm}(m_1, \ldots, \check{m}_i, \ldots m_n) = \operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots m'_n)c$.

$\Longleftarrow$: If j is a multiple of $\operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots m'_n)c$, then j is divisible by m_k ($k = 1, 2, \ldots, \check{i}, \ldots, n$), so $\dfrac{ja_k}{m_k}$ is an integer. Thus $e^{2\pi \mathrm{i} k a_k/m_k} = 1$ ($k = 1, 2, \ldots, \check{i}, \ldots, n$), so $\gamma^j \in \Gamma_i$. $\square$

From Lemma 16, the following holds:

Corollary 5. *Γ_i is generated by $\gamma_i := \gamma^{\operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)c}$.*

This element is explicitly given by

$$\gamma_i : \ (x_1, \ldots, x_i, \ldots, x_n, t) \longmapsto$$
$$\Big(x_1, \ \ldots, e^{2\pi \mathrm{i} a_i \operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)/m'_i} x_i, \ldots, x_n, \ e^{2\pi \mathrm{i} \operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)/m'_1 m'_2 \cdots m'_n} t\Big).$$

Here $e^{2\pi \mathrm{i} \operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)/m'_1 m'_2 \cdots m'_n} = e^{2\pi \mathrm{i}/m'_i l_i}$, because

$$\frac{\operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}{m'_1 m'_2 \cdots m'_n} = \frac{1}{m'_i} \frac{\operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}{m'_1 \cdots \check{m}'_i \cdots m'_n} = \frac{1}{m'_i l_i}.$$

Thus

$$\begin{aligned} \gamma_i : \ & (x_1, \ldots, x_i, \ldots, x_n, t) \\ & \longmapsto \ \Big(x_1, \ \ldots, e^{2\pi \mathrm{i} a_i \operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)/m'_i} x_i, \ldots, x_n, \ e^{2\pi \mathrm{i}/m'_i l_i} t\Big). \end{aligned}$$

Set $L_i := \operatorname{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)$, then

$$\gamma_i : \ (x_1, \ldots, x_n, t) \ \longmapsto \ \big(x_1, \ \ldots, e^{2\pi \mathrm{i} a_i L_i/m'_i} x_i, \ldots, x_n, \ e^{2\pi \mathrm{i}/m'_i l_i} t\big).$$

For $k \in \mathbb{Z}$,

$$\gamma_i^k : \ (x_1, \ldots, x_i, \ldots, x_n, t) \ \longmapsto \ \big(x_1, \ \ldots, e^{2\pi \mathrm{i} a_i L_i k/m'_i} x_i, \ldots, x_n, \ e^{2\pi \mathrm{i} k/m'_i l_i} t\big).$$

In particular,

$$\gamma_i^k = \mathrm{id} \text{ if and only if } e^{2\pi \mathrm{i} a_i L_i k/m'_i} = 1 \text{ and } e^{2\pi \mathrm{i} k/m'_i l_i} = 1.$$

Here:

(A) $e^{2\pi \mathrm{i} a_i L_i k/m'_i} = 1$ if and only if $\dfrac{L_i k}{m'_i}$ is an integer (because a_i and m'_i are relatively prime).

(B) $e^{2\pi \mathrm{i} k/m'_i l_i} = 1$ if and only if $\dfrac{k}{m'_i l_i}$ is an integer.

We restate (A). First write $\dfrac{L_i}{m'_i}$ as $\dfrac{L'_i}{m''_i}$ where L'_i and m''_i are relatively prime positive integers (or, $L'_i := \dfrac{L_i}{\gcd(L_i, m'_i)}$ and $m''_i := \dfrac{m'_i}{\gcd(L_i, m'_i)}$). Then $\dfrac{L_i k}{m'_i} \left(= \dfrac{L'_i k}{m''_i} \right)$ is an integer if and only if m''_i divides k. Thus (A) is restated as:

(A)' $e^{2\pi \mathrm{i} a_i L_i k/m'_i} = 1$ if and only if m''_i divides k.

From (A)' and (B),

$$\gamma_i^k = \mathrm{id} \text{ if and only if } k \text{ is a common multiple of } m''_i \text{ and } m'_i l_i.$$

Here m'_i is a multiple of m''_i (because $m''_i := \dfrac{m'_i}{\gcd(L_i, m'_i)}$). Thus any common multiple of m''_i and $m'_i l_i$ is necessarily a multiple of $m'_i l_i$. Therefore:

Lemma 17. $\gamma_i^k = \mathrm{id}$ *if and only if k is a multiple of $m'_i l_i$. In particular, the order of γ_i is $m'_i l_i$.*

We summarize the above results (Corollary 5 and Lemma 17) as follows:

Corollary 6. *For each $i = 1, 2, \ldots, n$, let Γ_i be the subgroup of Γ consisting of automorphisms of the form*

$$(x_1, \ldots, x_i, \ldots, x_n, t) \longmapsto (x_1, \ldots, e^{2\pi \mathrm{i} j a_i/m_i} x_i, \ldots, x_n, e^{2\pi \mathrm{i} j/m'_1 m'_2 \cdots m'_n c} t).$$

Then Γ_i is a cyclic group of order $m'_i l_i$ generated by the automorphism

$$\gamma_i : (x_1, \ldots, x_i, \ldots, x_n, t) \longmapsto \big(x_1, \ldots, e^{2\pi \mathrm{i} a_i L_i/m'_i} x_i, \ldots, x_n, e^{2\pi \mathrm{i}/m'_i l_i} t\big),$$

where $L_i := \mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)$ and $l_i := \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{L_i}$.

Let $p : \widetilde{A}_{d-1} \,(= \mathbb{C}^n) \to A_{d-1}$ be the covering of A_{d-1} given by

$$p(X_1, X_2, \ldots, X_n) = (X_1^d, X_2^d, \ldots, X_n^d, X_1 X_2 \cdots X_n),$$

and $\widetilde{\Gamma}$ be the lift of Γ with respect to p. Next let $\xi_i : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ be the automorphism given by

$$\xi_i : (X_1, \ldots, X_i, \ldots, X_n) \longmapsto (X_1, \ldots, e^{2\pi \mathrm{i}/m'_i l_i} X_i, \ldots, X_n). \tag{30}$$

Then:

Lemma 18. (1) *The order of ξ_i is $m'_i l_i$.* (The order of γ_i is also $m'_i l_i$ by Lemma 17.)

(2) $\xi_i \in \widetilde{\Gamma}$. *In fact, ξ_i is a lift of $\gamma_i \in \Gamma_i\,(\subset \Gamma)$, that is, the following diagram commutes*:

$$\begin{tikzcd} \widetilde{A}_{d-1} \arrow[r, "\xi_i"] \arrow[d, "p"'] & \widetilde{A}_{d-1} \arrow[d, "p"] \\ A_{d-1} \arrow[r, "\gamma_i"] & A_{d-1}. \end{tikzcd}$$

Proof. (1) is clear. We show (2). It suffices to show that $p \circ \xi_i = \gamma_i \circ p$. Note first that

$$\begin{aligned} &p \circ \xi_i(X_1, \ldots, X_i, \ldots, X_n) \\ &\qquad = (X_1^d, \ldots, e^{2\pi \mathrm{i} d/m_i' l_i} X_i^d, \ldots, X_n^d, e^{2\pi \mathrm{i}/m_i' l_i} X_1 X_2 \cdots X_n). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\gamma_i \circ p(X_1, \ldots, X_i, \ldots, X_n) \\ &\qquad = (X_1^d, \ldots, e^{2\pi \mathrm{i} a_i L_i/m_i'} X_i^d, \ldots, X_n^d, e^{2\pi \mathrm{i}/m_i' l_i} X_1 X_2 \cdots X_n). \end{aligned}$$

Thus to show that $p \circ \xi_i = \gamma_i \circ p$, it suffices to show that $e^{2\pi \mathrm{i} d/m_i' l_i} = e^{2\pi \mathrm{i} a_i L_i/m_i'}$, that is,

$$\frac{d}{m_i' l_i} \equiv \frac{a_i L_i}{m_i'} \mod \mathbb{Z}. \tag{31}$$

Since $d = m_1' m_2' \cdots m_n' c\Big(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa\Big)$ and $l_i = \dfrac{m_1' \cdots \check{m}_i' \cdots m_n'}{L_i}$, the right hand side of (31) is

$$\begin{aligned} \frac{d}{m_i' l_i} &= \frac{a_1 L_i}{m_1'} + \frac{a_2 L_i}{m_2'} + \cdots + \frac{a_n L_i}{m_n'} + c\kappa L_i \\ &\equiv \frac{a_1 L_i}{m_1'} + \frac{a_2 L_i}{m_2'} + \cdots + \frac{a_n L_i}{m_n'} \mod \mathbb{Z}. \end{aligned}$$

Here $L_i := \mathrm{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')$ is divisible by m_k' $(k = 1, 2, \ldots, \check{i}, \ldots, n)$, so $\dfrac{a_k L_i}{m_k'} \in \mathbb{Z}$, that is, $\dfrac{a_k L_i}{m_k'} \equiv 0 \mod \mathbb{Z}$ $(k = 1,\, 2, \ldots,\, \check{i}, \ldots, n)$, hence $\dfrac{d}{m_i' l_i} \equiv \dfrac{a_i L_i}{m_i'} \mod \mathbb{Z}$, confirming (31). $\square$

As we saw in Lemma 13,

$$\widetilde{\Gamma} = \Big\{\widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n} \,:\, (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)},\ j = 1, 2, \ldots, m_1' m_2' \cdots m_n' c\Big\},$$

where $\Lambda^{(j)} = \Big\{(p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n \,:\, 0 \le p_i < d,\ \sum_{i=1}^{n} \frac{p_i}{d} \equiv \frac{j\kappa}{d} \mod \mathbb{Z}\Big\}$ and $\widetilde{\gamma}^{(j)}_{p_1, p_2, \ldots, p_n} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ is the automorphism given by

$$\begin{aligned} &(X_1, X_2, \ldots, X_n) \longmapsto \\ &\qquad \big(e^{2\pi \mathrm{i}(j a_1 + m_1 p_1)/m_1 d} X_1,\ e^{2\pi \mathrm{i}(j a_2 + m_2 p_2)/m_2 d} X_2, \ldots, e^{2\pi \mathrm{i}(j a_n + m_n p_n)/m_n d} X_n\big). \end{aligned}$$

Here Corollary 4 states that (i) $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}$ descends to γ^j and (ii) moreover if $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}$ is of the form $(X_1,\dots,X_n)\mapsto(X_1,\dots,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}X_i,\dots,X_n)$, then it descends to γ^j of the form

$$(x_1,\dots,x_n,t)\longmapsto(x_1,\dots,e^{2\pi\mathrm{i}ja_i/m_i}x_i,\dots,x_n,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}t).$$

Note the following:

Lemma 19. *In the case of* (ii)*, there exists an integer s_i such that* $e^{2\pi\mathrm{i}ja_i/m_i}=e^{2\pi\mathrm{i}a_iL_is_i/m'_i}$ *and* $e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}=e^{2\pi\mathrm{i}s_i/m'_il_i}$.

Proof. Since the γ^j in (ii) is an element of Γ_i, and Γ_i is generated by γ_i (Corollary 6), there exists an integer s_i such that $\gamma^j=\gamma_i^{s_i}$. Here

$$\begin{cases}\gamma^j: (x_1,\dots,x_n,t)\mapsto(x_1,\dots,e^{2\pi\mathrm{i}ja_i/m_i}x_i,\dots,x_n,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}t),\\ \gamma_i^{s_i}: (x_1,\dots,x_n,t)\mapsto\big(x_1,\dots,e^{2\pi\mathrm{i}a_iL_is_i/m'_i}x_i,\dots,x_n,e^{2\pi\mathrm{i}s_i/m'_il_i}t\big),\end{cases}$$

so $e^{2\pi\mathrm{i}ja_i/m_i}=e^{2\pi\mathrm{i}a_iL_is_i/m'_i}$ and $e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}=e^{2\pi\mathrm{i}s_i/m'_il_i}$. □

Let $\xi_i:\widetilde{A}_{d-1}\to\widetilde{A}_{d-1}$ be the automorphism given by

$$\xi_i: (X_1,\dots,X_i,\dots,X_n)\longmapsto(X_1,\dots,e^{2\pi\mathrm{i}/m'_il_i}X_i,\dots,X_n).$$

Then $\xi_i\in\widetilde{\Gamma}$ (Lemma 18 (2)). In fact, $\xi_i\in\widetilde{\Gamma}\cap\Xi_i$, where Ξ_i $(i=1,2,\dots,n)$ is the multiplicative group of automorphisms consisting of scalar multiplication of the ith coordinate of $\widetilde{A}_{d-1}\,(=\mathbb{C}^n)$:

$$\Xi_i:=\Big\{(X_1,\dots,X_i,\dots,X_n)\longmapsto(X_1,\dots,\lambda X_i,\dots,X_n)\,:\,\lambda\in\mathbb{C}^\times\Big\}.$$

Setting $\widetilde{\Gamma}_i:=\widetilde{\Gamma}\cap\Xi_i$, we claim that ξ_i in fact generates $\widetilde{\Gamma}_i$, that is, any element of $\widetilde{\Gamma}_i$ is a power of ξ_i. To see this, note that $\widetilde{\Gamma}_i$ consists of $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}$ of the form

$$\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}: (X_1,\dots,X_i,\dots,X_n)\longmapsto(X_1,\dots,e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}X_i,\dots,X_n).$$

Here for each $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}\in\widetilde{\Gamma}_i$, there exists an integer s_i such that $e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}=e^{2\pi\mathrm{i}s_i/m'_il_i}$ (Lemma 19). Then

$$\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}: (X_1,\dots,X_i,\dots,X_n)\longmapsto(X_1,\dots,e^{2\pi\mathrm{i}s_i/m'_il_i}X_i,\dots,X_n),$$

so $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}=\xi_i^{s_i}$, confirming that ξ_i generates $\widetilde{\Gamma}_i$. Here the order of ξ_i is m'_il_i (Lemma 18 (1)), so the order of the cyclic group $\widetilde{\Gamma}_i$ is m'_il_i.

We formalize the above result as follows:

Proposition 4. *For each $i = 1, 2, \ldots, n$, let $\widetilde{\Gamma}_i$ be the subgroup of $\widetilde{\Gamma}$ consisting of $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}$ of the form*

$$(X_1, \ldots, X_i, \ldots, X_n) \longmapsto (X_1, \ldots, e^{2\pi \mathrm{i}(ja_i + m_i p_i)/m_i d} X_i, \ldots, X_n).$$

Then $\widetilde{\Gamma}_i$ is a cyclic group of order $m'_i l_i$ generated by the automorphism

$$\xi_i : (X_1, \ldots, X_i, \ldots, X_n) \longmapsto (X_1, \ldots, e^{2\pi \mathrm{i}/m'_i l_i} X_i, \ldots, X_n),$$

where $l_i := \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}$.

8.2 Cyclic subgroups H_i of H

We have described cyclic subgroups $\widetilde{\Gamma}_i$ $(i = 1, 2, \ldots, n)$ of $\widetilde{\Gamma}$. We next describe subgroups of H corresponding to them. Here H is the descent of $\widetilde{\Gamma}$ with respect to the covering map $q : \widetilde{A}_{d-1} \,(= \mathbb{C}^n) \to \mathbb{C}^n$ given by

$$q(X_1, X_2, \ldots, X_n) = (X_1^{m'_1},\, X_2^{m'_2}, \ldots, X_n^{m'_n}).$$

Explicitly H is given by (Lemma 15):

$$H = \Big\{ h^{(j)}_{p_1,p_2,\ldots,p_n} \,:\, (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)},\, j = 1, 2, \ldots, m'_1 m'_2 \cdots m'_n c \Big\},$$

where $h^{(j)}_{p_1,p_2,\ldots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ is the automorphism given by

$$(u_1, u_2, \ldots, u_n) \longmapsto \Big(e^{2\pi \mathrm{i}(ja_1 + m_1 p_1)/cd} u_1,\, e^{2\pi \mathrm{i}(ja_2 + m_2 p_2)/cd} u_2, \ldots, e^{2\pi \mathrm{i}(ja_n + m_n p_n)/cd} u_n \Big).$$

Now let H_i $(i = 1, 2, \ldots, n)$ be the subgroup of H consisting of $h^{(j)}_{p_1,p_2,\ldots,p_n}$ of the form

$$(u_1, \ldots, u_i, \ldots, u_n) \longmapsto (u_1, \ldots, e^{2\pi \mathrm{i}(ja_i + m_i p_i)/cd} u_i, \ldots, u_n). \tag{32}$$

Lemma 20. *Let $h_i : \mathbb{C}^n \to \mathbb{C}^n$ be the automorphism given by*

$$h_i : (u_1, \ldots, u_i, \ldots, u_n) \longmapsto (u_1, \ldots, e^{2\pi \mathrm{i}/l_i} u_i, \ldots, u_n). \tag{33}$$

Then $h_i \in H$. In fact, h_i is the descent of $\xi_i \in \widetilde{\Gamma}_i \,(\subset \widetilde{\Gamma})$, that is, the following diagram commutes:

$$\begin{tikzcd}
\widetilde{A}_{d-1} \arrow[r, "\xi_i"] \arrow[d, "q"'] & \widetilde{A}_{d-1} \arrow[d, "q"] \\
\mathbb{C}^n \arrow[r, "h_i"] & \mathbb{C}^n.
\end{tikzcd}$$

Proof. It suffices to show that $q \circ \xi_i = h_i \circ q$. This is confirmed as follows:

$$\begin{aligned} q \circ \xi_i(X_1, \dots, X_i, \dots, X_n) &= q(X_1, \dots, e^{2\pi \mathrm{i}/m'_i l_i} X_i, \dots, X_n) \\ &= (X_1^{m'_1}, \dots, e^{2\pi \mathrm{i}/l_i} X_i^{m'_i}, \dots, X_n^{m'_n}) \\ &= h_i(X_1^{m'_1}, \dots, X_i^{m'_i}, \dots, X_n^{m'_n}) \\ &= h_i \circ q(X_1, \dots, X_i, \dots, X_n). \end{aligned}$$

□

Since $\widetilde{\Gamma}_i$ is a cyclic group generated by ξ_i (Proposition 4) and h_i is the descent of ξ_i with respect to q (Lemma 20), the following holds:

Lemma 21. *The descent of $\widetilde{\Gamma}_i$ is a cyclic group generated by the automorphism $h_i : (u_1, \dots, u_n) \mapsto (u_1, \dots, e^{2\pi \mathrm{i}/l_i} u_i, \dots, u_n)$.*

In fact, as we show subsequently, this cyclic group coincides with the subgroup H_i of H consisting of $h^{(j)}_{p_1,p_2,\dots,p_n}$ of the form

$$(u_1, \dots, u_i, \dots, u_n) \longmapsto (u_1, \dots, e^{2\pi \mathrm{i}(ja_i + m_i p_i)/cd} u_i, \dots, u_n). \tag{34}$$

In other words, $\widetilde{\Gamma}_i$ *descends to H_i with respect to the covering map q.*

To show this, it suffices to show that for any $h^{(j)}_{p_1,p_2,\dots,p_n} \in H_i$, there exists an element of $\widetilde{\Gamma}_i$ that descends to $h^{(j)}_{p_1,p_2,\dots,p_n}$. Here

$$\begin{cases} h^{(j)}_{p_1,p_2,\dots,p_n} : \ (u_1, \dots, u_n) \longmapsto (u_1, \dots, e^{2\pi \mathrm{i}(ja_i + m_i p_i)/cd} u_i, \dots, u_n), \\ q : \ (X_1, X_2, \dots, X_n) \longmapsto (X_1^{m'_1}, X_2^{m'_2}, \dots, X_n^{m'_n}). \end{cases}$$

Thus an automorphism $\zeta : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ given by

$$\zeta : (X_1, \dots, X_n) \longmapsto (X_1, \dots, e^{2\pi \mathrm{i}\{(ja_i + m_i p_i)/m_i d\}} X_i, \dots, X_n) \tag{35}$$

descends to $h^{(j)}_{p_1,p_2,\dots,p_n}$. We show that in fact $\zeta \in \widetilde{\Gamma}$ (then from the form of ζ, $\zeta \in \widetilde{\Gamma}_i$, so ζ is a lift of $h^{(j)}_{p_1,p_2,\dots,p_n}$).

Step 1 Since $q(X_1, X_2, \dots, X_n) = (X_1^{m'_1}, X_2^{m'_2}, \dots, X_n^{m'_n})$, the set of all lifts of $h^{(j)}_{p_1,p_2,\dots,p_n} : (u_1, \dots, u_n) \mapsto (u_1, \dots, e^{2\pi \mathrm{i}(ja_i + m_i p_i)/cd} u_i, \dots, u_n)$ with respect to the covering q consists of automorphisms

$(X_1, \dots, X_n) \mapsto (e^{2\pi \mathrm{i} k_1/m'_1} X_1, \dots, e^{2\pi \mathrm{i}\{(ja_i + m_i p_i)/m_i d + k_i/m'_i\}} X_i, \dots, e^{2\pi \mathrm{i} k_n/m'_n} X_n)$,

where $k_1, k_2, \dots, k_n$ are integers.

Step 2 Since $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} \in \widetilde{\Gamma}$ is a lift of $h^{(j)}_{p_1,p_2,\dots,p_n}$ with respect to q (Lemma 14 (1)), $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}$ coincides with one of the automorphisms in Step 1. Namely for some integers $k_1, k_2, \dots, k_n$,

$$\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n} : (X_1, \dots, X_i, \dots, X_n) \longmapsto \\ (e^{2\pi \mathrm{i} k_1/m'_1} X_1, \dots, e^{2\pi \mathrm{i}\{(ja_i + m_i p_i)/m_i d + k_i/m'_i\}} X_i, \dots, e^{2\pi \mathrm{i} k_n/m'_n} X_n).$$

Next for each $k = 1, 2, \ldots, n$, take the automorphism

$$\xi_k : (X_1, \ldots, X_n) \mapsto (X_1, \ldots, e^{2\pi \mathrm{i}/m'_k l_k} X_k, \ldots, X_n).$$

The composite automorphism $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \xi_1^{-l_1 k_1} \xi_2^{-l_2 k_2} \cdots \xi_n^{-l_n k_n}$ is then given by

$$(X_1, \ldots, X_i, \ldots, X_n) \longmapsto (X_1, \ldots, e^{2\pi \mathrm{i}\{(ja_i + m_i p_i)/m_i d\}} X_i, \ldots, X_n).$$

This coincides with the automorphism ζ given by (35), thus

$$\zeta = \widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \xi_1^{-l_1 k_1} \xi_2^{-l_2 k_2} \cdots \xi_n^{-l_n k_n}.$$

Step 3 Since $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \in \widetilde{\Gamma}$ and $\xi_k \in \widetilde{\Gamma}$ $(k = 1, 2, \ldots, n)$, we have $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \xi_1^{-l_1 k_1} \xi_2^{-l_2 k_2} \cdots \xi_n^{-l_n k_n} \in \widetilde{\Gamma}$. Hence $\zeta \in \widetilde{\Gamma}$, confirming the assertion.

We thus obtained the following:

Lemma 22. *For each $h^{(j)}_{p_1,p_2,\ldots,p_n} \in H_i$, there exists an element of $\widetilde{\Gamma}_i$ that descends to it (with respect to the covering q). In fact, the automorphism $\zeta : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ given by $\zeta : (X_1, \ldots, X_n) \mapsto (X_1, \ldots, e^{2\pi \mathrm{i}\{(ja_i + m_i p_i)/m_i d\}} X_i, \ldots, X_n)$ is an element of $\widetilde{\Gamma}_i$ that descends to*

$$h^{(j)}_{p_1,p_2,\ldots,p_n} : (u_1, \ldots, u_n) \mapsto (u_1, \ldots, e^{2\pi \mathrm{i}(ja_i + m_i p_i)/cd} u_i, \ldots, u_n).$$

Corollary 7. *H_i is the descent of $\widetilde{\Gamma}_i$ with respect to the covering q.*

The descent of $\widetilde{\Gamma}_i$ with respect to the covering q is a cyclic group generated by h_i (Lemma 21). On the other hand, this descent coincides with H_i (Corollary 7). Thus:

Lemma 23. *H_i is a cyclic group generated by the automorphism $h_i : (u_1, \ldots, u_n) \mapsto (u_1, \ldots, e^{2\pi \mathrm{i}/l_i} u_i, \ldots, u_n)$.* (Here the order of h_i is l_i, so the order of H_i is l_i.)

We summarize the above results as follows:

Proposition 5. (1) *$\widetilde{\Gamma}_i$ is a cyclic group (of order $m'_i l_i$) generated by the automorphism $\xi_i : (X_1, \ldots, X_i, \ldots, X_n) \mapsto (X_1, \ldots, e^{2\pi \mathrm{i}/m'_i l_i} X_i, \ldots, X_n)$, where $l_i := \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}$.* (See Proposition 4.)

(2) *H_i is the descent of $\widetilde{\Gamma}_i$ with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$ given by $q(X_1, X_2, \ldots, X_n) = (X_1^{m'_1}, X_2^{m'_2}, \ldots, X_n^{m'_n})$.* (See Corollary 7.)

(3) *H_i is a cyclic group (of order l_i) generated by the automorphism $h_i : (u_1, \ldots, u_n) \mapsto (u_1, \ldots, e^{2\pi \mathrm{i}/l_i} u_i, \ldots, u_n)$.* (See Lemma 23.)

8.3 The pseudo-reflection subgroup of H

We keep the notation above. Let H be the descent of $\widetilde{\Gamma}$ with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$. Let H_i $(i = 1, 2, \ldots, n)$ be the subgroup of H consisting of $h^{(j)}_{p_1,p_2,\ldots,p_n}$ of the form

$$h^{(j)}_{p_1,p_2,\ldots,p_n} : (u_1, \ldots, u_i, \ldots, u_n) \mapsto (u_1, \ldots, e^{2\pi \mathrm{i}(ja_i+m_ip_i)/cd}u_i, \ldots, u_n). \quad (36)$$

In fact, H_i is a cyclic group of order l_i generated by the automorphism $h_i : (u_1, \ldots, u_n) \mapsto (u_1, \ldots, e^{2\pi \mathrm{i}/l_i}u_i, \ldots, u_n)$. See Proposition 5 (3). Note that if $i \neq j$, then $H_i \cap H_j = \{1\}$. In particular,

$$H_1H_2 \cdots H_n = H_1 \times H_2 \times \cdots \times H_n. \quad (37)$$

Note also that the set of all pseudo-reflections in H is given by $\left(\bigcup_{i=1}^{n} H_i\right) \setminus \{1\}$.

Now let P be the *pseudo-reflection subgroup* of H that is the subgroup generated by all pseudo-reflections in H, that is, by $\left(\bigcup_{i=1}^{n} H_i\right) \setminus \{1\}$. Here H_i $(i = 1, 2, \ldots, n)$ is a cyclic group generated by h_i, so P is generated by $h_1, h_2, \ldots, h_n$, thus $P = H_1H_2 \cdots H_n = H_1 \times H_2 \times \cdots \times H_n$ (see (37)). Since the order of H_i is l_i, the order of P is $l_1l_2 \cdots l_n$. This confirms the following:

Proposition 6. *Where H_i $(i = 1, 2, \ldots, n)$ is a cyclic subgroup of H generated by the automorphism $h_i : (u_1, \ldots, u_n) \mapsto (u_1, \ldots, e^{2\pi \mathrm{i}/l_i}u_i, \ldots, u_n)$, the pseudo-reflection subgroup P of H is the direct product $P = H_1 \times H_2 \times \cdots \times H_n$ and the order of P is $l_1l_2 \cdots l_n$.*

In particular, $P = \{1\}$ if and only if $l_1 = l_2 = \cdots = l_n = 1$. Thus:

Corollary 8. *H is small if and only if $l_1 = l_2 = \cdots = l_n = 1$.*

Now let G be the descent of H with respect to the $l_1l_2 \cdots l_n$-fold covering $r : \mathbb{C}^n \to \mathbb{C}^n$ given by $r(u_1, u_2, \ldots, u_n) = (u_1^{l_1}, u_2^{l_2}, \ldots, u_n^{l_n})$. Then $l_1 = l_2 = \cdots = l_n = 1$ *if and only if r is the identity map, or equivalently $H = G$.* This, combined with Corollary 8, gives the following:

Lemma 24.

$$\begin{aligned} H \textit{ is small} &\iff l_1 = l_2 = \cdots = l_n = 1 \\ &\iff r \textit{ is the identity map} \\ &\iff H = G. \end{aligned}$$

The following arithmetic results are proved later (Corollary 10):

(1) If $n = 2$, then $l_1 = l_2 = 1$.

(2) If $n \geq 3$, then $l_1 = l_2 = \cdots = l_n = 1$ if and only if $\gcd(m'_j, m'_k) = 1$ for any $j \neq k$.

This, combined with Lemma 24, yields the following:

Theorem 3 (**Numerical criterion of smallness**). (1) *If $n = 2$, then H is always small.*

(2) *If $n \geq 3$, then H is small if and only if* $\gcd(m'_i,\, m'_j) = 1$ *for any i, j such that $i \neq j$.*

Example 2. If $n = 3$, $a_1 = a_2 = a_3 = 1$, $m_1 = 2$, $m_2 = 4$, $m_3 = 6$ and $\kappa = 0$, then $c = \gcd(m_1, m_2, m_3) = 2$, $m'_1 = 1$, $m'_2 = 2$, $m'_3 = 3$ and $d = 2+3+6 = 11$. In this case, Γ is generated by the automorphism γ of A_{d-1} ($= A_{10}$) given by $\gamma(x_1, x_2, x_3, t) \mapsto (e^{2\pi \mathrm{i}/2}x_1,\, e^{2\pi \mathrm{i}/4}x_2,\, e^{2\pi \mathrm{i}/6}x_3,\, e^{2\pi \mathrm{i}/12}t)$. Let $\widetilde{\Gamma}$ be the lift of Γ with respect to the covering $p : \widetilde{A}_{10} \to A_{10}$, $p(X_1, X_2, X_3) = (X_1^{11}, X_2^{11}, X_3^{11}, X_1X_2X_3)$, and let H be the descent of $\widetilde{\Gamma}$ with respect to the covering $q : \widetilde{A}_{10} \to \mathbb{C}^3$, $q(X_1, X_2, X_3) = (X_1,\, X_2^2,\, X_3^3)$. Then, since $\gcd(m'_1, m'_2) = 1$, $\gcd(m'_1, m'_3) = 1$ and $\gcd(m'_2, m'_3) = 1$, Theorem 3 ensures that H is small.

8.4 Supplement: Arithmetic result

This section is devoted to proving an arithmetic result used in §8.3.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive integers such that $\gcd(\lambda_1, \lambda_2, \ldots, \lambda_n) = 1$, where $n \geq 2$. Set $l_i := \dfrac{\lambda_1 \cdots \check{\lambda}_i \cdots \lambda_n}{\mathrm{lcm}(\lambda_1, \ldots, \check{\lambda}_i, \ldots, \lambda_n)}$, where $\check{\lambda}_i$ means the omission of λ_i. Note that l_i is a positive integer (cf. Remark 7). We show that if $n \geq 3$, then $l_1 = l_2 = \cdots = l_n = 1$ if and only if $\gcd(\lambda_j, \lambda_k) = 1$ for any $j \neq k$.

Remark 9. If $n = 2$, this equivalence is vacuous, because $l_1 = l_2 = 1$ *always* holds (and $\gcd(\lambda_1, \lambda_2) = 1$ by assumption). In fact $l_1 = \dfrac{\lambda_1}{\gcd(\lambda_1)} = 1$ and $l_2 = \dfrac{\lambda_2}{\gcd(\lambda_2)} = 1$.

We begin with some preparation:

Lemma 25. *For any i, j, k such that i, j and k are distinct,* $l_i \geq \gcd(\lambda_j, \lambda_k)$.

Proof. We only show the assertion for $i = 1$, $j = 2$ and $k = 3$ (the assertion for other cases are similarly shown). Note first that $\lambda_2\lambda_3 = \gcd(\lambda_2, \lambda_3) \cdot \mathrm{lcm}(\lambda_2, \lambda_3)$. Multiplying $\lambda_4 \cdots \lambda_n$ to this yields:

$$\lambda_2\lambda_3\lambda_4 \cdots \lambda_n = \gcd(\lambda_2, \lambda_3) \cdot \mathrm{lcm}(\lambda_2, \lambda_3)\lambda_4 \cdots \lambda_n.$$

Here, since $\mathrm{lcm}(\lambda_2, \lambda_3)\lambda_4 \cdots \lambda_n \geq \mathrm{lcm}(\lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_n)$,

$$\lambda_2\lambda_3\lambda_4 \cdots \lambda_n \geq \gcd(\lambda_2, \lambda_3) \cdot \mathrm{lcm}(\lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_n).$$

Dividing this by $\mathrm{lcm}(\lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_n)$,

$$\frac{\lambda_2\lambda_3\lambda_4 \cdots \lambda_n}{\mathrm{lcm}(\lambda_2, \lambda_3, \lambda_4 \cdots \lambda_n)} \geq \gcd(\lambda_2, \lambda_3).$$

Since the left hand side is l_1, we have $l_1 \geq \gcd(\lambda_2, \lambda_3)$. (Note: If $n = 3$, then the equality holds. In fact, $l_1 = \dfrac{\lambda_2\lambda_3}{\mathrm{lcm}(\lambda_2, \lambda_3)} = \gcd(\lambda_2, \lambda_3)$.) □

We next show that:

Lemma 26. *For each $i = 1, 2, \ldots, n$, the following are equivalent*:

(1) $l_i = 1$. (2) $\gcd(\lambda_j, \lambda_k) = 1$ *for any $j \neq k$ (distinct from i).*

Proof. (1) $\Longrightarrow$ (2): By Lemma 25, for any i, j, k such that i, j and k are distinct, $l_i \geq \gcd(\lambda_j, \lambda_k)$. Hence if $l_i = 1$, then $\gcd(\lambda_j, \lambda_k) = 1$ for any $j \neq k$ such that j and k are distinct from i.

(2) $\Longrightarrow$ (1): If $\gcd(\lambda_j, \lambda_k) = 1$ for any $j \neq k$ such that j and k distinct from i, then $\mathrm{lcm}(\lambda_1, \ldots, \check{\lambda}_i, \ldots, \lambda_n) = \lambda_1 \cdots \check{\lambda}_i \cdots \lambda_n$, and thus $l_i = 1$. □

Lemma 26 yields (2) of the following:

Corollary 9. *Let $\lambda_1, \ldots, \lambda_n$ be positive integers such that* $\gcd(\lambda_1, \ldots, \lambda_n) = 1$, *where $n \geq 2$. Set $l_i := \dfrac{\lambda_1 \cdots \check{\lambda}_i \cdots \lambda_n}{\mathrm{lcm}(\lambda_1, \ldots, \check{\lambda}_i, \ldots, \lambda_n)}$, where $\check{\lambda}_i$ means the omission of λ_i.* (Note that l_i is a positive integer (cf. Remark 7).) *Then the following hold*:

(1) *If $n = 2$, then $l_1 = l_2 = 1$* (Remark 9).

(2) *If $n \geq 3$, then $l_1 = l_2 = \cdots = l_n = 1$ if and only if* $\gcd(\lambda_j, \lambda_k) = 1$ *for any $j \neq k$.*

Now let $m_1, m_2, \ldots, m_n$ be positive integers. Set $c := \gcd(m_1', m_2', \ldots, m_n')$ and $m_i' := \dfrac{m_i'}{c}$ $(i = 1, 2, \ldots, n)$. Then $m_1', m_2', \ldots, m_n'$ are positive integers such that $\gcd(m_1', m_2', \ldots, m_n') = 1$. So we may apply Corollary 9 to the case that m_1', $m_2', \ldots, m_n'$ are λ_1, $\lambda_2, \ldots, \lambda_n$, then we obtain the following:

Corollary 10. *Let $m_1, m_2, \ldots, m_n$ be positive integers. Set* $c := \gcd(m_1', m_2', \ldots, m_n')$, *$m_i' := \dfrac{m_i'}{c}$ and $l_i := \dfrac{m_1' \cdots \check{m}_i' \cdots m_n'}{\mathrm{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')}$, where $\check{m}_i'$ means the omission of m_i'.* (Note that l_i is a positive integer (cf. Remark 7).) *Then the following hold*:

(1) *If* $n = 2$, *then* $l_1 = l_2 = 1$.

(2) *If* $n \geq 3$, *then* $l_1 = l_2 = \cdots = l_n = 1$ *if and only if* $\gcd(m'_j, m'_k) = 1$ *for any* $j \neq k$.

9 Uniformization theorem for arbitrary dimension

9.1 Determination of G

Recall the diagram (25) between the coverings maps p, q, r:

$$\begin{tikzcd}[column sep=small] & & \widetilde{A}_{d-1} = \mathbb{C}^n \arrow[dl, "q"'] \arrow[dr, "p"] & \\ & \mathbb{C}^n \arrow[dl, "r"'] & & A_{d-1}. \\ \mathbb{C}^n & & & \end{tikzcd} \tag{38}$$

Then

- $\widetilde{\Gamma} = \coprod_{j=1}^{m'_1 m'_2 \cdots m'_n c} \mathrm{Lift}^{(j)}$ (disjoint union), where $\mathrm{Lift}^{(j)}$ is the set of all lifts of $\gamma^j \in \Gamma$ with respect to the covering p.
- $H = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} q_*\big(\mathrm{Lift}^{(j)}\big)$, where $q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of $\mathrm{Lift}^{(j)}$ with respect to the covering q.
- $G = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big)$, where $r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big)$ is the descent of $q_*\big(\mathrm{Lift}^{(j)}\big)$ with respect to the covering r.

Here $\mathrm{Lift}^{(j)} = \big\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\big\}$ (Corollary 3 (3)) and $q_*\big(\mathrm{Lift}^{(j)}\big) = \big\{h^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}\big\}$ (Lemma 14 (2)). We next determine $r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big)$. For $j = 1, 2, \ldots, m'_1 m'_2 \cdots m'_n c$ and $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$, define an automorphism $g^{(j)}_{p_1,p_2,\ldots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\begin{aligned}&(v_1, v_2, \ldots, v_n) \longmapsto \\ &\qquad \big(e^{2\pi i l_1 (j a_1 + m_1 p_1)/cd} v_1,\ e^{2\pi i l_2 (j a_2 + m_2 p_2)/cd} v_2, \ldots, e^{2\pi i l_n (j a_n + m_n p_n)/cd} v_n\big).\end{aligned}$$

Then

Lemma 27. (A) $g^{(j)}_{p_1,p_2,\ldots,p_n}$ *is the descent of* $h^{(j)}_{p_1,p_2,\ldots,p_n}$ *with respect to the covering* $r : \mathbb{C}^n \to \mathbb{C}^n$.

(B) $r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big) = \big\{g^{(j)}_{p_1,p_2,\ldots,p_n} \,:\, (p_1,p_2,\ldots,p_n) \in \Lambda^{(j)}\big\}$.

(C) $G = \Big\{g^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1,p_2,\ldots,p_n) \in \Lambda^{(j)},\ j = 1,2,\ldots,m'_1 m'_2 \cdots m'_n c\Big\}$.

Proof. (A): For each $i = 1,2,\ldots,n$, $\big(e^{2\pi\mathrm{i}(ja_i+m_ip_i)/cd}\big)^{l_i} = e^{2\pi\mathrm{i}l_i(ja_i+m_ip_i)/cd}$. The following diagram thus commutes:

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "h^{(j)}_{p_1,p_2,\ldots,p_n}"] \arrow[d, "r"'] & \mathbb{C}^n \arrow[d, "r"] \\ \mathbb{C}^n \arrow[r, "g^{(j)}_{p_1,p_2,\ldots,p_n}"] & \mathbb{C}^n. \end{tikzcd}$$

Hence $g^{(j)}_{p_1,p_2,\ldots,p_n}$ is the descent of $h^{(j)}_{p_1,p_2,\ldots,p_n}$ with respect to the covering r.

(B): Lemma 14 (2) and (A) imply

$$r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big) = \big\{g^{(j)}_{p_1,p_2,\ldots,p_n} \,:\, (p_1,p_2,\ldots,p_n) \in \Lambda^{(j)}\big\}.$$

(C): This follows from $G = \bigcup_{j=1}^{m'_1 m'_2 \cdots m'_n c} r_* \circ q_*\big(\mathrm{Lift}^{(j)}\big)$ and (B). $\square$

For later discussion, we restate (C) of Lemma 27 as follows:

Lemma 28. *The descent G of H consists of the automorphisms $g^{(j)}_{p_1,p_2,\ldots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ given by*

$$(v_1,v_2,\ldots,v_n) \longmapsto \big(e^{2\pi\mathrm{i}l_1(ja_1+m_1p_1)/cd}v_1,\ e^{2\pi\mathrm{i}l_2(ja_2+m_2p_2)/cd}v_2,\ldots,e^{2\pi\mathrm{i}l_n(ja_n+m_np_n)/cd}v_n\big),$$

where $(p_1,p_2,\ldots,p_n) \in \Lambda^{(j)}$ and $j = 1,2,\ldots m'_1 m'_2 \cdots m'_n c$. (In particular, any two elements of G commute, so G is abelian.)

9.2 Uniformization theorem

A *pseudo-reflection* is a diagonalizable matrix such that one of its eigenvalues is a root of unity (distinct from 1) and all other eigenvalues are 1. In particular, a pseudo-reflection is *not* the identity matrix.

Let H be the descent of $\widetilde{\Gamma}$ with respect to the $m'_1 m'_2 \cdots m'_n$-fold covering $q : \mathbb{C}^n \to \mathbb{C}^n$ and P be the pseudo-reflection subgroup of H, that is, P is generated by all pseudo-reflections in H. Let G be the descent of H with respect to the $l_1 l_2 \cdots l_n$-fold covering $r : \mathbb{C}^n \to \mathbb{C}^n$. We then define a group homomorphism $r_* : H \to G$ by $r_*(h) :=$ the descent of $h \in H$ with respect to r. This is surjective and $\mathrm{Ker}(r_*) = P$, hence by the first isomorphism theorem,

Lemma 29. $G \cong H/P$.

Moreover the following holds:

Proposition 7. *G contains no pseudo-reflections, that is, is a small group.* (In fact, G is obtained from H by *collapsing* the pseudo-reflections in H: precisely speaking $G \cong H/P$ (Lemma 29), where P denotes the pseudo-reflection subgroup of H.)

Proof. First recall Lemma 28 that states: G consists of automorphisms $g : \mathbb{C}^n \to \mathbb{C}^n$ given by

$$g:\ (v_1, v_2, \ldots, v_n)\ \longmapsto \\ \big(e^{2\pi \mathrm{i} l_1(ja_1+m_1p_1)/cd}v_1,\ e^{2\pi \mathrm{i} l_2(ja_2+m_2p_2)/cd}v_2, \ldots, e^{2\pi \mathrm{i} l_n(ja_n+m_np_n)/cd}v_n\big),$$

where $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$ and $j = 1, 2, \ldots m'_1 m'_2 \cdots m'_n c$.

We now show that G contains no pseudo-reflections by contradiction. If $g \in G$ is a pseudo-reflection, then it is necessarily of the form:

$$g:\ (v_1, v_2, \ldots, v_n)\ \longmapsto\ \big(v_1, \ldots, v_{i-1}, e^{2\pi \mathrm{i} l_i(ja_i+m_ip_i)/cd}v_i, v_{i+1}, \ldots, v_n\big).$$

But then Lemma 30 below implies that $g = 1$, contradicting the assertion that g is a pseudo-reflection. □

Lemma 30. *Any element $g \in G$ of the form*

$$g:\ (v_1, v_2, \ldots, v_n) \mapsto \big(v_1, \ldots, v_{i-1}, e^{2\pi \mathrm{i} l_i(ja_i+m_ip_i)/cd}v_i, v_{i+1}, \ldots, v_n\big) \tag{39}$$

is necessarily the identity map, that is, $e^{2\pi \mathrm{i} l_i(ja_i+m_ip_i)/cd} = 1$.

Proof. Recall that H is the lift of G with respect to the covering $r : \mathbb{C}^n \to \mathbb{C}^n$ given by $r\ :\ (u_1, u_2, \ldots, u_n) \longmapsto (v_1, v_2, \ldots, v_n) := (u_1^{l_1}, u_2^{l_2}, \ldots, u_n^{l_n})$. Let $\widetilde{g} \in H$ be a lift of (39) given by

$$\widetilde{g}\ :\ (u_1, u_2, \ldots, u_n)\ \longmapsto\ \big(u_1, \ldots, u_{i-1}, e^{2\pi \mathrm{i}(ja_i+m_ip_i)/cd}u_i, u_{i+1}, \ldots, u_n\big).$$

Notice that $\widetilde{g} \in H_i$, where H_i is the subgroup of H appearing above Lemma 20; in fact H_i is a cyclic group generated by h_i (Proposition 5 (3)). Thus $\widetilde{g} = h_i^s$ for some integer s. Here

$$h_i^s\ :\ (u_1, u_2, \ldots, u_n)\ \longmapsto\ \big(u_1, \ldots, u_{i-1}, e^{2\pi \mathrm{i} s/l_i}u_i, u_{i+1}, \ldots, u_n\big),$$

so $e^{2\pi \mathrm{i}(ja_i+m_ip_i)/cd} = e^{2\pi \mathrm{i} s/l_i}$. Hence $e^{2\pi \mathrm{i} l_i(ja_i+m_ip_i)/cd} = (e^{2\pi \mathrm{i} s/l_i})^{l_i}$, that is, $e^{2\pi \mathrm{i} l_i(ja_i+m_ip_i)/cd} = 1$, confirming the assertion. □

Recall now the diagram of covering maps:

$$
\begin{tikzcd}
 & & \widetilde{A}_{d-1} = \mathbb{C}^n \arrow[dl, "q"'] \arrow[dr, "p"] & \\
 & \mathbb{C}^n \arrow[dl, "r"'] & & A_{d-1}. \\
\mathbb{C}^n & & &
\end{tikzcd}
\tag{40}
$$

- $\widetilde{\Gamma}$ is the lift of Γ (with respect to the covering $p : \widetilde{A}_{d-1} \to A_{d-1}$),
- H is the descent of $\widetilde{\Gamma}$ (with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$),
- G is the descent of H (with respect to the covering $r : \mathbb{C}^n \to \mathbb{C}^n$).

Hence $A_{d-1}/\Gamma \cong \widetilde{A}_{d-1}/\widetilde{\Gamma} \cong \mathbb{C}^n/H \cong \mathbb{C}^n/G$. Here G is a finite abelian group (Proposition 3 (3)) and small (Proposition 7). Thus A_{d-1}/Γ is uniformized by the small finite abelian group G, so the following is established:

Theorem 4 (**Uniformization theorem**)**.** *Let Γ be the cyclic group generated by the automorphism $\gamma : A_{d-1} \to A_{d-1}$ given by*

$$\gamma :\ (x_1, \ldots, x_n, t)\ \longmapsto\ (e^{2\pi \mathrm{i} a_1/m_1} x_1, \ldots, e^{2\pi \mathrm{i} a_n/m_n} x_n,\ e^{2\pi \mathrm{i}/m_1' m_2' \cdots m_n' c} t).$$

Then there exists a small finite abelian group $G \subset GL(n, \mathbb{C})$ such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$.

We explicitly give the isomorphism $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ in the uniformization theorem. The covering maps p, q and r appearing in the diagram (40) induce isomorphisms $\overline{p} : \widetilde{A}_{d-1}/\widetilde{\Gamma} \to A_{d-1}/\Gamma$ and $\overline{q} : \widetilde{A}_{d-1}/\widetilde{\Gamma} \to \mathbb{C}^n/H$ and $\overline{r} : \mathbb{C}^n/H \to \mathbb{C}^n/G$. The isomorphism $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$ in the uniformization theorem (Theorem 4) is then given by

$$\Psi := \overline{r} \circ \overline{q} \circ \overline{p}^{\,-1} :\ A_{d-1}/\Gamma \xrightarrow{\ \cong\ } \mathbb{C}^n/G. \tag{41}$$

Explicitly:

Lemma 31. $\qquad \Psi\big([x_1, \ldots, x_n, t]\big) = \big[x_1^{m_1' l_1/d}, \ldots, x_n^{m_n' l_n/d}\big],$
where we denote the image of $(X_1, X_2, \ldots, X_n) \in \widetilde{A}_{d-1}$ under the quotient map $\widetilde{A}_{d-1} \to \widetilde{A}_{d-1}/\widetilde{\Gamma}$ by $[X_1, X_2, \ldots, X_n] \in \widetilde{A}_{d-1}/\widetilde{\Gamma}$ and so on.

Proof. Since $p(X_1, X_2, \ldots, X_n) = (X_1^d, X_2^d, \ldots, X_n^d, X_1 X_2 \cdots X_n)$, we have $\overline{p}\big([X_1, X_2, \ldots, X_n]\big) = [X_1^d, X_2^d, \ldots, X_n^d, X_1 X_2 \cdots X_n]$, so

$$\overline{p}^{\,-1}\big([x_1, x_2, \ldots, x_n, t]\big) = \big[x_1^{1/d},\ x_2^{1/d},\ \ldots,\ x_n^{1/d}\big].$$

Next since $q(X_1, X_2, \dots, X_n) = (X_1^{m'_1}, X_2^{m'_2}, \dots, X_n^{m'_n})$ and $r(u_1, u_2, \dots, u_n) = (u_1^{l_1}, u_2^{l_2}, \dots, u_n^{l_n})$, we have $\overline{q}\big([X_1, X_2, \dots, X_n]\big) = [X_1^{m'_1}, X_2^{m'_2}, \dots, X_n^{m'_n}]$ and $\overline{r}\big([u_1, u_2, \dots, u_n]\big) = [u_1^{l_1}, u_2^{l_2}, \dots, u_n^{l_n}]$, so

$$\begin{aligned}\overline{r} \circ \overline{q}\big([x_1^{1/d}, x_2^{1/d}, \dots, x_n^{1/d}]\big) &= \overline{r}\big([x_1^{m'_1/d}, x_2^{m'_2/d}, \dots, x_n^{m'_n/d}]\big)\\ &= \big[x_1^{m'_1 l_1/d}, x_2^{m'_2 l_2/d}, \dots, x_n^{m'_n l_n/d}\big].\end{aligned}$$

Hence $\Psi := \overline{r} \circ \overline{q} \circ \overline{p}^{\,-1}$ is explicitly given by

$$\Psi\big([x_1, x_2, \dots, x_n, t]\big) = \big[x_1^{m'_1 l_1/d},\ x_2^{m'_2 l_2/d}, \dots, x_n^{m'_n l_n/d}\big].$$

□

9.3 Correspondence between functions

We use the notation in §9.2. Besides, let $\Phi : A_{d-1} \to \mathbb{C}$ be a holomorphic map given by $\Phi(x_1, x_2, \dots, x_n, t) = t^{m'_1 m'_2 \cdots m'_n c}$. Then Φ is Γ-invariant, so induces a holomorphic map $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$. As we explained in § Introduction, the topological monodromy of $\overline{\Phi}$ is a $-\left(\frac{a_1}{m_1}, \frac{a_2}{m_2}, \dots, \frac{a_n}{m_n}, \kappa\right)$*-fractional Dehn twist*: If $n = 2$, then the topological monodromy of $\overline{\Phi}$ is the $-\left(\frac{a_1}{m_1}, \frac{a_2}{m_2}, \kappa\right)$-fractional Dehn twist.

Under the isomorphism $\Psi : A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^n/G$ in (41), $\overline{\Phi} : A_{d-1}/\Gamma \to \mathbb{C}$ corresponds to a holomorphic map on $\mathbb{C}^n/G$. We describe this map. To that end, we need the following:

Lemma 32. *For an element $g \in G$ given by*

$$(v_1, \dots, v_n) \ \longmapsto \ \big(e^{2\pi \mathrm{i} l_1 (j a_1 + m_1 p_1)/cd} v_1,\ \dots,\ e^{2\pi \mathrm{i} l_n (j a_n + m_n p_n)/cd} v_n\big),$$

write $\eta_i = e^{2\pi \mathrm{i} l_i (j a_i + m_i p_i)/cd}$ $(i = 1, 2, \dots, n)$. Next for $i = 1, 2, \dots, n$, set $k_i := \mathrm{lcm}(m'_1, \dots, \check{m}'_i, \dots, m'_n)c$, where $\check{m}'_i$ means the omission of m'_i. Then $\eta_1^{k_1} \eta_2^{k_2} \cdots \eta_n^{k_n} = 1$.

Proof. Since $l_i = \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\mathrm{lcm}(m'_1, \dots, \check{m}'_i, \dots, m'_n)}$, we have $k_i l_i = m'_1 \cdots \check{m}'_i \cdots m'_n c$, so

$$\begin{aligned}\eta_1^{k_1} \eta_2^{k_2} \cdots \eta_n^{k_n} &= e^{2\pi \mathrm{i} k_1 l_1 (j a_1 + m_1 p_1)/cd} e^{2\pi \mathrm{i} k_2 l_2 (j a_2 + m_2 p_2)/cd} \cdots e^{2\pi \mathrm{i} k_n l_n (j a_n + m_n p_n)/cd}\\ &= e^{2\pi \mathrm{i} m'_1 m'_2 \cdots m'_n c \sum_{i=1}^n (j a_i/m_i + p_i)/d}.\end{aligned}$$

Here $\sum_{i=1}^n p_i/d = j\kappa/d$ (because $(p_1, p_2, \dots p_n) \in \Lambda^{(j)}$), so

$$\begin{aligned}e^{2\pi \mathrm{i} m'_1 m'_2 \cdots m'_n c \sum_{i=1}^n (j a_i/m_i + p_i)/d} &= e^{2\pi \mathrm{i} j m'_1 m'_2 \cdots m'_n c \left(\sum_{i=1}^n a_i/m_i + \kappa\right)/d}\\ &= e^{2\pi \mathrm{i} j} \qquad \text{by (21).}\end{aligned}$$

Hence $\eta_1^{k_1} \eta_2^{k_2} \cdots \eta_n^{k_n} = e^{2\pi \mathrm{i} j} = 1$. □

We next show the following (this generalizes Lemma 6):

Theorem 5. *Let $\phi : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic map given by $\phi(v_1, v_2, \dots v_n) = v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n}$, where $k_i := \mathrm{lcm}(m'_1, \dots, \check{m}'_i, \dots, m'_n)c$. Then:*

(1) *ϕ is G-invariant. In particular, this induces a holomorphic map $\overline{\phi} : \mathbb{C}^n/G \to \mathbb{C}$.*

(2) *Under the isomorphism $\Psi : A_{d-1}/\Gamma \xrightarrow{\cong} \mathbb{C}^n/G$ in (41), $\overline{\Phi}$ corresponds to $\overline{\phi}$, that is, $\overline{\Phi} = \overline{\phi} \circ \Psi$.*

Proof. (1): For $(v_1, v_2, \dots, v_n) \in \mathbb{C}^n$ and an element $g \in G$ given by $g : (v_1, v_2, \dots, v_n) \mapsto (\eta_1 v_1, \eta_2 v_2, \dots, \eta_n v_n)$,

$$\begin{aligned}\phi \circ g(v_1, v_2, \dots, v_n) &= \phi(\eta_1 v_1,\ \eta_2 v_2, \dots, \eta_n v_n) = (\eta_1^{k_1} \eta_2^{k_2} \cdots \eta_n^{k_n}) v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n} \\ &= \eta_1^{k_1} \eta_2^{k_2} \cdots \eta_n^{k_n} \phi(v_1, v_2, \dots, v_n) \\ &= \phi(v_1, v_2, \dots, v_n) \qquad \text{by Lemma 32.}\end{aligned}$$

Thus $\phi \circ g = \phi$, confirming the assertion.

(2): Note first that

$$\begin{aligned}\overline{\phi} \circ \Psi\big([x_1, x_2, \dots, x_n, t]\big) &= \overline{\phi}\big([x_1^{m'_1 l_1/d},\ x_2^{m'_2 l_2/d},\ \dots, x_n^{m'_n l_n/d}]\big) \quad \text{(Lemma 31)} \\ &= x_1^{m'_1 l_1 k_1/d} x_2^{m'_2 l_2 k_2/d} \cdots x_n^{m'_n l_n k_n/d}.\end{aligned}$$

Here since $k_i l_i = m'_1 \cdots \check{m}'_i \cdots m'_n c$, we have $m'_i l_i k_i = m'_1 m'_2 \cdots m'_n c$. Thus the last expression is rewritten as

$$\begin{aligned}x_1^{m'_1 l_1 k_1/d} x_2^{m'_2 l_2 k_2/d} \cdots x_n^{m'_n l_n k_n/d} &= (x_1 x_2 \cdots x_n)^{m'_1 m'_2 \cdots m'_n/d} \\ &= t^{m'_1 m'_2 \cdots m'_n c} \qquad \text{because } x_1 x_2 \cdots x_n = t^d.\end{aligned}$$

Hence $\overline{\phi} \circ \Psi\big([x_1, x_2, \dots, x_n, t]\big) = \overline{\Phi}\big([x_1, x_2, \dots, x_n, t]\big)$. $\square$

9.4 Equi-smallness theorem

Let Γ be the cyclic group generated by the automorphism $\gamma : A_{d-1} \to A_{d-1}$ given by

$$\gamma :\ (x_1, \dots, x_n, t)\ \longmapsto\ (e^{2\pi \mathrm{i} a_1/m_1} x_1,\ \dots, e^{2\pi \mathrm{i} a_n/m_n} x_n,\ e^{2\pi \mathrm{i}/m'_1 m'_2 \cdots m'_n c} t),$$

where $d := \sum_{k=1}^{n} a_k m'_1 \cdots \check{m}'_k \cdots m'_n + m'_1 m'_2 \cdots m'_n c\kappa$. Here κ is an integer satisfying $(*)$ $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa > 0$. Then $\kappa \geq -n+1$ (Lemma 7).

Let $\widetilde{\Gamma}$ be the lift of Γ and H is the descent of $\widetilde{\Gamma}$. The pseudo-reflection subgroup P of H is generated by the automorphisms $h_i : \mathbb{C}^n \to \mathbb{C}^n$ ($i = 1, 2, \ldots, n$) given by $h_i : (u_1, \ldots, u_i, \ldots, u_n) \mapsto (u_1, \ldots, e^{2\pi \mathrm{i}/l_i} u_i, \ldots, u_n)$ (Proposition 6). Here $l_i = \dfrac{m'_1 \cdots \check{m}'_i \cdots m'_n}{\mathrm{lcm}(m'_1, \ldots, \check{m}'_i, \ldots, m'_n)}$ does not depend on κ. Thus:

Lemma 33. *The pseudo-reflection subgroup P of H does not depend on κ.*

In what follows, regarding κ as a 'parameter', write $\widetilde{\Gamma}$, H, P as $\widetilde{\Gamma}_\kappa$, H_κ, P_κ. These are subgroups of $GL(n, \mathbb{C})$. From Lemma 33,

$$P_{\kappa_0} = P_{\kappa_0+1} = \cdots = P_\kappa = \cdots \,, \tag{42}$$

where κ_0 denotes the least integer in the set S of integers κ satisfying $(*)$. If H_{κ_0} is small, then $P_{\kappa_0} = \{1\}$ and by (42), $P_{\kappa_0} = P_{\kappa_0+1} = \cdots = P_\kappa = \cdots = \{1\}$. Thus H_κ is small for any $\kappa \in S$. This confirms the following:

Theorem 6 (Equi-smallness). *Let S be the set of integers κ satisfying $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \cdots + \frac{a_n}{m_n} + \kappa > 0$, and let κ_0 denote the least integer in S. Then H_{κ_0} is small $\Longleftrightarrow$ H_κ is small for any $\kappa \in S$. (In other words, H_{κ_0} is not small $\Longleftrightarrow$ H_κ is not small for any $\kappa \in S$.)*

Example 3. (i): When $n = 3$, $a_1 = a_2 = a_3 = 1$, $m_1 = 2$, $m_2 = 4$ and $m_3 = 6$, $c = \gcd(m_1, m_2, m_3) = 2$, $m'_1 = 1$, $m'_2 = 2$ and $m'_3 = 3$. Then $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{11}{12}$, and thus $\frac{11}{12} + \kappa > 0$. Hence $\kappa_0 = 0$. Here by Example 2, H_{κ_0} is small. Thus by Theorem 6, H_κ is small for any integer κ such that $\kappa \geq 0$.

(ii): When $n = 3$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $m_1 = 2$, $m_2 = 3$ and $m_3 = 4$, $c = \gcd(m_1, m_2, m_3) = 1$, $m'_1 = 2$, $m'_2 = 3$ and $m'_3 = 4$. Then $\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{23}{12}$, and thus $\frac{23}{12} + \kappa > 0$. Hence $\kappa_0 = -1$. Here since $\gcd(m'_1, m'_3) = 2$, Theorem 3 ensures that H_{κ_0} is not small. Thus by Theorem 6, H_κ is not small for any integer κ such that $\kappa \geq -1$.

10 Generators of $\widetilde{\Gamma}$, H and G

Let $\widetilde{\Gamma}$ be the lift of Γ with respect to the covering p. Let H be the descent of $\widetilde{\Gamma}$ with respect to the covering q, and G be the descent of H with respect to the covering r. Then G is a small finite abelian group such that $A_{d-1}/\Gamma \cong \mathbb{C}^n/G$. We explicitly give generators of $\widetilde{\Gamma}$, H, G.

10.1 Generators of $\widetilde{\Gamma}$

Recall that (see Lemma 13)

$$\widetilde{\Gamma}=\left\{\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}\ :\ (p_1,p_2,\dots,p_n)\in\Lambda^{(j)},\ j=1,2,\dots,m_1'm_2'\cdots m_n'c\right\},$$

where $\Lambda^{(j)}=\left\{(p_1,p_2,\dots,p_n)\in\mathbb{Z}^n\ :\ 0\le p_i<d,\ \sum_{i=1}^{n}\frac{p_i}{d}\equiv\frac{j\kappa}{d}\bmod\mathbb{Z}\right\}$ and $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}:\widetilde{A}_{d-1}\to\widetilde{A}_{d-1}$ is an automorphism given by

$$(X_1,X_2,\dots,X_n)\ \longmapsto$$
$$\left(e^{2\pi\mathrm{i}(ja_1+m_1p_1)/m_1d}X_1,\ e^{2\pi\mathrm{i}(ja_2+m_2p_2)/m_2d}X_2,\dots,e^{2\pi\mathrm{i}(ja_n+m_np_n)/m_nd}X_n\right).$$

Recall that Γ is generated by the automorphism $\gamma:A_{d-1}\to A_{d-1}$ given by

$$\gamma:\ (x_1,\dots,x_n,t)\in A_{d-1}\ \longmapsto\ (e^{2\pi\mathrm{i}a_1/m_1}x_1,\dots,e^{2\pi\mathrm{i}a_n/m_n}x_n,e^{2\pi\mathrm{i}/(m_1'm_2'\cdots m_n'c)}t).$$

The automorphism $\delta:\widetilde{A}_{d-1}\to\widetilde{A}_{d-1}$ given by

$$(X_1,X_2,\dots,X_n)\ \longmapsto\ (e^{2\pi\mathrm{i}a_1/m_1d}X_1,e^{2\pi\mathrm{i}a_2/m_2d}X_2,\dots,e^{2\pi\mathrm{i}(a_n+m_n\kappa)/m_nd}X_n)$$

is a lift of $\gamma\in\Gamma$ with respect to the covering $p:\widetilde{A}_{d-1}\to A_{d-1}$. Hence $\delta\in\widetilde{\Gamma}$. The automorphism $\eta_i:\widetilde{A}_{d-1}\to\widetilde{A}_{d-1}$ $(i=1,2,\dots,n-1)$ given by

$$(X_1,X_2,\dots,X_n)\ \longmapsto\ (X_1,\dots,X_{i-1},e^{2\pi\mathrm{i}/d}X_i,X_{i+1},\dots,e^{-2\pi\mathrm{i}/d}X_n)$$

is a lift of the identity $1\in\Gamma$ with respect to the covering p. Hence $\eta_i\in\widetilde{\Gamma}$ $(i=1,2,\dots,n-1)$.

Lemma 34. *Any element of $\widetilde{\Gamma}$ is expressed by δ, η_1, $\eta_2,\dots,\eta_{n-1}\in\widetilde{\Gamma}$. In fact, $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}\in\widetilde{\Gamma}$ is expressed as $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}=\delta^j\eta_1^{p_1}\eta_2^{p_2}\cdots\eta_{n-1}^{p_{n-1}}$.*

Proof. It suffices to show that $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}\delta^{-j}\eta_1^{-p_1}\eta_2^{-p_2}\cdots\eta_{n-1}^{-p_{n-1}}$ is the identity. For brevity, express $\widetilde{\gamma}^{(j)}_{p_1,p_2,\dots,p_n}(\vec{x})=A\vec{x}$, $\delta(\vec{x})=B\vec{x}$ and $\eta_i(\vec{x})=C_i\vec{x}$, where

$$A=\begin{pmatrix} e^{2\pi\mathrm{i}(ja_1+m_1p_1)/m_1d} & & & O\\ & e^{2\pi\mathrm{i}(ja_2+m_2p_2)/m_2d} & & \\ & & \ddots & \\ O & & & e^{2\pi\mathrm{i}(ja_n+m_np_n)/m_nd}\end{pmatrix},$$

$$B=\begin{pmatrix} e^{2\pi\mathrm{i}a_1/m_1d} & & & & O\\ & e^{2\pi\mathrm{i}a_2/m_2d} & & & \\ & & \ddots & & \\ & & & e^{2\pi\mathrm{i}a_{n-1}/m_{n-1}d} & \\ O & & & & e^{2\pi\mathrm{i}(a_n+m_n\kappa)/m_nd}\end{pmatrix},$$

$$C_i = \begin{array}{cc} \begin{array}{cccccccc} & & & \overset{i}{\vee} & & & & \end{array} & \\ \begin{pmatrix} 1 & & & \vdots & & & & O \\ & \ddots & & \vdots & & & & \\ & & 1 & \vdots & & & & \\ & & & e^{2\pi\mathrm{i}/d} & \cdots & \cdots & \cdots & \cdots \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ O & & & & & & & e^{-2\pi\mathrm{i}/d} \end{pmatrix} & \leftarrow i \end{array} .$$

Accordingly $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n}\delta^{-j}\eta_1^{-p_1}\eta_2^{-p_2}\cdots\eta_{n-1}^{-p_{n-1}}(\vec{x}) = AB^{-j}C_1^{-p_1}C_2^{-p_2}\cdots\, C_{n-1}^{-p_{n-1}}\vec{x}$. It suffices to show that the matrix $D := AB^{-j}C_1^{-p_1}C_2^{-p_2}\cdots\, C_{n-1}^{-p_{n-1}}$ is the identity matrix. Since A, B, C_i are diagonal, D is also diagonal, so it suffices to show that any of its diagonal entries is 1. This is confirmed as follows:

- For $i = 1, 2, \ldots, n-1$, the (i,i) entry of D is
$$e^{2\pi\mathrm{i}(ja_i+m_ip_i)/m_id}\big(e^{2\pi\mathrm{i}a_i/m_id}\big)^{-j}\big(e^{2\pi\mathrm{i}/d}\big)^{-p_i} = 1.$$
- The (n,n) entry of D is
$e^{2\pi\mathrm{i}(ja_n+m_np_n)/m_nd}\big(e^{2\pi\mathrm{i}(a_n+m_n\kappa)/m_nd}\big)^{-j}\big(e^{2\pi\mathrm{i}/d}\big)^{p_1+\cdots+p_{n-1}} = e^{2\pi\mathrm{i}(p_1+\cdots+p_n-j\kappa)/d}$.
Here since $(p_1, p_2, \ldots, p_n) \in \Lambda^{(j)}$, we have $\sum_{i=1}^{n}\dfrac{p_i}{d} \equiv \dfrac{j\kappa}{d} \bmod \mathbb{Z}$, and thus $e^{2\pi\mathrm{i}(p_1+\cdots+p_n-j\kappa)/d} = 1$. $\square$

Lemma 34 implies that:

Corollary 11. *$\widetilde{\Gamma}$ is generated by $\delta, \eta_1, \eta_2, \ldots, \eta_{n-1}$, or as a subgroup of $GL(n,\mathbb{C})$, generated by the matrices B, C_1, $C_2, \ldots, C_{n-1}$ appearing in the proof of Lemma 34.*

10.2 Relations among generators of $\widetilde{\Gamma}$

Recall that $\widetilde{\Gamma}$ is a finite abelian group of order $m_1'm_2'\cdots m_n'cd^{n-1}$ (Proposition 3 (1)) and is generated by δ, η_1, η_2, …, η_{n-1} (Corollary 11). These generators are generally *not* independent. In fact, the following holds (the proof is the same as that of Lemma 34):

Lemma 35. $\delta^{m_1'm_2'\cdots m_n'c} = \eta_1^{a_1m_2'm_3'\cdots m_n'}\eta_2^{a_2m_1'm_3'\cdots m_n'}\cdots\eta_{n-1}^{a_{n-1}m_1'\cdots m_{n-2}'m_n'}$.

Caution: As we will see in (2) of Corollary 12, if the order of δ is $m_1'm_2'\cdots m_n'c$, then this relation is vacuous. To show this, we need some preparation:

Lemma 36. (1) *Express $\delta(\vec{x}) = B\vec{x}$, where B is the matrix appearing in the proof of Lemma 34. Then* $\det B = e^{2\pi\mathrm{i}/m_1'm_2'\cdots m_n'c}$.

(2) *If $\delta^k = 1$, then k is a multiple of $m_1'm_2'm_3'c$. In particular, the order of δ is a multiple of $m_1'm_2'\cdots m_n'c$.*

(3) $\mathrm{lcm}(m_1', m_2', \dots, m_n')cd$ *is a multiple of* $m_1'm_2'\cdots m_n'c$, *and* $\delta^{\mathrm{lcm}(m_1', m_2', \dots, m_n')cd} = 1$.

(4) *Write* $\mathrm{lcm}(m_1', m_2', \dots, m_n')cd = Nm_1'm_2'\cdots m_n'c$ *where N is a positive integer. Then the order of δ is $lm_1'm_2'\cdots m_n'c$ for some positive integer l ($1 \le l \le N$).*

Proof. We show the assertions only for $n = 3$ (the proof is the same for any n).

(1): Since $B = \begin{pmatrix} e^{2\pi \mathrm{i} a_1/m_1 d} & 0 & 0 \\ 0 & e^{2\pi \mathrm{i} a_2/m_2 d} & 0 \\ 0 & 0 & e^{2\pi \mathrm{i}(a_3+m_3\kappa)/m_3 d} \end{pmatrix}$, we have $\det B = e^{2\pi \mathrm{i} a_1/m_1 d}\, e^{2\pi \mathrm{i} a_2/m_2 d}\, e^{2\pi \mathrm{i}(a_3+m_3\kappa)/m_3 d} = e^{2\pi \mathrm{i}(a_1/m_1 + a_2/m_2 d + a_3/m_3 + \kappa)/d}$. Here $(a_1/m_1 + a_2/m_2 + a_3/m_3 + \kappa)/d = 1/m_1'm_2'm_3'c$ (because $d := m_1'm_2'm_3'c(a_1/m_1 + a_2/m_2 + a_3/m_3 + \kappa)$), confirming the assertion.

(2): If $\delta^k = 1$, then $B^k = I$ (the identity matrix), so $\det(B^k) = 1$. Then $e^{2\pi \mathrm{i} k/m_1'm_2'm_3'c} = 1$ by (1). Thus k is a multiple of $m_1'm_2'm_3'c$.

(3): We first show that $\mathrm{lcm}(m_1', m_2', m_3')cd$ is a multiple of $m_1'm_2'm_3'c$, for which it is sufficient to demonstrate that $\dfrac{\mathrm{lcm}(m_1', m_2', m_3')cd}{m_1'm_2'm_3'c}$ is an integer. Using $d := m_1'm_2'm_3'c\left(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} + \kappa\right)$, we rewrite:

$$\begin{aligned}\frac{\mathrm{lcm}(m_1', m_2', m_3')cd}{m_1'm_2'm_3'c} &= \mathrm{lcm}(m_1', m_2', m_3')c\left(\frac{a_1}{m_1} + \frac{a_2}{m_2} + \frac{a_3}{m_3} + \kappa\right) \\ &= \frac{\mathrm{lcm}(m_1', m_2', m_3')c}{m_1}a_1 + \frac{\mathrm{lcm}(m_1', m_2', m_3')c}{m_2}a_2 \\ &\quad + \frac{\mathrm{lcm}(m_1', m_2', m_3')c}{m_3}a_3 + \mathrm{lcm}(m_1', m_2', m_3')c\kappa.\end{aligned}$$

Since $m_i = m_i'c$, the last expression is equal to

$$\begin{aligned}&\frac{\mathrm{lcm}(m_1', m_2', m_3')}{m_1'}a_1 + \frac{\mathrm{lcm}(m_1', m_2', m_3')}{m_2'}a_2 \\ &\quad + \frac{\mathrm{lcm}(m_1', m_2', m_3')}{m_3'}a_3 + \mathrm{lcm}(m_1', m_2', m_3')c\kappa.\end{aligned}$$

This is an integer, because

$$\frac{\mathrm{lcm}(m_1', m_2', m_3')}{m_1'},\ \frac{\mathrm{lcm}(m_1', m_2', m_3')}{m_2'},\ \frac{\mathrm{lcm}(m_1', m_2', m_3')}{m_3'} \text{ are integers.}$$

Thus $\dfrac{\mathrm{lcm}(m_1', m_2', m_3')cd}{m_1'm_2'm_3'c}$ is an integer, confirming the assertion.

We next show that $\delta^{\mathrm{lcm}(m_1', m_2', m_3')cd} = 1$. For an integer k, the automorphism $\delta^k : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ is given by

$$(X_1, X_2, X_3) \longmapsto (e^{2\pi \mathrm{i} a_1 k/m_1 d}X_1, e^{2\pi \mathrm{i} a_2 k/m_2 d}X_2, e^{2\pi \mathrm{i}(a_3+m_3\kappa)k/m_3 d}X_3).$$

Here if $k = \mathrm{lcm}(m_1', m_2', m_3')cd$, then

$$\begin{aligned} &k/m_1 d = \mathrm{lcm}(m_1', m_2', m_3')/m_1', \qquad k/m_2 d = \mathrm{lcm}(m_1', m_2', m_3')/m_2', \\ &k/m_3 d = \mathrm{lcm}(m_1', m_2', m_3')/m_3', \end{aligned}$$

hence $k/m_1 d$, $k/m_2 d$, $k/m_3 d$ are integers, consequently $\delta^{\mathrm{lcm}(m_1', m_2', m_3')cd} : (X_1, X_2, X_3) \longmapsto (X_1, X_2, X_3)$, so $\delta^{\mathrm{lcm}(m_1', m_2', m_3')cd} = 1$.

(4): This follows from (2) and (3). $\square$

Since $\eta_i : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ $(i = 1, 2, \ldots, n-1)$ is given by

$$(X_1, X_2, \ldots, X_n) \longmapsto (X_1, \ldots, X_{i-1}, e^{2\pi \mathrm{i}/d}X_i, X_{i+1}, \ldots, e^{-2\pi \mathrm{i}/d}X_n),$$

the following is clear:

Lemma 37. *The order of η_i $(i = 1, 2, \ldots, n-1)$ is d.*

We next show that:

Lemma 38. (1) *There is no nontrivial relation among $\eta_1, \eta_2, \ldots, \eta_{n-1}$. Namely if $\eta_1^{k_1}\eta_2^{k_2}\cdots\eta_{n-1}^{k_{n-1}} = 1$, then all of $\eta_1^{k_1}$, $\eta_2^{k_2}$, $\ldots$, $\eta_{n-1}^{k_{n-1}}$ are 1.*

(2) *Let k be an integer such that $\delta^k \neq 1$. If δ^k is expressed by $\eta_1, \eta_2, \ldots, \eta_{n-1}$, that is, $\delta^k = \eta_1^{l_1}\eta_2^{l_2}\cdots\eta_{n-1}^{l_{n-1}}$ for some integers $l_1, l_2, \ldots, l_{n-1}$, then k is a multiple of $m_1'm_2'\cdots m_n'c$.*

(3) *If an integer k is not a multiple of $m_1'm_2'\cdots m_n'c$, then $\delta^k \neq 1$. Moreover δ^k cannot be expressed by $\eta_1, \eta_2, \ldots, \eta_{n-1}$.*

(4) *Let $\langle\delta\rangle$ and $\langle\eta_1, \eta_2, \ldots, \eta_{n-1}\rangle$ denote the subgroups of $GL(n, \mathbb{C})$ generated by δ and $\eta_1, \eta_2, \ldots, \eta_{n-1}$ respectively. If the order of δ is $m_1'm_2'\cdots m_n'c$, then $\langle\delta\rangle \cap \langle\eta_1, \eta_2, \ldots, \eta_{n-1}\rangle = \{1\}$.*

Proof. We show the assertions only for $n = 3$ (the proof is the same for any n).

(1): The automorphism $\eta_1^{k_1}\eta_2^{k_2} : \widetilde{A}_{d-1} \to \widetilde{A}_{d-1}$ is given by $(X_1, X_2, X_3) \mapsto (e^{2\pi \mathrm{i} k_1/d}X_1,\ e^{2\pi \mathrm{i} k_2/d}X_2,\ e^{-2\pi \mathrm{i}(k_1+k_2)/d}X_n)$. If $\eta_1^{k_1}\eta_2^{k_2} = 1$, then $e^{2\pi \mathrm{i} k_1/d} = 1$, $e^{2\pi \mathrm{i} k_2/d} = 1$, $e^{-2\pi \mathrm{i}(k_1+k_2)/d} = 1$. Accordingly $\eta_1^{k_1} = 1$ and $\eta_2^{k_2} = 1$ hold.

(2): Suppose that $\delta^k = \eta_1^{l_1}\eta_2^{l_2}$. Here since $\delta \in \widetilde{\Gamma}$ is a lift of $\gamma \in \Gamma$, $\delta^k \in \widetilde{\Gamma}$ is a lift of $\gamma^k \in \Gamma$ and since $\eta_1, \eta_2 \in \widetilde{\Gamma}$ are lifts of $1 \in \Gamma$, $\eta_1^{l_1}\eta_2^{l_2} \in \widetilde{\Gamma}$ is a lift of

$1 \in \Gamma$. The relation $\delta^k = \eta_1^{l_1}\eta_2^{l_2}$ thus implies that δ^k is a lift of both γ^k and 1, so $\gamma^k = 1$. Since the order of γ is $m_1'm_2'm_3'c$, this implies that k is a multiple of $m_1'm_2'm_3'c$.

(3): Since the order of δ is a multiple of $m_1'm_2'm_3'c$ (Lemma 36 (2)), if an integer k is not a multiple of $m_1'm_2'm_3'c$, then $\delta^k \neq 1$. The rest is a restatement of (2).

(4): This can be shown by contradiction. If $\langle\delta\rangle \cap \langle\eta_1, \eta_2\rangle \neq \{1\}$, then there exist elements $\delta^k \neq 1$ of $\langle\delta\rangle$ and $\eta_1^{l_1}\eta_2^{l_2} \neq 1$ of $\langle\eta_1, \eta_2\rangle$ such that $\delta^k = \eta_1^{l_1}\eta_2^{l_2}$. Then (2) implies that k is a multiple of $m_1'm_2'm_3'c$. But $\delta^{m_1'm_2'm_3'c} = 1$ by assumption, accordingly $\delta^k = 1$. This contradicts that $\delta^k \neq 1$. $\square$

By (4) of Lemma 36, the order of δ is $lm_1'm_2'\cdots m_n'c$ for some positive integer l $(1 \leq l \leq N)$, where $N = \dfrac{\mathrm{lcm}(m_1', m_2', \ldots, m_n')cd}{m_1'm_2'\cdots m_n'c}$. The following holds:

Corollary 12. *Let $\widetilde{\Gamma}$ be the lift of the cyclic group Γ with respect to the covering $p : \widetilde{A}_{d-1} \to A_{d-1}$. Recall that $\delta, \eta_i \in \widetilde{\Gamma}$ $(i = 1, 2, \ldots, n-1)$ are automorphisms of $\widetilde{A}_{d-1}$ given by*

$$\delta : (X_1, X_2, \ldots, X_n) \longmapsto (e^{2\pi \mathrm{i} a_1/m_1 d}X_1, e^{2\pi \mathrm{i} a_2/m_2 d}X_2, \ldots, e^{2\pi \mathrm{i}(a_n + m_n\kappa)/m_n d}X_n),$$
$$\eta_i : (X_1, X_2, \ldots, X_n) \longmapsto (X_1, \ldots, X_{i-1}, e^{2\pi \mathrm{i}/d}X_i, X_{i+1}, \ldots, e^{-2\pi \mathrm{i}/d}X_n).$$

Then the following hold:

(1) $\delta^{m_1'm_2'\cdots m_n'c} = \eta_1^{a_1m_2'm_3'\cdots m_n'}\eta_2^{a_2m_1'm_3'\cdots m_n'}\cdots\eta_{n-1}^{a_{n-1}m_1'\cdots m_{n-2}'m_n'}$.

(2) *If the order of δ is $m_1'm_2'\cdots m_n'c$, then the relation in* (1) *is vacuous, that is, $\delta^{m_1'm_2'\cdots m_n'c}=1$, $\eta_1^{a_1m_2'm_3'\cdots m_n'}=$, $\ldots$, $\eta_{n-1}^{a_{n-1}m_1'\cdots m_{n-2}'m_n'}=1$.*

(3) *If the order of δ is $m_1'm_2'\cdots m_n'c$, then $\widetilde{\Gamma}$ is isomorphic to the product of cyclic groups $\langle\delta\rangle \times \langle\eta_1\rangle \times \langle\eta_2\rangle \times \cdots \times \langle\eta_{n-1}\rangle$, where $\langle\delta\rangle$ and $\langle\eta_i\rangle$ denote the cyclic groups generated by δ and η_i respectively.*

Proof. (1): This is already shown in Lemma 35.

(2): We show the assertion only for $n = 3$. If the order of δ is $m_1'm_2'm_3'c$, then $\delta^{m_1'm_2'm_3'c} = 1$. Here by (1), $\eta_1^{a_1m_2'm_3'}\eta_2^{a_2m_1'm_3'} = 1$. Thus by Lemma 38 (1), $\eta_1^{a_1m_2'm_3'} = 1$ and $\eta_2^{a_2m_1'm_3'} = 1$ hold, confirming the assertion.

(3): We also show the assertion only for $n = 3$. By Lemma 38 (4), if the order of δ is $m_1'm_2'm_3'c$, then $\langle\delta\rangle \cap \langle\eta_1, \eta_2\rangle = \{1\}$. Since $\widetilde{\Gamma}$ is generated by δ, η_1, η_2 (Corollary 11), we obtain $\widetilde{\Gamma} \cong \langle\delta\rangle \times \langle\eta_1, \eta_2\rangle$. Here $\langle\eta_1, \eta_2\rangle = \langle\eta_1\rangle \times \langle\eta_2\rangle$ because there is no nontrivial relation between η_1 and η_2 (Lemma 38 (1)). Hence $\widetilde{\Gamma} \cong \langle\delta\rangle \times \langle\eta_1\rangle \times \langle\eta_2\rangle$, confirming the assertion. $\square$

Remark 10. *If the order of δ is greater than $m'_1 m'_2 \cdots m'_n c$, then $\widetilde{\Gamma}$ is not isomorphic to $\langle\delta\rangle \times \langle\eta_1\rangle \times \langle\eta_2\rangle \times \cdots \times \langle\eta_{n-1}\rangle$, because there is a nontrivial relation among δ, η_1, $\eta_2, \ldots, \eta_{n-1}$ (Corollary 12 (1)).*

10.3 Generators of H and relations among them

Recall that (see Lemma 15)

$$H = \left\{ h^{(j)}_{p_1,p_2,\ldots,p_n} : (p_1, p_2, \ldots, p_n) \in \Lambda^{(j)},\ j = 1, 2, \ldots, m'_1 m'_2 \cdots m'_n c \right\},$$

where $h^{(j)}_{p_1,p_2,\ldots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ is an automorphism given by

$$(u_1, u_2, \ldots, u_n) \longmapsto \left(e^{2\pi \mathrm{i}(ja_1+m_1p_1)/cd} u_1,\ e^{2\pi \mathrm{i}(ja_2+m_2p_2)/cd} u_2, \ldots, e^{2\pi \mathrm{i}(ja_n+m_np_n)/cd} u_n \right).$$

Recall that $\widetilde{\Gamma}$ is generated by δ, η_1, $\eta_2, \ldots, \eta_{n-1}$ (Corollary 11), where

$$\delta : (X_1, X_2, \ldots, X_n) \longmapsto (e^{2\pi \mathrm{i} a_1/m_1 d} X_1, e^{2\pi \mathrm{i} a_2/m_2 d} X_2, \ldots, e^{2\pi \mathrm{i}(a_n+m_n\kappa)/m_n d} X_n),$$

$$\eta_i : (X_1, X_2, \ldots, X_n) \longmapsto (X_1, \ldots, X_{i-1}, e^{2\pi \mathrm{i}/d} X_i, X_{i+1}, \ldots, e^{-2\pi \mathrm{i}/d} X_n).$$

Let α, β_i $(i = 1, 2, \ldots, n-1)$ be automorphisms of $\mathbb{C}^n$ given by

$$\alpha : (u_1, u_2, \ldots, u_n) \longmapsto (e^{2\pi \mathrm{i} a_1/cd} u_1, e^{2\pi \mathrm{i} a_2/cd} u_2, \ldots, e^{2\pi \mathrm{i}(a_n+m_n\kappa)/cd} u_n),$$

$$\beta_i : (u_1, u_2, \ldots, u_n) \longmapsto (u_1, \ldots, u_{i-1}, e^{2\pi \mathrm{i} m'_i/d} u_i, u_{i+1}, \ldots, e^{-2\pi \mathrm{i} m'_n/d} u_n).$$

They are respectively the descents of δ, $\eta_i \in \widetilde{\Gamma}$ (with respect to the covering $q : \widetilde{A}_{d-1} \to \mathbb{C}^n$), hence α, $\beta_i \in H$.

Lemma 39. *Any element of H is expressed by α, β_1, $\beta_2, \ldots, \beta_{n-1}$. In fact, $h^{(j)}_{p_1,p_2,\ldots,p_n} \in H$ is expressed as $h^{(j)}_{p_1,p_2,\ldots,p_n} = \alpha^j \beta_1^{p_1} \beta_2^{p_2} \cdots \beta_{n-1}^{p_{n-1}}$.*

Proof. Since α, $\beta_i \in H$ are the descents of δ, $\eta_i \in \widetilde{\Gamma}$ respectively, $\alpha^j \beta_1^{p_1} \beta_2^{p_2} \cdots \beta_{n-1}^{p_{n-1}} \in H$ is the descent of $\delta^j \eta_1^{p_1} \eta_2^{p_2} \cdots \eta_{n-1}^{p_{n-1}} \in \widetilde{\Gamma}$. On the other hand, $h^{(j)}_{p_1,p_2,\ldots,p_n} \in H$ is the descent of $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} \in \widetilde{\Gamma}$ (Lemma 14 (1)). The relation $\widetilde{\gamma}^{(j)}_{p_1,p_2,\ldots,p_n} = \delta^j \eta_1^{p_1} \eta_2^{p_2} \cdots \eta_{n-1}^{p_{n-1}}$ (in Lemma 34) then implies $h^{(j)}_{p_1,p_2,\ldots,p_n} = \alpha^j \beta_1^{p_1} \beta_2^{p_2} \cdots \beta_{n-1}^{p_{n-1}}$. □

Lemma 39 implies that H is generated by α, β_1, $\beta_2, \ldots, \beta_{n-1}$. Here α and β_i are expressed by the following matrices:

$$S = \begin{pmatrix} e^{2\pi \mathrm{i} a_1/cd} & & & & O \\ & e^{2\pi \mathrm{i} a_2/cd} & & & \\ & & \ddots & & \\ & & & e^{2\pi \mathrm{i} a_{n-1}/m_{n-1} d} & \\ O & & & & e^{2\pi \mathrm{i}(a_n+m_n\kappa)/cd} \end{pmatrix},$$

$$
T_i = \begin{array}{c@{}c}
\begin{array}{cccccccc} & & & \overset{i}{\vee} & & & & \end{array} & \\
\begin{pmatrix}
1 & & & \vdots & & & & O \\
& \ddots & & \vdots & & & & \\
& & 1 & \vdots & & & & \\
& & & e^{2\pi \mathrm{i} m'_i/d} & \cdots & \cdots & \cdots & \cdots \\
& & & & 1 & & & \\
& & & & & \ddots & & \\
& & & & & & 1 & \\
O & & & & & & & e^{-2\pi \mathrm{i} m'_n/d}
\end{pmatrix}
& \begin{array}{c} \\ \\ \\ < i \\ \\ \\ \\ \\ \end{array}
\end{array} .
$$

Thus:

Corollary 13. *H is generated by α, β_1, $\beta_2, \ldots, \beta_{n-1}$, or as a subgroup of $GL(n, \mathbb{C})$, generated by the matrices S, T_1, $T_2, \ldots, T_{n-1}$.*

Here $\alpha, \beta_1, \beta_2, \ldots, \beta_{n-1}$ are actually *not* independent. In fact, there are relations among them:

Lemma 40. *The generators α, β_1, $\beta_2, \ldots, \beta_{n-1}$ of H satisfy the following equations*:

(a) $\alpha^{m'_1 m'_2 \cdots m'_{n-1} c} = \beta_1^{a_1 m'_2 m'_3 \cdots m'_{n-1}} \beta_2^{a_2 m'_1 m'_3 \cdots m'_{n-1}} \cdots \beta_{n-1}^{a_{n-1} m'_1 m'_2 \cdots m'_{n-2}}$.

(b) *For $i = 1, 2, \ldots, n-1$,*

$$\alpha^{m'_1 \cdots \check{m}'_i \cdots m'_n c} = \beta_1^{a_1 m'_2 \cdots \check{m}'_i \cdots m'_n} \cdots \beta_i^{(a_i m'_1 \cdots \check{m}'_i \cdots m'_n - d)/m'_i} \cdots \beta_{n-1}^{a_{n-1} m'_1 \cdots \check{m}'_i \cdots m'_{n-2} m'_n},$$

where note that $(a_i m'_1 \cdots \check{m}'_i \cdots m'_n - d)/m'_i$ is an integer. (Observe that the exponent of $\beta_i^{(a_i m'_1 \cdots \check{m}'_i \cdots m'_n - d)/m'_i}$ in (b) is slightly different from that of $\beta_i^{a_i m'_1 \cdots \check{m}'_i \cdots m'_{n-1}}$ in (a).)

The existence of nontrivial relations among δ, η_1, $\eta_2, \ldots, \eta_{n-1}$ implies that $H = \langle \delta, \eta_1, \eta_2, \ldots, \eta_{n-1} \rangle$ is *not* isomorphic to the product of cyclic groups $\langle \alpha \rangle \times \langle \beta_1 \rangle \times \langle \beta_2 \rangle \times \cdots \times \langle \beta_{n-1} \rangle$.

We summarize properties of H:

Proposition 8. (1) *The descent H of $\widetilde{\Gamma}$ is a finite abelian group of order cd^{n-1}* (Proposition 3 (2)).

(2) *H is, as a subgroup of $GL(n, \mathbb{C})$, generated by matrices S, T_1, $T_2, \ldots, T_{n-1}$* (Corollary 13).

10.4 Generators of G and relations among them

Recall that (see Lemma 28)

$$G = \left\{ g^{(j)}_{p_1,p_2,\ldots,p_n} \,:\, (p_1,p_2,\ldots,p_n) \in \Lambda^{(j)},\ j = 1,2,\ldots,m'_1 m'_2 \cdots m'_n c \right\},$$

where $g^{(j)}_{p_1,p_2,\ldots,p_n} : \mathbb{C}^n \to \mathbb{C}^n$ is an automorphism given by

$$\begin{aligned}&(v_1,v_2,\ldots,v_n) \longmapsto \\ &\qquad \left(e^{2\pi \mathrm{i} l_1(ja_1+m_1p_1)/cd} v_1,\ e^{2\pi \mathrm{i} l_2(ja_2+m_2p_2)/cd} v_2, \ldots, e^{2\pi \mathrm{i} l_n(ja_n+m_np_n)/cd} v_n\right).\end{aligned}$$

Recall that H is generated by α, β_1, $\beta_2, \ldots, \beta_{n-1}$ (Corollary 13), where

$$\begin{aligned}\alpha &: (u_1,u_2,\ldots,u_n) \longmapsto (e^{2\pi \mathrm{i} a_1/cd} u_1, e^{2\pi \mathrm{i} a_2/cd} u_2, \ldots, e^{2\pi \mathrm{i}(a_n+m_n\kappa)/cd} u_n),\\ \beta_i &: (u_1,u_2,\ldots,u_n) \longmapsto (u_1,\ldots,u_{i-1}, e^{2\pi \mathrm{i} m'_i/d} u_i, u_{i+1}, \ldots, e^{-2\pi \mathrm{i} m'_n/d} u_n).\end{aligned}$$

Let $f, g_i\, (i = 1,2,\ldots,n-1)$ be automorphisms of $\mathbb{C}^n$ given by

$$\begin{aligned}f &: (v_1,v_2,\ldots,v_n) \longmapsto (e^{2\pi \mathrm{i} l_1 a_1/cd} v_1, e^{2\pi \mathrm{i} l_2 a_2/cd} v_2, \ldots, e^{2\pi \mathrm{i} l_n(a_n+m_n\kappa)/cd} v_n),\\ g_i &: (v_1,v_2,\ldots,v_n) \longmapsto (v_1,\ldots,v_{i-1}, e^{2\pi \mathrm{i} l_i m'_i/d} v_i, v_{i+1}, \ldots, e^{-2\pi \mathrm{i} l_n m'_n/d} v_n).\end{aligned}$$

They are respectively the descents of α, $\beta_i \in H$ (with respect to the covering $r : \mathbb{C}^n \to \mathbb{C}^n$), hence $f, g_i \in G$.

Lemma 41. *Any element of G is expressed by f, g_1, $g_2, \ldots, g_{n-1}$. In fact, $g^{(j)}_{p_1,p_2,\ldots,p_n} \in G$ is expressed as $g^{(j)}_{p_1,p_2,\ldots,p_n} = f^j g_1^{p_1} g_2^{p_2} \cdots g_{n-1}^{p_{n-1}}$.*

Proof. Since f, $g_i \in G$ are the descents of α, $\beta_i \in H$ respectively, $f^j g_1^{p_1} g_2^{p_2} \cdots g_{n-1}^{p_{n-1}} \in G$ is the descent of $\alpha^j \beta_1^{p_1} \beta_2^{p_2} \cdots \beta_{n-1}^{p_{n-1}} \in H$. On the other hand, $g^{(j)}_{p_1,p_2,\ldots,p_n} \in G$ is the descent of $h^{(j)}_{p_1,p_2,\ldots,p_n} \in H$ (Lemma 27 (A)). The relation $h^{(j)}_{p_1,p_2,\ldots,p_n} = \alpha^j \beta_1^{p_1} \beta_2^{p_2} \cdots \beta_{n-1}^{p_{n-1}}$ (in Lemma 39) then implies $g^{(j)}_{p_1,p_2,\ldots,p_n} = f^j g_1^{p_1} g_2^{p_2} \cdots g_{n-1}^{p_{n-1}}$. □

Lemma 41 implies that G is generated by $f, g_1, g_2, \ldots, g_{n-1}$. Here f and g_i are expressed by the following matrices:

$$Q = \begin{pmatrix} e^{2\pi \mathrm{i} l_1 a_1/cd} & & & & O \\ & e^{2\pi \mathrm{i} l_2 a_2/cd} & & & \\ & & \ddots & & \\ & & & e^{2\pi \mathrm{i} l_{n-1} a_{n-1}/m_{n-1} d} & \\ O & & & & e^{2\pi \mathrm{i} l_n(a_n+m_n\kappa)/cd} \end{pmatrix},$$

$$R_i = \begin{array}{cc} \begin{array}{ccccccccc} & & & \check{i} & & & & & \end{array} & \\ \left(\begin{array}{ccccccccc} 1 & & & \vdots & & & & & O \\ & \ddots & & \vdots & & & & & \\ & & 1 & \vdots & & & & & \\ & & & e^{2\pi\mathrm{i}l_i m_i'/d} & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ O & & & & & & & & e^{-2\pi\mathrm{i}l_n m_n'/d} \end{array}\right) & \begin{array}{c} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \\ \end{array} \end{array}.$$

Thus:

Corollary 14. *G is generated by $f, g_1, g_2, \ldots, g_{n-1}$, or as a subgroup of $GL(n, \mathbb{C})$, generated by the matrices Q, R_1, $R_2, \ldots, R_{n-1}$.*

Here $f, g_1, g_2, \ldots, g_{n-1}$ are actually *not* independent. In fact, there are relations among them:

Lemma 42. *The generators f, g_1, $g_2, \ldots, g_{n-1}$ of G satisfy the following equations*:

(a) $f^{\mathrm{lcm}(m_1', m_2', \ldots, m_{n-1}')c} = g_1^{a_1 \mathrm{lcm}(m_1', m_2', \ldots, m_{n-1}')/m_1'} \cdots g_{n-1}^{a_{n-1}\mathrm{lcm}(m_1', m_2', \ldots, m_{n-1}')/m_{n-1}'}$, *where note that $a_k \mathrm{lcm}(m_1', \ldots, m_{n-1}')/m_k'$ $(k = 1, 2, \ldots, n-1)$ is an integer* (*because m_k' divides* $\mathrm{lcm}(m_1', m_2', \ldots, m_{n-1}')$).

(b) *For $i = 1, 2, \ldots, n-1$,*

$$\begin{aligned} & f^{\mathrm{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')c} \\ = & g_1^{a_1 \mathrm{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')/m_1'} \cdots g_i^{(a_i m_1' \cdots \check{m}_i' \cdots m_n' - d)/l_i m_i'} \cdots g_{n-1}^{a_{n-1}\mathrm{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')/m_{n-1}'}, \end{aligned}$$

where note that $(a_i m_1' \cdots \check{m}_i' \cdots m_n' - d)/l_i m_i'$ is an integer, and that for $k = 1, 2, \ldots, \check{i}, \ldots, n$, $a_k \mathrm{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')/m_k'$ is an integer (*because m_k' divides* $\mathrm{lcm}(m_1', \ldots, \check{m}_i', \ldots, m_n')$).

(Observe that the exponent of $g_i^{(a_i m_1' \cdots \check{m}_i' \cdots m_n' - d)/l_i m_i'}$ in (b) is slightly different from that of $g_i^{a_i \mathrm{lcm}(m_1', \ldots, m_{n-1}')/m_i'}$ in (a).)

The existence of nontrivial relations among f, g_1, $g_2, \ldots, g_{n-1}$ implies that $G = \langle f, g_1, g_2, \ldots, g_{n-1} \rangle$ is *not* isomorphic to the product of cyclic groups $\langle f \rangle \times \langle g_1 \rangle \times \langle g_2 \rangle \times \cdots \times \langle g_{n-1} \rangle$.

We summarize properties of G:

Proposition 9. (1) *The descent G of H is a finite abelian group of order* $\dfrac{cd^{n-1}}{l_1 l_2 \cdots l_n}$ (Proposition 3 (3)).

(2) *G is, as a subgroup of $GL(n,\mathbb{C})$, generated by matrices Q, R_1, R_2, $\dots$, R_{n-1}* (Corollary 14).

The case $n = 2$

Let a_1^* $(0 < a_1^* < m_1)$ be the integer such that $a_1 a_1^* \equiv 1 \bmod m_1$. If $n = 2$, then G is a cyclic group generated by $g : (u_1, u_2) \mapsto (e^{2\pi \mathrm{i}/cd} u_1,\ e^{2\pi \mathrm{i}\mathsf{q}/cd} u_2)$, where q $(0 < \mathsf{q} < cd)$ is the integer such that $\mathsf{q} \equiv \dfrac{a_1^* d - m_2'}{m_1'} \bmod cd$ (Theorem 1). Note that $\dfrac{a_1^* d - m_2'}{m_1'}$ is an integer (cf. Lemma 4 (1)). Here the automorphism g is expressed by the matrix $P := \begin{pmatrix} e^{2\pi \mathrm{i}/cd} & 0 \\ 0 & e^{2\pi \mathrm{i}\mathsf{q}/cd} \end{pmatrix}$, and as a subgroup of $GL(2,\mathbb{C})$, G is generated by P. On the other hand by Corollary 14, G is generated by two matrices $Q = \begin{pmatrix} e^{2\pi \mathrm{i}a/cd} & \\ 0 & e^{2\pi \mathrm{i}(b+n\kappa)/cd} \end{pmatrix}$ and $R_1 = \begin{pmatrix} e^{2\pi \mathrm{i}m'/d} & \\ 0 & e^{-2\pi \mathrm{i}n'/d} \end{pmatrix}$. Note that when $n = 2$, $l_1 = l_2 = 1$ and thus $G = H$, $f = \alpha$, $g_1 = \beta_1$. We denote the relation between P and Q, R_1.

For simplicity, write m_1, m_2, a_1, a_2, a_1^*, β_1, R_1 as m, n, a, b, a^*, β, R, and set $c := \gcd(m,n)$, $m' := \frac{m}{c}$, $n' := \frac{n}{c}$ and $d := an' + bm' + m'n'c\kappa$.

Proposition 10. *The matrices P, Q, $R \in GL(2,\mathbb{C})$ that express the automorphisms g, α, β are related as follows*:

(1) $P^a = Q$, $P^m = R$.

(2) *Noting that $\dfrac{1-aa^*}{m}$ is an integer* (*because $aa^* \equiv 1 \bmod m$*), *let l $(0 < l < cd)$ be the integer such that $l \equiv \dfrac{1-aa^*}{m} \bmod cd$. Then $Q^{a^*} R^l = P$.*

Proof. (1): We first show $P^a = Q$. Since $a\mathsf{q} \equiv \dfrac{a(a^*d - n')}{m'} \equiv \dfrac{d - an'}{m'} \equiv b + n\kappa \bmod cd$,

$$P^a = \begin{pmatrix} e^{2\pi \mathrm{i}a/cd} & 0 \\ 0 & e^{2\pi \mathrm{i}a\mathsf{q}/cd} \end{pmatrix} = \begin{pmatrix} e^{2\pi \mathrm{i}a/cd} & 0 \\ 0 & e^{2\pi \mathrm{i}(b+n\kappa)/cd} \end{pmatrix} = Q.$$

We next show $P^m = R$. Since $m\mathsf{q} \equiv \dfrac{m(a^*d - n')}{m'} \equiv a^*cd - cn' \equiv -cn' \bmod cd$,

$$P^m = \begin{pmatrix} e^{2\pi \mathrm{i}m/cd} & 0 \\ 0 & e^{2\pi \mathrm{i}m\mathsf{q}/cd} \end{pmatrix} = \begin{pmatrix} e^{2\pi \mathrm{i}m'/d} & 0 \\ 0 & e^{-2\pi \mathrm{i}n'/d} \end{pmatrix} = R.$$

(2): We first show $P^{aa^*+ml} = P$. Since $l \equiv \dfrac{1-aa^*}{m} \bmod cd$ and $aa^* + m\dfrac{1-aa^*}{m} = 1$, we have $aa^* + ml \equiv 1 \bmod cd$. Hence

$$e^{2\pi \mathrm{i}(aa^*+ml)/cd} = e^{2\pi \mathrm{i}/cd}, \qquad e^{2\pi \mathrm{i}(aa^*+ml)\mathsf{q}/cd} = e^{2\pi \mathrm{i}\mathsf{q}/cd}.$$

Accordingly, $P^{aa^*+ml} = P$. Then $(P^a)^{a^*}(P^m)^l = P$. Here since $P^a = Q$ and $P^m = R$ hold by (1), $Q^{a^*}R^l = P$. The assertion is thus confirmed. $\square$

Corollary 15. *The automorphisms g, α, $\beta : \mathbb{C}^2 \to \mathbb{C}^2$ are related as follows*:

(1) $g^a = \alpha$, $g^m = \beta$.

(2) *Noting that* $\dfrac{1-aa^*}{m}$ *is an integer* (*because* $aa^* \equiv 1 \bmod m$), *let* l $(0 < l < cd)$ *be the integer such that* $l \equiv \dfrac{1-aa^*}{m} \bmod cd$. *Then* $\alpha^{a^*}\beta^l = g$.

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